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Special Cases of Generalized Leonardo Numbers: Modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo Numbers

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Abstract. In this paper, we define and investigate modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo sequences as special cases of the generalized Leonardo sequence. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between modified p-Leonardo, p-Leonardo-Lucas, p-Leonardo numbers and Fibonacci, Lucas numbers.

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1. Introduction

The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \ F_1 = 1,$$

and the sequence of Lucas numbers $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \ L_1 = 1.$$

The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences.

In [3], Catarino and Borges introduced a new sequence of numbers called Lenonardo numbers. They defined Lenonardo numbers as

$$l_n = l_{n-1} + l_{n-2} + 1, \quad n \geq 2,$$

with $l_0 = 1, l_1 = 1.$ Lenonardo sequences has been studied by many authors, see for example, [1,2,4,6,7].

In Soykan [6], two sequences related to Lenonardo sequence were defined: Modified Leonardo and Leonardo-Lucas numbers are defined as

$$G_n = G_{n-1} + G_{n-2} + 1, \quad \text{with} \ \ G_0 = 0, G_1 = 1, \ \ n \geq 2,$$

and

$$H_n = H_{n-1} + H_{n-2} - 1, \quad \text{with} \ \ H_0 = 3, H_1 = 2, \quad n \geq 2,$$

respectively. The sequences $\{G_n\}$, $\{H_n\}$ and $\{l_n\}$ satisfy the following third order linear recurences:

$$\begin{aligned} G_n &= 2G_{n-1} - G_{n-3}, \qquad G_0 = 0, G_1 = 1, G_2 = 2, \\ H_n &= 2H_{n-1} - H_{n-3}, \qquad H_0 = 3, H_1 = 2, H_2 = 4, \\ l_n &= 2l_{n-1} - l_{n-3}, \qquad l_0 = 1, l_1 = 1, l_2 = 3. \end{aligned}$$

In Soykan [6], generalized Leonardo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$W_n = 2W_{n-1} - W_{n-3} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-2)} - W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

Note that the sequences $\{G_n\}$, $\{H_n\}$ and $\{l_n\}$ are the special cases of the generalized Leonardo sequence $\{W_n\}$. Next, we list some properties of the sequence $\{W_n\}$, $\{G_n\}$, $\{H_n\}$ and $\{l_n\}$ as follows (see [6] for details).

- Binet formula of generalized Leonardo numbers can be given as

$$\begin{aligned} W_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{z_1\alpha^{n+1} - z_2\beta^{n+1}}{(\alpha-\beta)} - z_3 \end{aligned} \tag{1.2}$$

where

$$z_1 = W_2 - (2-\alpha)W_1 + (1-\alpha)W_0, \tag{1.3}$$

$$z_2 = W_2 - (2-\beta)W_1 + (1-\beta)W_0, \tag{1.4}$$

$$z_3 = W_2 - W_1 - W_0. \tag{1.5}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 2x^2 + 1 = \left(x^2 - x - 1\right)(x - 1) = 0.$$

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Moreover,

$$\begin{aligned}\alpha &= \frac{1+\sqrt{5}}{2},\\ \beta &= \frac{1-\sqrt{5}}{2},\\ \gamma &= 1.\end{aligned}$$

Note that

$$\begin{aligned}\alpha+\beta+\gamma &= 2,\\ \alpha\beta+\alpha\gamma+\beta\gamma &= 0,\\ \alpha\beta\gamma &= -1,\end{aligned}$$

or

$$\alpha+\beta=1,\ \alpha\beta=-1.$$

- For all integers n, modified Leonardo, Leonardo-Lucas and Leonardo numbers can be expressed using Binet's formulas as

$$\begin{aligned}G_n &= \frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}-1,\\ H_n &= \alpha^n+\beta^n+1,\\ l_n &= \frac{2(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta}-1,\end{aligned}$$

respectively. Note that Binet's formulas of Fibonacci and Lucas numbers, respectively, are

$$\begin{aligned}F_n &= \frac{\alpha^n}{\alpha-\beta}+\frac{\beta^n}{\beta-\alpha}=\frac{\alpha^n-\beta^n}{\alpha-\beta},\\ L_n &= \alpha^n+\beta^n,\end{aligned}$$

and so

$$G_n = F_{n+2}-1, \tag{1.6}$$

$$H_n = L_n+1, \tag{1.7}$$

$$l_n = 2F_{n+1}-1. \tag{1.8}$$

- We have the following interrelations:

$$5G_n = 3L_{n+1}+L_n-5, \tag{1.9}$$

$$H_n = 2F_{n+1}-F_n+1, \tag{1.10}$$

$$5l_n = 2L_{n+1}+4L_n-5. \tag{1.11}$$

- The ordinary generating function $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ of the generalized Leonardo sequence is given by

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1)x^2}{1 - 2x + x^3}.$$

- (Simson Formula of Generalized Leonardo Numbers) For all integers n, we have

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = (-1)^n(-W_2^3 + W_1^3 - W_0^3 + 2(2W_1W_2^2 + W_0W_1^2 - 2W_1^2W_2 + W_0^2W_2) - 3W_0W_1W_2).$$

- We now present a few special identities for the generalized Leonardo sequence $\{W_n\}$. For all integers n and m, the following identities hold:
  - (Catalan's identity of the generalized Leonardo sequence)
    $W_{n+m}W_{n-m} - W_n^2 = ((W_1 - W_0)F_{n+m+2} + (W_2 - 2W_1 + W_0)F_{n+m+1} - (W_2 - W_1 - W_0))((W_1 - W_0)F_{n-m+2} + (W_2 - 2W_1 + W_0)F_{n-m+1} - (W_2 - W_1 - W_0)) - ((W_1 - W_0)F_{n+2} + (W_2 - 2W_1 + W_0)F_{n+1} - (W_2 - W_1 - W_0))^2.$
  - (Cassini's identity of the generalized Leonardo sequence)
    $W_{n+1}W_{n-1} - W_n^2 = (-W_2^2 - W_1^2 + W_0^2 + 3W_1W_2 - W_0W_2 - W_0W_1)F_{n+1}^2 + (W_2^2 + W_1^2 - W_0^2 - 3W_1W_2 + W_0W_2 + W_0W_1)F_n^2 + (W_2 - 3W_1 + 2W_0)(W_2 - W_1 - W_0)F_{n+1} + (2W_2 - 5W_1 + 3W_0)(-W_2 + W_1 + W_0)F_n + (W_2^2 + W_1^2 - W_0^2 - 3W_1W_2 + W_0W_2 + W_0W_1)F_{n+1}F_n.$
  - (d'Ocagne's identity)
    $W_{m+1}W_n - W_mW_{n+1} = (W_0 - W_1)(W_0 + W_1 - W_2)(F_{n+1} - F_{m+1}) + (W_0 - 2W_1 + W_2)(W_0 + W_1 - W_2)(F_m - F_n) + (W_0^2 - W_1^2 - W_2^2 - W_0W_1 - W_0W_2 + 3W_1W_2)(F_nF_{m+1} - F_mF_{n+1}).$
  - (Melham's identity)
    $W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = (W_0^2 - W_1^2 - W_2^2 - W_0W_1 - W_0W_2 + 3W_1W_2)(-(W_1 - W_2)F_{n+1}^3 + (W_0 - W_1)F_n^3 + (W_0 - W_2)F_{n+1}F_n^2 - (W_0 - 2W_1 + W_2)F_{n+1}^2F_n) + (W_0 + W_1 - W_2)((5W_0 + 2W_1 - 7W_2)(W_0 + W_1 - 2W_2)F_{n+1}^2 + (W_0 - W_2)(2W_0 + 3W_1 - 5W_2)F_n^2 + (7W_0^2 + 3W_1^2 + 17W_2^2 + 7W_0W_1 - 21W_0W_2 - 13W_1W_2)F_{n+1}F_n) + (W_0 + W_1 - W_2)^2(-(4W_0 + 3W_1 - 7W_2)F_{n+1} - (3W_0 + W_1 - 4W_2)F_n).$
- For $n \in \mathbb{Z}$, generalized Leonardo numbers have the following identity:

$$W_{-n} = (-1)^{-n}(W_{2n} - H_nW_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

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- If we define

$$\begin{aligned} A &= \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ B_n &= \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}, \end{aligned}$$

  then

$$B_n = A^n$$

  i.e.,

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}.$$

- For all integers m, n, we have

$$W_{n+m} = W_n G_{m+1} - W_{n-1} G_{m-1} - W_{n-2} G_m. \tag{1.12}$$

- For all integers m, n, we have

$$\begin{aligned} & (W_0 + W_1 - W_2)(W_0^2 - W_1^2 - W_2^2 - W_0W_1 - W_0W_2 + 3W_1W_2)W_{n+m} \\ = \; & W_n(-(W_0^2 - 2W_1^2 + W_1W_2)W_{m+3} + (W_2^2 + W_0W_1 - 2W_1W_2)W_{m+2} + (-W_1^2 + W_0W_2)W_{m+1}) \\ & -W_{n-1}(-(W_0^2 - 2W_1^2 + W_1W_2)W_{m+1} + (W_2^2 + W_0W_1 - 2W_1W_2)W_m + (-W_1^2 + W_0W_2)W_{m-1}) \\ & -W_{n-2}(-(W_0^2 - 2W_1^2 + W_1W_2)W_{m+2} + (W_2^2 + W_0W_1 - 2W_1W_2)W_{m+1} + (-W_1^2 + W_0W_2)W_m). \end{aligned}$$

2. Special Cases of Generalized Leonardo Sequence

Now, we define new three sequences related to modified Leonardo, Leonardo-Lucas and Leonardo sequences: Modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers are defined as

$$\begin{aligned} G_{p,n} &= G_{p,n-1} + G_{p,n-2} + p, \quad G_{p,0} = 0, G_{p,1} = 1, \quad n \geq 2, \\ H_{p,n} &= H_{p,n-1} + H_{p,n-2} - p, \quad H_{p,0} = 3, H_{p,1} = p+1, \quad n \geq 2, \\ l_{p,n} &= l_{p,n-1} + l_{p,n-2} + p, \quad l_{p,0} = 1, l_{p,1} = 1 \quad n \geq 2, \end{aligned}$$

where p is an arbitrary non-zero complex (or real) number.

The sequences $\{G_{p,n}\}$, $\{H_{p,n}\}$ and $\{l_{p,n}\}$ satisfy the following third order linear recurences:

$$\begin{aligned} G_{p,n} &= 2G_{p,n-1} - G_{p,n-3}, \quad G_{p,0} = 0, G_{p,1} = 1, G_{p,2} = p+1, \\ H_{p,n} &= 2H_{p,n-1} - H_{p,n-3}, \quad H_{p,0} = 3, H_{p,1} = p+1, H_{p,2} = 4, \\ l_{p,n} &= 2l_{p,n-1} - l_{p,n-3}, \quad l_{p,0} = 1, l_{p,1} = 1, l_{p,2} = p+2. \end{aligned}$$

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Note that

$$\begin{aligned} G_{1,n} &= G_n, \\ H_{1,n} &= H_n, \\ l_{1,n} &= l_n, \end{aligned}$$

and note also that the sequences $\{G_{p,n}\}$, $\{H_{p,n}\}$ and $\{l_{p,n}\}$ are the special cases of the generalized Leonardo sequence $\{W_n\}$. So, we can use all the results given in [6].

Next, we present the first few values of the modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers with positive and negative subscripts:

Table 1. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9
$G_{p,n}$	0	1	$p+1$	$2p+2$	$4p+3$	$7p+5$	$12p+8$	$20p+13$	$33p+21$	$54p+34$
$G_{p,-n}$		$1-p$	-1	$2-2p$	$p-3$	$5-4p$	$4p-8$	$13-9p$	$12p-21$	$34-22p$
$H_{p,n}$	3	$p+1$	4	5	$9-p$	$14-2p$	$23-4p$	$37-7p$	$60-12p$	$97-20p$
$H_{p,-n}$		$2p-2$	$5-p$	$4p-7$	$12-4p$	$9p-19$	$31-12p$	$22p-50$	$81-33p$	$56p-131$
$l_{p,n}$	1	1	$p+2$	$2p+3$	$4p+5$	$7p+8$	$12p+13$	$20p+21$	$33p+34$	$54p+55$
$l_{p,-n}$		$-p$	1	$-2p-1$	$p+2$	$-4p-3$	$4p+5$	$-9p-8$	$12p+13$	$-22p-21$

For all integers n, modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers can be expressed using Binet's formulas as

$$\begin{aligned} G_{p,n} &= \frac{(\alpha+p-1)}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{(\beta+p-1)}{(\beta-\alpha)(\beta-1)}\beta^n - p, \\ H_{p,n} &= \frac{5-2\alpha+p(\alpha-2)}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{5-2\beta+p(\beta-2)}{(\beta-\alpha)(\beta-1)}\beta^n + p, \\ l_{p,n} &= \frac{1+p}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{1+p}{(\beta-\alpha)(\beta-1)}\beta^n - p, \end{aligned}$$

respectively.

We now present generating functions.

Corollary 1. *The ordinary generating functions of modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers are given as*

$$\begin{aligned} \sum_{n=0}^{\infty} G_{p,n}x^n &= \frac{x+(p-1)\,x^2}{1-2x+x^3}, \\ \sum_{n=0}^{\infty} H_{p,n}x^n &= \frac{3+(p-5)\,x+(2-2p)\,x^2}{1-2x+x^3}, \\ \sum_{n=0}^{\infty} l_{p,n}x^n &= \frac{1-x+px^2}{1-2x+x^3}, \end{aligned}$$

respectively.

Next, we give Simson's formulas.

COROLLARY 2. *For all integers n, Simson's formulas of modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers are given as*

$$\begin{vmatrix} G_{p,n+2} & G_{p,n+1} & G_{p,n} \\ G_{p,n+1} & G_{p,n} & G_{p,n-1} \\ G_{p,n} & G_{p,n-1} & G_{p,n-2} \end{vmatrix} = p\left(-p^2+p+1\right)(-1)^n,$$

$$\begin{vmatrix} H_{p,n+2} & H_{p,n+1} & H_{p,n} \\ H_{p,n+1} & H_{p,n} & H_{p,n-1} \\ H_{p,n} & H_{p,n-1} & H_{p,n-2} \end{vmatrix} = p\left(p^2-7p+11\right)(-1)^n,$$

$$\begin{vmatrix} l_{p,n+2} & l_{p,n+1} & l_{p,n} \\ l_{p,n+1} & l_{p,n} & l_{p,n-1} \\ l_{p,n} & l_{p,n-1} & l_{p,n-2} \end{vmatrix} = p\left(p+1\right)^2(-1)^{n+1},$$

respectively.

3. Some Identities

In this section, we obtain some identities of modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{G_{p,n}\}$.

LEMMA 3. *The following equalities are true:*

- **(a):** $(-p^3+p^2+p)W_n = (-W_0p^2+W_1p+W_0+W_1-W_2)G_{p,n+2} - (W_0+W_1-W_2+2pW_1-pW_2-2p^2W_0+p^2W_1)G_{p,n+1} - (W_0+W_1-W_2+pW_0-2p^2W_1+p^2W_2)G_{p,n}$.
- **(b):** $(W_0^3-2W_0^2W_2-2W_0W_1^2+3W_0W_1W_2-W_1^3+4W_1^2W_2-4W_1W_2^2+W_2^3)G_{p,n} = -(W_0^2-W_1^2-W_0W_2+W_1W_2-pW_1^2+pW_0W_2)W_{n+2}+(W_0^2+W_2^2+W_0W_1-2W_0W_2-W_1W_2-pW_0^2+2pW_0W_2-pW_1W_2)W_{n+1} - (W_1^2+W_2^2+W_0W_1-W_0W_2-2W_1W_2-pW_2^2-pW_0W_1+2pW_1W_2)W_n$.

Proof. Note that all the identities hold for all integers p, n. We prove (a). To show (a), writing

$$W_n = a\times G_{p,n+2} + b\times G_{p,n+1} + c\times G_{p,n}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a\times G_{p,2} + b\times G_{p,1} + c\times G_{p,0} \\ W_1 &= a\times G_{p,3} + b\times G_{p,2} + c\times G_{p,1} \\ W_2 &= a\times G_{p,4} + b\times G_{p,3} + c\times G_{p,2} \end{aligned}$$

we find that

$$\begin{aligned} a &= \frac{1}{-p^3+p^2+p}(-W_0p^2+W_1p+W_0+W_1-W_2), \\ b &= -\frac{1}{-p^3+p^2+p}(W_0+W_1-W_2+2pW_1-pW_2-2p^2W_0+p^2W_1), \\ c &= -\frac{1}{-p^3+p^2+p}(W_0+W_1-W_2+pW_0-2p^2W_1+p^2W_2). \end{aligned}$$

The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{H_{p,n}\}$.

Lemma 4. *The following equalities are true:*

(a): $p(p^2-7p+11)W_n = -(11W_0+11W_1-11W_2-5pW_0-4pW_1+2pW_2+p^2W_2)H_{p,n+2}+(11W_0+11W_1-11W_2-8pW_0-3pW_1+4pW_2+p^2W_0)H_{p,n+1}+(11W_0+11W_1-11W_2-4pW_0-8pW_1+5pW_2+p^2W_1)H_{p,n}$.

(b): $(-W_0^3+2W_0^2W_2+2W_0W_1^2-3W_0W_1W_2+W_1^3-4W_1^2W_2+4W_1W_2^2-W_2^3)H_{p,n} = (W_0^2-4W_1^2-3W_2^2-3W_0W_1+2W_0W_2+7W_1W_2+pW_0^2-2pW_0W_2+pW_1W_2)W_{n+2}+(2W_0^2+3W_1^2+5W_2^2+5W_0W_1-7W_0W_2-8W_1W_2-2pW_0^2-pW_2^2-pW_0W_1+4pW_0W_2)W_{n+1}+(W_1^2-3W_0^2-2W_2^2-2W_0W_1+5W_0W_2+W_1W_2+pW_1^2+2pW_2^2+2pW_0W_1-pW_0W_2-4pW_1W_2)W_n$

Now, we give a few basic relations between $\{W_{p,n}\}$ and $\{l_{p,n}\}$.

Lemma 5. *The following equalities are true:*

(a): $p(p+1)W_n = (-W_2+W_1+(1+p)W_0)l_{p,n+2}+(W_2+(p-1)W_1-(2p+1)W_0)l_{p,n+1}+((p+1)W_2-(2p+1)W_1-W_0)l_{p,n}$.

(b): $(W_0^3-2W_0^2W_2-2W_0W_1^2+3W_0W_1W_2-W_1^3+4W_1^2W_2-4W_1W_2^2+W_2^3)l_{p,n} = (2W_1^2-W_0^2+W_2^2+W_0W_1-3W_1W_2+pW_1^2-pW_0W_2)W_{n+2}-(W_1^2+W_2^2+W_0W_1-W_0W_2-2W_1W_2+pW_0^2-2pW_0W_2+pW_1W_2)W_{n+1}+(W_0^2-W_1^2-W_0W_2+W_1W_2+pW_2^2+pW_0W_1-2pW_1W_2)W_n$.

Next, we present a few basic relations between $\{G_{p,n}\}$ and $\{H_{p,n}\}$.

Lemma 6. *The following equalities are true*

$$\begin{aligned} (p^2-7p+11)G_{p,n} &= -2(2p-5)H_{p,n+4}+2(p-3)H_{p,n+3}-(p^2-9p+15)H_{p,n+2}, \\ (p^2-7p+11)G_{p,n} &= -2(3p-7)H_{p,n+3}-(p^2-9p+15)H_{p,n+2}+(4p-10)H_{p,n+1}, \\ (p^2-7p+11)G_{p,n} &= -(p^2+3p-13)H_{p,n+2}+(4p-10)H_{p,n+1}+(6p-14)H_{p,n}, \\ (p^2-7p+11)G_{p,n} &= -2(p^2+p-8)H_{p,n+1}+(6p-14)H_{p,n}+(p^2+3p-13)H_{p,n-1}, \\ (p^2-7p+11)G_{p,n} &= 2(-2p^2+p+9)H_{p,n}+(p^2+3p-13)H_{p,n-1}+2(p^2+p-8)H_{p,n-2}, \end{aligned}$$

and

$$\begin{aligned}
(-p^2+p+1)H_{p,n} &= -(-p^2+3p+1)G_{p,n+4}+(-4p^2+8p+6)G_{p,n+3}-2(-2p^2+3p+3)G_{p,n+2}, \\
(-p^2+p+1)H_{p,n} &= 2(-p^2+p+2)G_{p,n+3}-2(-2p^2+3p+3)G_{p,n+2}+(-p^2+3p+1)G_{p,n+1}, \\
(-p^2+p+1)H_{p,n} &= -2(p-1)G_{p,n+2}+(-p^2+3p+1)G_{p,n+1}-2(-p^2+p+2)G_{p,n}, \\
(-p^2+p+1)H_{p,n} &= -(p^2+p-5)G_{p,n+1}-2(-p^2+p+2)G_{p,n}+2(p-1)G_{p,n-1}, \\
(-p^2+p+1)H_{p,n} &= -2(2p-3)G_{p,n}+2(p-1)G_{p,n-1}+(p^2+p-5)G_{p,n-2}.
\end{aligned}$$

Next, we give a few basic relations between $\{G_{p,n}\}$ and $\{l_{p,n}\}$.

LEMMA 7. *The following equalities are true*

$$\begin{aligned}
(p+1)G_{p,n} &= -2l_{p,n+4}-(p-4)l_{p,n+3}+(2p-1)l_{p,n+2}, \\
(p+1)G_{p,n} &= -pl_{p,n+3}+(2p-1)l_{p,n+2}+2l_{p,n+1}, \\
(p+1)G_{p,n} &= -l_{p,n+2}+2l_{p,n+1}+pl_{p,n}, \\
(p+1)G_{p,n} &= pl_{p,n}+l_{p,n-1},
\end{aligned}$$

and

$$\begin{aligned}
(-p^2+p+1)l_{p,n} &= -(2p+1)G_{p,n+4}+(p^2+4p+2)G_{p,n+3}-p(2p+1)G_{p,n+2}, \\
(-p^2+p+1)l_{p,n} &= p^2G_{p,n+3}-p(2p+1)G_{p,n+2}+(2p+1)G_{p,n+1}, \\
(-p^2+p+1)l_{p,n} &= -pG_{p,n+2}+(2p+1)G_{p,n+1}-p^2G_{p,n}, \\
(-p^2+p+1)l_{p,n} &= G_{p,n+1}-p^2G_{p,n}+pG_{p,n-1}, \\
(-p^2+p+1)l_{p,n} &= -(p^2-2)G_{p,n}+pG_{p,n-1}-G_{p,n-2}.
\end{aligned}$$

Now, we present a few basic relations between $\{H_{p,n}\}$ and $\{l_{p,n}\}$.

LEMMA 8. *The following equalities are true*

$$\begin{aligned}
(p+1)H_{p,n} &= -(p-6)l_{p,n+4}+(4p-13)l_{p,n+3}-2(2p-3)l_{p,n+2}, \\
(p+1)H_{p,n} &= (2p-1)l_{p,n+3}-2(2p-3)l_{p,n+2}+(p-6)l_{p,n+1}, \\
(p+1)H_{p,n} &= 4l_{p,n+2}+(p-6)l_{p,n+1}-(2p-1)l_{p,n}, \\
(p+1)H_{p,n} &= (p+2)l_{p,n+1}-(2p-1)l_{p,n}-4l_{p,n-1}, \\
(p+1)H_{p,n} &= 5l_{p,n}-4l_{p,n-1}-(p+2)l_{p,n-2},
\end{aligned}$$

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and

$$\begin{aligned}
(p^2-7p+11)l_{p,n} &= -(5p-14)H_{p,n+4}+(4p-15)H_{p,n+3}-(p^2-8p+10)H_{p,n+2},\\
(p^2-7p+11)l_{p,n} &= -(6p-13)H_{p,n+3}-(p^2-8p+10)H_{p,n+2}+(5p-14)H_{p,n+1},\\
(p^2-7p+11)l_{p,n} &= -(p^2+4p-16)H_{p,n+2}+(5p-14)H_{p,n+1}+(6p-13)H_{p,n},\\
(p^2-7p+11)l_{p,n} &= -(2p^2+3p-18)H_{p,n+1}+(6p-13)H_{p,n}+(p^2+4p-16)H_{p,n-1},\\
(p^2-7p+11)l_{p,n} &= -(4p^2-23)H_{p,n}+(p^2+4p-16)H_{p,n-1}+(2p^2+3p-18)H_{p,n-2}.
\end{aligned}$$

We can give a few basic identities by using Lemma 3.

Corollary 9. *The following identities hold:*

- **(a):** $(-p^3+p^2+p)G_n=(p-1)\,G_{p,n+2}-\left(p^2-1\right)G_{p,n+1}+G_{p,n}$.
- **(b):** $G_{p,n}=-(p-1)\,G_{n+2}+(2p-2)\,G_{n+1}+G_n$.
- **(c):** $(-p^3+p^2+p)H_n=(-3p^2+2p+1)G_{p,n+2}+(4p^2-1)G_{p,n+1}-(3p+1)G_{p,n}$.
- **(d):** $5G_{p,n}=(8p+1)H_{n+2}-(7p-1)H_{n+1}-(6p+2)H_n$.
- **(e):** $(-p^3+p^2+p)l_n=-(p^2-p+1)G_{p,n+2}+(p^2+p+1)G_{p,n+1}-(p^2+p-1)G_{p,n}$.
- **(f):** $4G_{p,n}=-2pl_{n+2}+(2p+2)l_{n+1}+(4p-2)l_n$.

Proof.

- **(a) and (b):** Take $W_n=G_{1,n}=G_n$ in Lemma 3.
- **(c) and (d):** Take $W_n=H_{1,n}=H_n$ in Lemma 3.
- **(e) and (f):** Take $W_n=l_{1,n}=l_n$ in Lemma 3. □

We now present a few basic identities as a result of Lemma 4.

Corollary 10. *The following identities hold:*

- **(a):** $p(p^2-7p+11)G_n=-(2p^2-11)H_{p,n+2}+(5p-11)H_{p,n+1}+(p^2+2p-11)H_{p,n}$.
- **(b):** $H_{p,n}=(2p-2)G_{n+2}-(4p-7)G_{n+1}+(p-5)G_n$.
- **(c):** $p(p^2-7p+11)H_n=-(4p^2-15p+11)H_{p,n+2}+(3p^2-14p+11)H_{p,n+1}+(2p^2-8p+11)H_{p,n}$.
- **(d):** $5H_{p,n}=-(7p-7)H_{n+2}+(8p-8)H_{n+1}+(4p+1)H_n$.
- **(e):** $p(p^2-7p+11)l_n=(-3p^2+3p+11)H_{p,n+2}+(p^2+p-11)H_{p,n+1}+(p^2+3p-11)H_{p,n}$.
- **(f):** $2H_{p,n}=(p+3)l_{n+2}-5l_{n+1}-(3p-2)l_n$.

Proof.

- **(a) and (b):** Take $W_n=G_{1,n}=G_n$ in Lemma 4.
- **(c) and (d):** Take $W_n=H_{1,n}=H_n$ in Lemma 4.
- **(e) and (f):** Take $W_n=l_{1,n}=l_n$ in Lemma 4. □

We next give a few basic identities by using Lemma 5.

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COROLLARY 11. *The following identities hold:*

(a): $p(p+1)G_n = -l_{p,n+2} + (p+1)\, l_{p,n+1} + l_{p,n}$.

(b): $l_{p,n} = -pG_{n+2} + (2p+1)G_{n+1} - G_n$.

(c): $p(p+1)H_n = (3p+1)l_{p,n+2} - (4p+1)l_{p,n+1} - l_{p,n}$.

(d): $5l_{p,n} = (8p+3)H_{n+2} - (7p+2)H_{n+1} - (6p+1)H_n$.

(e): $p(p+1)l_n = (p-1)l_{p,n+2} - (p-1)l_{p,n+1} + (p+1)l_{p,n}$.

(f): $4l_{p,n} = -(2p-2)l_{n+2} + (2p-2)l_{n+1} + 4pl_n$.

Proof.

(a) and (b): Take $W_n = G_{1,n} = G_n$ in Lemma 5.

(c) and (d): Take $W_n = H_{1,n} = H_n$ in Lemma 5.

(e) and (f): Take $W_n = l_{1,n} = l_n$ in Lemma 5. $\square$

4. Relations Betwen Special Numbers

In this section, we present identities on p-modified Leonardo, p-Leonardo-Lucas and p-Leonardo numbers and Fibonacci and Lucas numbers.

LEMMA 12. *For all integers n, we have the following identities:*

(a):

$$G_{p,n} = pF_{n+1} + F_n - p, \tag{4.1}$$

$$5G_{p,n} = (p+2)L_{n+1} + (2p-1)L_n - 5p. \tag{4.2}$$

(b):

$$H_{p,n} = (3-p)F_{n+1} + (p-2)F_n + p, \tag{4.3}$$

$$5H_{p,n} = (p-1)L_{n+1} + (8-3p)L_n + 5p. \tag{4.4}$$

(c):

$$l_{p,n} = pF_{n+1} + F_{n+1} - p, \tag{4.5}$$

$$5l_{p,n} = (p+1)L_{n+1} + 2(p+1)L_n - 5p. \tag{4.6}$$

Proof. Note that

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n, \\ F_{n+3} &= 2F_{n+1} + F_n, \\ F_{n+4} &= 3F_{n+1} + 2F_n, \\ L_{n+2} &= L_{n+1} + L_n, \\ L_{n+3} &= 2L_{n+1} + L_n. \end{aligned}$$

We know that

$$\begin{aligned} G_n &= F_{n+2} - 1, \\ H_n &= L_n + 1, \\ l_n &= 2F_{n+1} - 1. \end{aligned}$$

We also know from Corollaries 9, 10 and 11 that

$$\begin{aligned} G_{p,n} &= -(p-1)\,G_{n+2} + (2p-2)\,G_{n+1} + G_n, \\ 5G_{p,n} &= (8p+1)H_{n+2} - (7p-1)H_{n+1} - (6p+2)H_n, \\ 4G_{p,n} &= -2pl_{n+2} + (2p+2)l_{n+1} + (4p-2)l_n, \end{aligned}$$

and

$$\begin{aligned} H_{p,n} &= (2p-2)G_{n+2} - (4p-7)G_{n+1} + (p-5)G_n, \\ 5H_{p,n} &= -(7p-7)H_{n+2} + (8p-8)H_{n+1} + (4p+1)H_n, \\ 2H_{p,n} &= (p+3)l_{n+2} - 5l_{n+1} - (3p-2)l_n, \end{aligned}$$

and

$$\begin{aligned} l_{p,n} &= -pG_{n+2} + (2p+1)G_{n+1} - G_n, \\ 5l_{p,n} &= (8p+3)H_{n+2} - (7p+2)H_{n+1} - (6p+1)H_n, \\ 4l_{p,n} &= -(2p-2)l_{n+2} + (2p-2)l_{n+1} + 4pl_n, \end{aligned}$$

Now, use the above identities. □

5. Special Identities

As special cases of Catalan's identity of the generalized Leonardo sequence, we have the following corollary.

Corollary 13. *For all integers n and m, the following identities hold:*

- **(a):** $G_{p,n+m}G_{p,n-m} - G_{p,n}^2 = (F_{n+m+2} + (p-1)F_{n+m+1} - p)(F_{n-m+2} + (p-1)F_{n-m+1} - p) - (F_{n+2} + (p-1)F_{n+1} - p)^2$
- **(b):** $H_{p,n+m}H_{p,n-m} - H_{p,n}^2 = ((p-2)F_{n+m+2} + (5-2p)F_{n+m+1} + p)((p-2)F_{n-m+2} + (5-2p)F_{n-m+1} + p) - ((p-2)F_{n+2} + (5-2p)F_{n+1} + p)^2$
- **(c):** $l_{p,n+m}l_{p,n-m} - l_{p,n}^2 = -(p+1)(pF_{m+n+1} + pF_{n-m+1} - (p+1)\,F_{-m+n+1}F_{m+n+1} + (p+1)\,F_{n+1}^2 - 2pF_{n+1})$

As special cases of Cassini's identity of the generalized Leonardo sequence, we have the following corollary.

Corollary 14. *For all integers n, the following identities hold:*

(a): $G_{p,n+1}G_{p,n-1}-G_{p,n}^2=(-p^2+p+1)F_{n+1}^2+(p^2-p-1)F_n^2+(p^2-p-1)F_nF_{n+1}+p(p-2)F_{n+1}-p(2p-3)F_n$.

(b): $H_{p,n+1}H_{p,n-1}-H_{p,n}^2=(p^2-7p+11)F_n^2+(p^2-7p+11)F_nF_{n+1}-p(5p-12)F_n+((-p^2+7p-11)F_{n+1}^2+p(3p-7)F_{n+1}$.

(c): $l_{p,n+1}l_{p,n-1}-l_{p,n}^2=-(p+1)^2F_{n+1}^2+(p+1)^2F_n^2+(p+1)^2F_nF_{n+1}+p(p+1)F_{n+1}-p(2p+2)F_n$.

As special cases of the d'Ocagne's and Melham's identities, we have the following three corollaries. First one presents d'Ocagne's and Melham's identities of modified p-Leonardo sequence $\{G_{p,n}\}$.

Corollary 15. *Let n and m be any integers. Then the following identities are true:*

(a): *(d'Ocagne's identity)*

$G_{p,m+1}G_{p,n}-G_{p,m}G_{p,n+1}=(-p^2+p+1)(F_nF_{m+1}-F_mF_{n+1})-p(F_{m+1}-F_{n+1})-p(p-1)(F_m-F_n)$.

(b): *(Melham's identity)*

$G_{p,n+1}G_{p,n+2}G_{p,n+6}-G_{p,n+3}^3=-p(p^2-p-1)F_{n+1}^3+(p^2-p-1)F_n^3-p(7p+5)(2p+1)F_{n+1}^2-p(p+1)(5p+2)F_n^2+(p-1)(p^2-p-1)F_{n+1}^2F_n+(p+1)(p^2-p-1)F_{n+1}F_n^2-p(17p^2+21p+7)F_{n+1}F_n+p^2(7p+4)F_{n+1}+p^2(4p+3)F_n$.

Second one presents d'Ocagne's and Melham's identities of p-Leonardo-Lucas sequence $\{H_{p,n}\}$.

Corollary 16. *Let n and m be any integers. Then the following identities are true:*

(a): *(d'Ocagne's identity)*

$H_{p,m+1}H_{p,n}-H_{p,m}H_{p,n+1}=-p(p-2)(F_{n+1}-F_{m+1})-p(2p-5)(F_m-F_n)+(-p^2+7p-11)(F_nF_{m+1}-F_mF_{n+1})$.

(b): *(Melham's identity)*

$H_{p,n+1}H_{p,n+2}H_{p,n+6}-H_{p,n+3}^3=(p-3)(p^2-7p+11)F_{n+1}^3+(p-2)(p^2-7p+11)F_n^3+p(p-4)(2p-11)F_{n+1}^2-p(3p-11)F_n^2-(2p-5)(p^2-7p+11)F_{n+1}^2F_n+(p^2-7p+11)F_{n+1}F_n^2+p(3p^2-25p+55)F_{n+1}F_n-p^2(3p-13)F_{n+1}-p^2(p-6)F_n$.

Third one presents d'Ocagne's and Melham's identities of p-Leonardo sequence $\{l_{p,n}\}$.

Corollary 17. *Let n and m be any integers. Then the following identities are true:*

(a): *(d'Ocagne's identity)*

$l_{m+1}l_{p,n}-l_ml_{p,n+1}=-(p+1)^2(F_nF_{m+1}-F_mF_{n+1})-p(p+1)(F_m-F_n)$.

(b): *(Melham's identity)*

$l_{p,n+1}l_{p,n+2}l_{p,n+6}-l_{p,n+3}^3=-(p+1)((p+1)^2F_{n+1}^3+14p(p+1)F_{n+1}^2+5p(p+1)F_n^2-(p+1)^2F_{n+1}^2F_n-(p+1)^2F_{n+1}F_n^2+17p(p+1)F_{n+1}F_n-7p^2F_{n+1}-4p^2F_n)$.

6. On the Recurrence Properties of Special Cases of Generalized Leonardo Sequence

We present modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers at the negative index. Here, $H_n = H_{1,n}$.

Corollary 18. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *modified p-Leonardo sequence:*

$$G_{p,-n} = (-1)^{-n}(G_{p,2n} - H_n G_{p,n}).$$

(b): *p-Leonardo-Lucas sequence:*

$$H_{p,-n} = (-1)^{-n}(H_{p,2n} - H_n H_{p,n} + \frac{3}{2}(H_n^2 - H_{2n})).$$

(c): *p-Leonardo sequence:*

$$l_{p,-n} = \frac{1}{2}(-1)^{-n}\left(H_n^2 - H_{2n} + 2l_{p,2n} - 2H_n l_{p,n}\right).$$

Note that since

$$\begin{aligned} G_{p,n} &= pF_{n+1} + F_n - p, \\ H_{p,n} &= (3-p)F_{n+1} + (p-2)F_n + p, \\ l_{p,n} &= pF_{n+1} + F_{n+1} - p, \end{aligned}$$

and

$$F_{-n} = (-1)^{n+1} F_n,$$

we get

$$\begin{aligned} G_{p,-n} &= (-1)^n (pF_{n-1} - F_n) - p, \\ H_{p,-n} &= (-1)^n ((3-p)F_{n-1} - (p-2)F_n) + p, \\ l_{p,-n} &= (p+1)(-1)^n F_{n-1} - p. \end{aligned}$$

7. Sums

The following Corollary gives sum formulas of Fibonacci and Lucas numbers.

Corollary 19. *For $n \geq 0$, Fibonacci and Lucas numbers have the following properties:*

(1)

(a): $\sum_{k=0}^{n} F_k = 2F_n + F_{n-1} - 1$.

(b): $\sum_{k=0}^{n} F_{2k} = 2F_{2n} - F_{2n-2} - 1$.

(c): $\sum_{k=0}^{n} F_{2k+1} = 2F_{2n+1} - F_{2n-1}$.

(2)

(a): $\sum_{k=0}^{n} L_k = 2L_n + L_{n-1} - 1.$

(b): $\sum_{k=0}^{n} L_{2k} = 2L_{2n} - L_{2n-2} + 1.$

(c): $\sum_{k=0}^{n} L_{2k+1} = 2L_{2n+1} - L_{2n-1} - 2.$

Proof. It is given in Soykan [5, Corollary 4.5.]. □

The following Corollary presents sum formulas of modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers.

Corollary 20. *For $n \geq 0$, modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers have the following properties:*

(1)

(a): $\sum_{k=0}^{n} G_{p,k} = (2p+1)F_{n+1} + (p+1)F_n - (n+2)p - 1.$

(b): $\sum_{k=0}^{n} G_{p,2k} = (p+1)F_{2n+1} + pF_{2n} - (n+1)p - 1.$

(c): $\sum_{k=0}^{n} G_{p,2k+1} = (2p+1)F_{2n+1} + (p+1)F_{2n} - (n+2)p.$

(2)

(a): $5\sum_{k=0}^{n} H_{p,k} = (6-p)L_{n+1} + (7-2p)L_n + 5(n+1)p - 5.$

(b): $5\sum_{k=0}^{n} H_{p,2k} = (7-2p)L_{2n+1} + (p-1)L_{2n} + 5np + 10.$

(c): $5\sum_{k=0}^{n} H_{p,2k+1} = (6-p)L_{2n+1} + (7-2p)L_{2n} + 5(n+2)p - 15.$

(3)

(a): $\sum_{k=0}^{n} l_{p,k} = 2(p+1)F_{n+1} + (p+1)F_n - (n+2)p - 1.$

(b): $\sum_{k=0}^{n} l_{p,2k} = (p+1)F_{2n+1} + (p+1)F_{2n} - (n+1)p.$

(c): $\sum_{k=0}^{n} l_{p,2k+1} = 2(p+1)F_{2n+1} + (p+1)F_{2n} - (n+2)p - 1.$

Proof. The proof follows from Corollary 19 and the identities (4.2), (4.4) and (4.6), i.e.,

$$\begin{aligned} G_{p,n} &= pF_{n+1} + F_n - p, \\ 5H_{p,n} &= (p-1)L_{n+1} + (8-3p)L_n + 5p, \\ l_{p,n} &= pF_{n+1} + F_{n+1} - p. \quad \square \end{aligned}$$

8. Matrices and Some Identities Related With Generalized p-Leonardo Numbers

Next, we present formulas for the p-modified Leonardo, p-Leonardo-Lucas and p-Leonardo numbers.

Corollary 21. *For all integers n, we have the following formulas for the p-modified Leonardo, p-Leonardo-Lucas and p-Leonardo numbers.*

(a): *Modified p-Leonardo Numbers.*

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{-p^3+p^2+p}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$\begin{aligned} a_{11} &= (p-1)\,G_{p,n+3} - \left(p^2-1\right)G_{p,n+2} + G_{p,n+1} \\ a_{21} &= (p-1)\,G_{p,n+2} - \left(p^2-1\right)G_{p,n+1} + G_{p,n} \\ a_{31} &= (p-1)\,G_{p,n+1} - \left(p^2-1\right)G_{p,n} + G_{p,n-1} \end{aligned}$$

$$\begin{aligned} a_{12} &= -((p-1)\,G_{p,n+1} - \left(p^2-1\right)G_{p,n} + G_{p,n-1}) \\ a_{22} &= -((p-1)\,G_{p,n} - \left(p^2-1\right)G_{p,n-1} + G_{p,n-2}) \\ a_{32} &= -((p-1)\,G_{p,n-1} - \left(p^2-1\right)G_{p,n-2} + G_{p,n-3}) \end{aligned}$$

$$\begin{aligned} a_{13} &= -((p-1)\,G_{p,n+2} - \left(p^2-1\right)G_{p,n+1} + G_{p,n}) \\ a_{23} &= -((p-1)\,G_{p,n+1} - \left(p^2-1\right)G_{p,n} + G_{p,n-1}) \\ a_{33} &= -((p-1)\,G_{p,n} - \left(p^2-1\right)G_{p,n-1} + G_{p,n-2}) \end{aligned}$$

(b): *p-Leonardo-Lucas Numbers.*

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{p(p^2-7p+11)}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$$\begin{aligned} b_{11} &= -(2p^2-11)H_{p,n+3} + (5p-11)H_{p,n+2} + (p^2+2p-11)H_{p,n+1} \\ b_{21} &= -(2p^2-11)H_{p,n+2} + (5p-11)H_{p,n+1} + (p^2+2p-11)H_{p,n} \\ b_{31} &= -(2p^2-11)H_{p,n+1} + (5p-11)H_{p,n} + (p^2+2p-11)H_{p,n-1} \end{aligned}$$

$$\begin{aligned} b_{12} &= -(-(2p^2-11)H_{p,n+1} + (5p-11)H_{p,n} + (p^2+2p-11)H_{p,n-1}) \\ b_{22} &= -(-(2p^2-11)H_{p,n} + (5p-11)H_{p,n-1} + (p^2+2p-11)H_{p,n-2}) \\ b_{32} &= -(-(2p^2-11)H_{p,n-1} + (5p-11)H_{p,n-2} + (p^2+2p-11)H_{p,n-3}) \end{aligned}$$

$$\begin{aligned} b_{13} &= -(-(2p^2-11)H_{p,n+2} + (5p-11)H_{p,n+1} + (p^2+2p-11)H_{p,n}) \\ b_{23} &= -(-(2p^2-11)H_{p,n+1} + (5p-11)H_{p,n} + (p^2+2p-11)H_{p,n-1}) \\ b_{33} &= -(-(2p^2-11)H_{p,n} + (5p-11)H_{p,n-1} + (p^2+2p-11)H_{p,n-2}) \end{aligned}$$

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(c): *p-Leonardo Numbers.*

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{p(p+1)} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

where

$$\begin{aligned} c_{11} &= -l_{p,n+3} + (p+1)\, l_{p,n+2} + l_{p,n+1} \\ c_{21} &= -l_{p,n+2} + (p+1)\, l_{p,n+1} + l_{p,n} \\ c_{31} &= -l_{p,n+1} + (p+1)\, l_{p,n} + l_{p,n-1} \end{aligned}$$

$$\begin{aligned} c_{12} &= -(-l_{p,n+1} + (p+1)\, l_{p,n} + l_{p,n-1}) \\ c_{22} &= -(-l_{p,n} + (p+1)\, l_{p,n-1} + l_{p,n-2}) \\ c_{32} &= -(-l_{p,n-1} + (p+1)\, l_{p,n-2} + l_{p,n-3}) \end{aligned}$$

$$\begin{aligned} c_{13} &= -(-l_{p,n+2} + (p+1)\, l_{p,n+1} + l_{p,n}) \\ c_{23} &= -(-l_{p,n+1} + (p+1)\, l_{p,n} + l_{p,n-1}) \\ c_{33} &= -(-l_{p,n} + (p+1)\, l_{p,n-1} + l_{p,n-2}) \end{aligned}$$

Proof. We know that

$$A^n = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}.$$

Now, use the following identities respectively:

$$\begin{aligned} (-p^3+p^2+p)G_n &= (p-1)\, G_{p,n+2} - \left(p^2-1\right) G_{p,n+1} + G_{p,n} \quad \text{(Corollary 9)}, \\ p(p^2-7p+11)G_n &= -(2p^2-11)H_{p,n+2} + (5p-11)H_{p,n+1} + (p^2+2p-11)H_{p,n} \quad \text{(Corollary 10)}, \\ p(p+1)G_n &= -l_{p,n+2} + (p+1)\, l_{p,n+1} + l_{p,n} \quad \text{(Corollary 11)}. \ \square \end{aligned}$$

In the following corollary, we give identities for the p-modified Leonardo, p-Leonardo-Lucas and p-Leonardo numbers.

Corollary 22. *For all integers m, n, we have*

$$\begin{aligned} G_{p,n+m} &= G_{p,n}G_{m+1} - G_{p,n-1}G_{m-1} - G_{p,n-2}G_m, \\ H_{p,n+m} &= H_{p,n}G_{m+1} - H_{p,n-1}G_{m-1} - H_{p,n-2}G_m, \\ l_{p,n+m} &= l_{p,n}G_{m+1} - l_{p,n-1}G_{m-1} - l_{p,n-2}G_m, \end{aligned}$$

where

$$\begin{aligned} G_{m-1} &= G_{1,m-1}, \\ G_{m} &= G_{1,m}, \\ G_{m+1} &= G_{1,m+1}, \end{aligned}$$

and

$$\begin{aligned} (p^3-p^2-p)G_{p,n+m} &= -((p-1)G_{p,m+3}+(1-p^2)G_{p,m+2}+G_{p,m+1})G_{p,n} \\ &\quad +((p-1)G_{p,m+1}+(1-p^2)G_{p,m}+G_{p,m-1})G_{p,n-1} \\ &\quad +((p-1)\,G_{p,m+2}+(1-p^2)G_{p,m+1}+G_{p,m})G_{p,n-2}, \end{aligned}$$

and

$$\begin{aligned} -p(p^2-7p+11)H_{p,n+m} &= -((11-2p^2)H_{p,m+3}+(5p-11)H_{p,m+2}+(p^2+2p-11)H_{p,m+1})H_{p,n} \\ &\quad +((11-2p^2)H_{p,m+1}+(5p-11)H_{p,m}+(p^2+2p-11)H_{p,m-1})H_{p,n-1} \\ &\quad +((11-2p^2)H_{p,m+2}+(5p-11)\,H_{p,m+1}+(p^2+2p-11)H_{p,m})H_{p,n-2}, \end{aligned}$$

and

$$\begin{aligned} p\,(p+1)^2\,l_{p,n+m} &= -\,(p+1)\,(l_{p,m+3}-(p+1)l_{p,m+2}-l_{p,m+1})l_{p,n} \\ &\quad +(p+1)\,(l_{p,m+1}-(p+1)l_{p,m}-l_{p,m-1})l_{p,n-1} \\ &\quad +(p+1)\,(l_{p,m+2}-(p+1)l_{p,m+1}-l_{p,m})l_{p,n-2}. \end{aligned}$$

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