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The Proof of a Conjecture for a Continuous Golomb Ruler Model

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Abstract

In this paper we study a conjecture proposed by P.Duxbury, C.laror, L.Leduino de Salles Neto in 2021[6] on the Golomb ruler problem which is a classical optimization model in the scope of discrete optimization. In [6] the authors constructed a continuous model for the Golomb Ruler Problem attached to the discrete case and conjectured that the optimal values of both cases are equal. We give a proof of this conjecture by using algebraic-geometry-theoretical methods.

Keywords: The Golomb Ruler Problem , Brauer Groups , Rational Approximation

Contents

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1 Introduction

The Golomb Ruler Problem is a classical optimization problem, named for Solomon W. Golomb and discovered independently by Sidon (1932)[25] and Babcock (1953)[1], a Golomb ruler is a finite set of marked points in a ruler at integer positions such that the distances between any two points are different. Furthermore, we say that a Golomb Ruler problem is to find the smallest length of the ruler such that the n marked points form a Golomb ruler. Next, we give the exact definition of a Golomb ruler.

Definition 1.1 (Golomb Rulers)**.** *Given a set of positive integers $G = \{a_1, a_2, \ldots, a_n\}$ where $a_1 = 0$ and $a_1 < a_2 < \cdots < a_n$, we say that G is a Golomb ruler if and only if for all $i, j, k, l \in \{1, 2, \ldots, n\}$ satisfying $a_i - a_j = a_k - a_l \iff i = j$ and $k = l$. And we call $L_n := a_n - a_0 = a_n$ the length of the given Golomb ruler.*

Given an order n and a length L_n of the Golomb ruler, the **optimal Golomb ruler** is referred to the Golomb ruler that is optimally short and exhibits minimal n for the specific value of L_n.

Definition 1.2 (Golomb Ruler Problem)**.** *Given $n \in \mathbb{Z}_{>0}$. The Golomb ruler problem asks for the minimum length $\tilde{L}_n$ of all Golomb rulers of order n. We call the Golomb ruler with the minimum length for n the* ***optimal Golomb ruler****.*

For example, for $n = 2$, the optimal Golomb ruler is $\{0, 1\}$ and for $n = 3$ the optimal Golomb ruler is $\{0, 1, 3\}$. One can find more examples of optimal Golomb rulers on Wikipedia. Moreover, Golomb ruler problem is not merely a purely combinatorial problem, it has many profound applications in areas such as astronomy [3], information theory helping to error correcting codes [1] [20], radio frequency selection helping to select radio frequencies [2][8]. It is also studied by number theorists like Paul Erdös out of pure mathematical interest [7].

To better understand the Golomb ruler problem, a general way is to reinterpret it as a discrete optimization problem; then we can make use of various powerful tools in mathematical programming[26][9]. One can solve the Golomb ruler problem in a brute-force way, that is, to apply the greedy algorithm to search the optimal Golomb ruler[5]. However, although not proven to be NP-hard, solving the Golomb ruler problem is believed by experts to be very challenging and sophisticated [5][18][24].

In [6] P.Duxbury and his cooperators established a discrete optimization model and a continuous one for the Golomb ruler problem. More surprisingly, they proposed a conjecture which predicated that the optimal value of the discrete model is equal to that of the continuous one. In this paper we study this conjecture utilizing algebraic geometrical techniques and some elementary arguments on rational approximation of real numbers.

In [14], a discrete model for the Golomb ruler problem was first presented but it will not be discussed in this paper. Given a positive integer n and an upper bound L_n for the length of the ruler, P. Duxbury, C.Lavor, L. Leduino de Salles-Neto [6] modify the form of modeling the Golomb ruler as follows:

$$\min_{x_i \in \{0,1\}} t$$

$$\begin{cases} ix_i \le t, \\ \sum_{i=1}^{L_n} x_i = n-1, \\ x_j + \sum_{i=1}^{L_n-j-1} x_i x_{i+j} \le 1, \quad j = 1, \dots, L_n - 1, \\ x_i \in \{0,1\}, i = 1, \dots, L_n. \end{cases} \tag{1}$$

In [6] a continuous model for the Golomb ruler is also established which can be viewed as a nonlinear relaxation for the above model1:

$$\min_{x_i \in [0,1]} t$$

$$\begin{cases} ix_i \le tx_i, \\ \sum_{i=1}^{L_n} x_i = n-1, \\ x_j + \sum_{i=1}^{L_n-j-1} x_i x_{i+j} \le 1, \quad j = 1, \dots, L_n - 1, \\ x_i \in [0,1], i = 1, \dots, L_n. \end{cases} \tag{2}$$

Moreover, they propose a fascinating conjecture:

Conjecture 1.3 (P. Duxbury, C.Lavor, L. Leduino de Salles-Neto [6])**.** *The optimal value of model (1) is identical to that of model (2) .*

To illustrate our strategy, it is inevitable to use some algebraic geometry and algebraic number theory terminologies. Our naive insight is to add some non-negative auxiliary parameters to each inequality in 1 to make them equal. Then we can use these equations to define a non-singular algebraic variety $\mathcal{X}_{L_n}$ that contains all the arithmetic and geometric information of 1, where the notation $\mathcal{X}(A)$ denotes the A-points of an arbitrary algebraic variety $\mathcal{X}$ and A is some $\mathbb{Q}$-algebra. Set L the optimal value of 1, and we directly know that $\mathcal{X}_{L-1}(\mathbb{Z}) = \varnothing$. If we further prove that $\mathcal{X}_{L-1}(\mathbb{R}) = \varnothing$, then Conjecture 1.3 is implied directly.

We divide the proof of this conjecture into three parts: (1)We regard the feasible regions of 2 and 1 as algebraic schemes after substituting $x_i = y_i^2$ and for n and L_n we write $\mathcal{X}_{L_n}$ to denote the associated schemes; (2) We intend to prove that $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$ where L is the optimal value for the discrete model 1 ; (3) Lastly we need to verify that $\mathcal{X}_{L-1}(\mathbb{Q})$ is dense in $\mathcal{X}_{L-1}(\mathbb{R})$ and it follows immediately that $\mathcal{X}_{L-1}(\mathbb{R}) = \varnothing$ which suffices to prove conjecture 1.3.

The most difficult and technical part turns out to be part (2), in which we utilize algebraic-geometry-theoretical techniques and tools. Given a global field k, the adèle ring of k is defined as the restricted product $\mathbb{A}_k := \prod_v' (k_v, \mathcal{O}_v) \subseteq \prod_v k_v$ consisting of tuples (a_v) where $a_v \in \mathcal{O}_v$ for all but finitely many places v of k, where $\mathcal{O}_v$ is the algebraic integer subring of k_v. More specifically, it suffices to prove that $\mathcal{X}_{L-1}(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \varnothing$ (we will explain the notation in Definition 2.18) which implies that $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$ (here $\mathbb{A}_\mathbb{Q}$ denotes the adèle ring of $\mathbb{Q}$, for a further theory of the adèle ring of a local field, see A. Weil's textbook [28] or J. Neukirch's lecture [17]). Hence we hope to find some Azumaya algebras $\{A_s\}_{s \in S}$ indexed by a finite index S such that $\bigcap_{s \in S} \mathcal{X}_{L-1}(\mathbb{A}_\mathbb{Q})^{A_s} = \varnothing$ (this notation will also be defined in Definition 2.18).

The idea of tackling part (3) is simple. If an $\mathbb{R}$-points P exists in $\mathcal{X}_{L-1}$, we can pick out a $\mathbb{Q}$-points Q very close to P which also satisfies constraints in 2. This leads to a contraction of the temporarily assumed fact that $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$.

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Organization. This paper is organized as follows : In **Section 2** we recall some notions and properties in the basic Brauer group theory and realize the goal of Part 2. In **Section 3** we finish Part 3.

2 The Manin-Brauer Obstruction for $\mathcal{X}_{L-1}$

The goal of this section is to prove that $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$. As mentioned in the Introduction part, the key insight is to find a finite set $S \subseteq \mathrm{Br}\mathcal{X}_{L-1}$ of Azumaya algebras over $\mathcal{X}_{L-1}$ such that $\bigcap_{A \subseteq S} \mathcal{X}_{L-1}(\mathbb{A}_{\mathbb{Q}})^A = \varnothing$.

2.1 Preliminary

Firstly, let us recall some basic algebraic geometry notions. The main references for this subsection are Hartshorne's textbook [12] and Poonen's lecture note [19]. The readers can also check [16], [23] [4] for more detailed illustrations and advanced topics on algebraic geometry and arithmetic theory. For more detailed and interesting descriptions of Brauer groups and Brauer-Manin obstruction, cf. [27], [21], and Grothendieck's original research on Brauer groups [11] and [10].

For a precise definition of schemes, the readers can check **Section I , II** in **Chapter 2** in [12].

2.1.1 A Brief Review of Brauer Group

The Brauer group of a field (and more generally a scheme) is an important invariant for studying the arithmetic and geometric information of the field(resp. the scheme). It was originally introduced by Richard Brauer and aims to classify the central simple algebras over a field. And it plays an important role in the construction of *Manin-Brauer obstruction.*

Definition 2.1 (Azumaya Algebras over Fields). *An Azumaya algebra1 over a field F is a F-algebra A (associative and containing the identity element, but not necessarily noncommutative) such that $A \otimes_F F^{sep}$ is isomorphic to the matrix algebra $M_n(F^{sep})$ as a F^{sep}-algebra for some $n \geq 1$ where F^{sep} denotes the separable closure of F. And let Az_F be the category of Azumaya algebras over F.*

Definition 2.2 (Brauer Group of Fields). *For a field k, two Azumaya algebras over k are called **similar** if there exist $m, n \geq 1$ such that $M_n(A) \simeq M_m(B)$ as k-algebras. And we write $A \sim B$ if they are similar. Then the Brauer group of a field k is defined as follows:*

$$\mathrm{Br}\, k := Az_k / \sim$$

Definition 2.3. *Let X be a scheme, the **(cohomological) Brauer group** of X is defined as*

$$\mathrm{Br}X := \mathrm{H}^2_{et}(X, \mathbb{G}_m)$$

However, although the cohomological definition of the Brauer group of schemes is elegant and general, it is not easy to compute and apply explicitly. We will introduce another definition using **Azumaya algebra**.

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Definition 2.4 (Azumaya Algebra over Schemes)**.** *An Azumaya algebra on a scheme X is an $\mathscr{O}_X$-algebra $\mathcal{A}$ that is coherent as a $\mathscr{O}_X$-module with $\mathcal{A}_x \neq 0$ for all $x \in X$, that satisfies one of the following conditions:*

(i) There is an open covering $\{U_i \to X\}$ in the étale topology such that for each i there exists $r_i \in \mathbb{Z}_{\geq o}$ such that $\mathcal{A} \otimes_{\mathscr{O}_X} \mathscr{O}_{U_i} \cong M_{r_i}(\mathscr{O}_{U_i})$.

(ii) $\mathcal{A}$ is locally free as an $\mathscr{O}_X$-module , and the fibre $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathscr{O}_X} k(x)$ is an Azumaya algebra over the residue field $k(x)$ for each $x \in X$.

Definition 2.5. *Two Azumaya algebra $\mathcal{A}$ and $\mathcal{A}'$ on X are **similar** if there exist locally free coherent $\mathscr{O}_X$-modules $\mathscr{E}$ and $\mathscr{E}'$ of positive rank at each $x \in X$ such that*

$$\mathcal{A} \otimes_{\mathscr{O}_X} \mathbf{End}_{\mathscr{O}_x}(\mathscr{E}) \simeq \mathcal{A}' \otimes_{\mathscr{O}_X} \mathbf{End}_{\mathscr{O}_x}(\mathscr{E}')$$

Definition 2.6. *Given a scheme X, the **Azumaya Brauer group** $\mathrm{Br}_{\mathrm{Az}}X$ is the set of similarity classes of Azumaya algebras on X. The multiplication is induced by $\mathcal{A}, \mathcal{B} \to \mathcal{A} \otimes_{\mathscr{O}_X} \mathcal{B}$, the inverse is induced by $\mathcal{A} \to \mathcal{A}^{\mathrm{opp}}$, and the identity is the class of $\mathscr{O}_X$.*

Fact 1 (Theorem.6.6.17 in [19])**.** *a) For any scheme X , the following map*

$$\mathrm{Br}_{\mathrm{Az}}X \to \mathrm{Br}X$$

is an injective homomorphism.

b) An Azumaya algebra $\mathcal{A}$ on X is locally free of rank n^2 defines an element of $\mathrm{Br}_{\mathrm{Az}}X$ that is killed by n. In particular, if x has at most finitely many connected components, then $\mathrm{Br}_{\mathrm{Az}}X$ is torsion.

2.1.2 Residue Homomorphism

There are two common way to compute the Brauer group. The first is using the Hochschild-Serre spectral sequence in étale cohomology. The second is to apply the **residue homomorphism** . For practical reason, our proof of rational emptiness is inspired by Iskovskikh's construction of Mannin-Brauer obstruction to his conic bundle with 4 singular fibers (cf. [13]), we only need to utilize some properties of the residue homomorphism.

Let us recall some important facts for residue homomorphisms .

Proposition 2.7. *Let X be a regular integral notherian scheme . Then the sequence*

$$0 \to \mathrm{Br}X \to \mathrm{Br}K(X) \xrightarrow{res} \bigoplus_{x \in X^{(1)}} \mathrm{H}^1(K(X), \mathbb{Q}/\mathbb{Z})$$

is exact, where $K(X)$ is the function field of X, with the caveat that the primary p part of all groups must be excluded if X is of dimension$\leqslant 1$ and some $k(x)$ is imperfect of characteristic p, or X is of dimension$\geqslant 2$ and $k(x)$ is of characteristic p.

2.2 Algebraic-geometry-theoretical Arguments of the Proof

In this subsection we explain how to choose a substitution to translate the feasible region of models 1 and 2 and reformulate the conjecture 1.3 in the language of algebraic geometry.

2.2.1 Step I: Substitution

For an integer $n \in \mathbb{Z}_{>0}$, let L be the optimal value of the optimization model 1, and let $x_i = \sum_{s=1}^4 z_{i,s}^2$, where x_i is with additional restriction $x_i \geq 0$, the same as given in the models 2 and 1, for $i = 1, \dots, L-1$. And we allow

$$R := \mathbb{R}[z_{1,1}, z_{2,1}, z_{3,1}, z_{4,1}, \dots z_{1,L-1}, \dots, z_{4,L-1}, w_{1,1}, \dots, w_{4,1} \dots, w_{1,L-2}, \dots, w_{4,L-2}]$$

$$\mathcal{X}_{L-1} := \mathrm{Spec} R/(f_0, f_1, \dots, f_{L-2})$$

where the $L-1$ equations are given as:

$$\begin{cases} f_0 = \sum_{i=1}^{L-1} \sum_{s=1}^4 z_{i,s}^2 - n + 1, \\ f_j = \sum_{s=1}^4 z_{s,j}^2 + \sum_{s=1}^{L-1-j} (\sum_{s=1}^4 z_{s,i}^2)(\sum_{s=1}^4 z_{s,i+j}^2) + \sum_{s=1}^4 w_{s,j}^2 - 1, j = 1 \dots, L-2. \end{cases} \tag{3}$$

As a convention we use $\mathcal{X}_{L-1}(M)$ to denote the M-valued points on $\mathcal{X}_{L-1}$ where M is any $\mathbb{Q}$-algebra. And we observe that:

Observation 2.8. *If $\mathcal{X}_{L-1}(\mathbb{R}) = \varnothing$, then conjeture 1.3 holds.*

Proof. Let $\mathcal{F}_{L-1}$ to denote the feasible region determined by the following constraints:

$$\begin{cases} \sum_{i=1}^{L-1} x_i = n-1, \\ x_j + \sum_{i=1}^{L-1-j} x_i x_{i+j} \leqslant 1. \\ 0 \leqslant x_i \leqslant 1, i = 1, \dots, L-1. \end{cases} \tag{4}$$

On the one hand, since L is the optimal value of (1), one can easily verify that 1 and 4 has no integer feasible solutions for $L_n = L-1$. We now give the following explanation: If 4 has an integer solution, denoted as $\mathbf{a} = (a_1, \dots, a_{L-1})$, then $(a_1, \dots, a_{L-1}, 0, \dots, 0) \in \mathbb{R}^{L_n}$ is a feasible solution for 1 which implies that the optimal value of 1 is not greater than $L-1$. This yields a contradiction. Similarly, if 4 has a real solution, then so does 2. The above discussion tells us that to check whether L is the optimal value for the continuous model 2 we only need to know whether 4 has a real solution.

On the other hand, if $\mathcal{X}_{L-1}(\mathbb{R}) = \varnothing$ which is equivalent to say that the system 3 has no real solution, then 4 is also free of real solutions for the reason that every positive real number can be decomposed as the sum of four squares of real numbers (the same argument also holds for positive rational numbers, i.e. every positive rational number can be decomposed into the sum of four squares of rational numbers) . This implies the conjecture 1.3.

□

The residue part of this section is devoted to proving that the **Rational Emptiness** of $\mathcal{X}_{L-1}$, that is,

2.2.2 Step II : Local Arguments on $\mathcal{X}_{L-1}$

The following fact inspires us to study the $\mathbb{Q}$-points on $\mathcal{X}_{L-1}$.

Proposition 2.9. *If $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$, then (4) has no rational solutions .*

Proof. This is a straightforward result by Lagrange's four-square theorem (cf. Landau's classical textbook [15] for a detailed proof). If there exists a rational solution of (4) , then we can definitely find four rational numbers $z_{1,i}, z_{2,i}, z_{3,i}, z_{4,i} \in \mathbb{Q}$ such that $x_i = \sum_{s=1}^{4} z_{s,i}^2$ for each $i \in \{1, \ldots, L-1\}$, which means that $\mathcal{X}_{L-1}(\mathbb{Q}) \neq \varnothing$. This leads to a contradiction. □

According to the above proposition, we only need to work on $\mathcal{X}_{L-1}$.

Theorem 2.10. *Let $\mathcal{X}_{L-1}$ be as above. Then $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$.*

Before proving this theorem, let us recall some basic results of invariant map of Brauer groups.

Lemma 2.11. *Let k be a local field. Then we have the following properties:*

(i) There is an injection $\text{inv} : \text{Br}k \to \mathbb{Q}/\mathbb{Z}$ *, whose image is :*

$$\begin{cases} \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } k = \mathbb{R}, \\ 0 & \text{if } k = \mathbb{C}, \\ \mathbb{Q}/\mathbb{Z} & \text{if } k \text{ is nonarchimedean.} \end{cases} \tag{5}$$

(ii) Every element of $\text{Br}k$ *has period equal to index . Especially, if $A \in \text{Br}k$ is a quaternion algebra , then $\text{inv}_p A$ is equal to $\frac{1}{2}$ if A is not split or 0 if A is split.*

Proof. One can find the proof of (i) in p.130 of [22] and (ii) in p.25 of [19]. □

Lemma 2.12 (Proposition 1.5.23. in [19])**.** *Given a field k, let $L \supseteq k$ be a cyclic extension of k of degree n and let $\chi : \text{Gal}(\bar{k}/k) \to \mathbb{Z}/n\mathbb{Z}$ be a continuous homomorphism. Then for $a \in k^\times$, the k-algebra (a, χ) is split if and only if $a \in N_{L/k}(L^\times)$.*

Proof. The proof can be found in [19]. □

Remark 2.13. *Notice that a cyclic algebra (a, χ) given above is an Azumaya algebra in* $\text{Br}k$*. (a, χ) being split over k means that it is an identity element in* $\text{Br}k$*, hence $\text{inv}_v(a, \chi) = 0$ for every place v of k.*

Lemma 2.14. *Given a field k . And let $a, b, c, d \in k^\times$. We have*

$$(a, b) \simeq (ac^2, bd^2)$$

Proof. See Exercise 4.(2) in **Chapter 2** of [27]. □

Next, we intend to introduce an important notion : the **Brauer-Manin obsruction** for a k-variety where k is a given field.

Definition 2.15 (**Evaluation of the Brauer group**)**.** *Let $A \in \text{Br}X$. If L is a k-algebra and $x \in X(L)$, then $\text{Spec}L \xrightarrow{x} X$ induces a homomorphism $\text{Br}X \to \text{Br}L$, which maps A to an element of $\text{Br}L$ that we call $A(x)$. For more detailed explanations , c.f. Section 8.1.1. in [19].*

Now assume that k is a global field and let $A \in \text{Br}X$. Then we have:

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Proposition 2.16. *If* $(x_v) \in X(\mathbb{A}_k)$*, then we have* $A(x_v) = 0$ *for almost all places* v *of* k*.*

Proof. See Proposition 8.2.1 in [19]. □

Proposition 2.17. *If* $x \in X(k) \subseteq X(\mathbb{A}_k)$ *, then we have* $(A, x) = 0$ *.*

Proof. See Proposition 8.2.2 in [19]. □

Definition 2.18. *For* $A \in \mathrm{Br}X$*, define that*

$$X(\mathbb{A}_k)^A := \{(x_v) \in X(\mathbb{A}_k) : (A, (x_v)) = 0\} \quad .$$

Furthermore we define

$$X(\mathbb{A}_k)^{\mathrm{Br}} := \bigcap_{A \in \mathrm{Br}X} X(\mathbb{A}_k)^A \quad .$$

Proposition 2.19. *It is clear that* $X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}}$*.*

Proof. cf. Corollary 8.2.6 in [19]. □

Before beginning the proof of Theorem2.10 we need to define some necessary quaternion algebras. Let $A^1_{ij} := (1 - z_i^2 z_{i+j}^2, -1)$, $A^2_{ij} := (1 + z_i^2 z_{i+j}^2, -1)$ $B^1_{ij} := (\frac{1}{z_i^2} - z_{i+j}^2, -1)$, $B^2_{ij} := (\frac{1}{z_{i+j}^2} - z_i^2, -1)$, $C^1_{ij} := (\frac{1}{z_i^2} + z_{i+j}^2, -1)$, $C^2_{ij} := (\frac{1}{z_{i+j}^2} + z_i^2, -1)$. It is clear that there is a pair $(i.j)$ such that the above quaternion algebras are all in $\mathrm{Br}K(\mathcal{X}_{L-1})$ according to the assumption that $\mathcal{X}_{L-1}(\mathbb{Z}) = \varnothing$.

We will show that:

Lemma 2.20. $A^\delta_{ij} = B^\delta_{ij} = C^\delta_{ij}$*, here* $\delta \in \{0,1\}$ *and* $A^1_{ij} = A^2_{ij}$*. In addition, we have* $A^1_{ij} \in \mathrm{Br}\mathcal{X}_{L-1}$*.*

Proof. The first argument is merely the corollary of Lemma 2.12 and Lemma 2.14. And it is clear that $A^1_{ij} \in \mathrm{Br}K(\mathcal{X}_{L-1})$. And by Proposition 2.7 we only need to find out a Zariski open covering $\{U_i\}$ of $\mathcal{X}_{L-1}$ such that A^i_{ij} extends to an element of $\mathrm{Br}U_i$ for each i. Let $P_{1-z_i^2 z_{i+j}^2}$ and $P_{1+z_i^2 z_{i+j}^2}$ denote the closed points in $\mathbb{P}^2_{\mathbb{Q}}$ that $1 - z_i^2 z_{i+j}^2$ and $1 + z_i^2 z_{i+j}^2$ vanish, respectively. Then we define

$$U^1_{ij} := \mathcal{X}_{L-1} - \text{(fibre above } \infty) - \text{(fibre above } P_{1-z_i^2 z_{i+j}^2}),$$
$$U^2_{ij} := \mathcal{X}_{L-1} - \text{(fibre above } \infty) - \text{(fibre above } P_{1+z_i^2 z_{i+j}^2}),$$
$$U^3_{ij} := \mathcal{X}_{L-1} - \text{(fibre above } z_{i+j} = \infty) - \text{(fibre above } z_i = 0\text{)} - \text{(fibre above } P_{1-z_i^2 z_{i+j}^2}),$$
$$U^4_{ij} := \mathcal{X}_{L-1} - \text{(fibre above } z_i = \infty) - \text{(fibre above } z_{i+j} = 0\text{)} - \text{(fibre above } P_{1-z_i^2 z_{i+j}^2}),$$
$$U^5_{ij} := \mathcal{X}_{L-1} - \text{(fibre above } z_{i+j} = \infty) - \text{(fibre above } z_i = 0\text{)} - \text{(fibre above } P_{1+z_i^2 z_{i+j}^2}),$$
$$U^6_{ij} := \mathcal{X}_{L-1} - \text{(fibre above } z_i = \infty) - \text{(fibre above } z_{i+j} = 0\text{)} - \text{(fibre above } P_{1+z_i^2 z_{i+j}^2}),$$

One immediately sees that $\bigcup_{i=1}^6 U^i_{ij} = \mathcal{X}_{L-1}$. And $A^1_{ij} \in \mathrm{Br}U^1_{ij}$, $A^2_{ij} \in \mathrm{Br}U^2_{ij}$, $B^1_{ij} \in \mathrm{Br}U^3_{ij}$, $B^2_{ij} \in \mathrm{Br}U^4_{ij}$, $C^1_{ij} \in \mathrm{Br}U^5_{ij}$, $C^2_{ij} \in \mathrm{Br}U^6_{ij}$. Therefore, A^1_{ij} extends to an element of $\mathrm{Br}\mathcal{X}_{L-1}$. □

Lemma 2.21. *In a similar process, we can prove that* $D^2_{ij} := (2 - z_i^2 z_{i+j}^2, -1)$*,* $D^3_{ij} := (3 - z_i^2 z_{i+j}^2, -1)$*,* $D^4_{ij} := (4 - z_i^2 z_{i+j}^2, -1)$ *are in* $\mathrm{Br}\mathcal{X}_{L-1}$*.*

Proof. The proof is just repeating the proof of Lemma 2.20. □

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Proof of Theorem2.10. As stated in the Introduction section, we want to find finitely many quaternion algebras to obstruct $\mathcal{X}_{L-1}$. It suffices to show that

$$\bigcap_{i,j}\left(\mathcal{X}_{L-1}(\mathbb{A}_{\mathbb{Q}})^{A^1_{ij}}\bigcap_{i=2}^{4}\mathcal{X}_{L-1}(\mathbb{A}_{\mathbb{Q}})^{E^i_{ij}}\right)=\varnothing$$

Next, we need to calculate $\mathrm{inv}_p A^1_{ij}(P)$, and $\mathrm{inv}_p E^k_{ij}(P)$, for all places p of $\mathbb{Q}$ and each $P\in\mathcal{X}_{L-1}(\mathbb{Q}_p)$.

(1) If $p\notin\{2,\infty\}$, we will show that $\mathrm{inv}_p A^1_{ij}(P)=0$. If $v_p(z_i z_{i+j})<0$, then $v_p(\frac{1}{z_i^2 z_{i+j}^2}-1)=0$ which means that $\frac{1}{z_i^2 z_{i+j}^2}-1\in\mathbb{Z}_p^\times$. If $v_p(z_i z_{i+j})\ge 0$ then at least one of $1-z_i^2 z_{i+j}^2$ or $1+z_i^2 z_{i+j}^2$ is in $\mathbb{Z}_p^\times$. Therefore $A^1_{ij}\in\mathrm{Br}\mathbb{Z}_p$ applying the assumption $p\neq 2$. However, for any local field K, its Brauer group is equal to zero. Hence $\mathrm{Br}\mathbb{Z}_p=0$. It follows that $\mathrm{inv}_p A^1_{ij}(P)=0$ when $p\neq 2,\infty$. The same argument also holds for E^k_{ij}, $k=2,3,4$.

(2) If $p=\infty$, it is obvious that at least $s+z_i^2 z_{i+j}^2>0$ or $s-z_i^2 z_{i+j}^2>0$ for $s=1,2,3,4$, which means that they are all in $N_{\mathbb{Q}(\sqrt{-1})/\mathbb{R}}(\mathbb{R}(\sqrt{-1})^\times)=\mathbb{R}_{>0}$. This implies $\mathrm{inv}_\infty A^1_{ij}(P)=\mathrm{inv}_\infty D^2_{ij}(P)=\mathrm{inv}_\infty D^3_{ij}(P)=\mathrm{inv}_\infty D^4_{ij}(P)=0$.

(3) If $p=2$, the discussion is different from that of $p\neq 2$.

(a) We first study this case when $v_2(z_i z_{i+j})>0$. One immediately obtains $k-z_i^2 z_{i+j}^2\equiv k \pmod 4$ for $k=1,2,3,4$. Therefore when $k=3$, $3-z_i^2 z_{i+j}^2\equiv -1 \pmod 4$ which means that $3-z_i^2 z_{i+j}^2\notin N_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2}(\mathbb{Q}_2(\sqrt{-1})^\times)$ and $\mathrm{inv}_2 D^3_{ij}=1/2$. However, $1-z_i^2 z_{i+j}^2\equiv 1 \pmod 4$ can always be written as a norm form of $N_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2}(\mathbb{Q}_2(\sqrt{-1})^\times)$ and it is direct to see that $\mathrm{inv}_2 A^1_{ij}=0$.

(b) If $v_2(z_i z_{i+j})=0$, then $1-z_i^2 z_{i+j}^2$, $2-z_i^2 z_{i+j}^2$, $3-z_i^2 z_{i+j}^2$, $4-z_i^2 z_{i+j}^2$ also range over the residue class modulo 4, similarly we have exactly one of $A^1_{ij}, D^2_{ij}, D^3_{ij}, D^4_{ij}$, whose evaluation at 2 is 1/2 and one of them is 0.

(c) If $v_2(z_i z_{i+j})<0$, we have $v_2(\frac{k}{z_i^2 z_{i+j}^2}-1)=0$ and then $\frac{k}{z_i^2 z_{i+j}^2}-1\equiv -1 \pmod 4$ for all $k=1,2,3,4$. That is to say, $\frac{k}{z_i^2 z_{i+j}^2}-1$ is always not a norm form in $N_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2}(\mathbb{Q}_2(\sqrt{-1})^\times)$. Therefore, $\mathrm{inv}_2 A^1_{ij}=\mathrm{inv}_2 D^2_{ij}=\mathrm{inv}_2 D^3_{ij}=\mathrm{inv}_2 D^4_{ij}=1/2$.

Combining the computation results from (1), (2) and (3), we achieve that

$$\bigcap_{i,j}\left(\mathcal{X}_{L-1}(\mathbb{A}_{\mathbb{Q}})^{A^1_{ij}}\bigcap_{i=2}^{4}\mathcal{X}_{L-1}(\mathbb{A}_{\mathbb{Q}})^{E^i_{ij}}\right)=\varnothing$$

. □

3 Approximating the Real Points

3.1 Outline of the Proof

The proof of the emptiness of real-valued points depends only on the original form of 1.3 and will not involve any algebraic-geometry-theoretical notions and techniques. If a real point exists

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in $\mathcal{X}_{L-1}$, it is equivalent to claim that 2 has a real-valued feasible point B. Hence our strategy is to return to model 2 and find a feasible rational point close enough to B that also lies in the feasible region of 2. This will contradict with $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$ we have proven in **Section 2**.

We let $\mathbf{x} = (x_1, \ldots, x_{L-1})$, $g_j(\mathbf{x}) = x_j + \sum_{i=1}^{L-1-j} x_i x_{i+j}$ for $j = 1, \ldots, L-2$. And explicitly we write $B = (t_1, \ldots, t_{L-1}) \in \mathbb{R}^{L-1}$ which is the hypothetical real-valued feasible solution of the following equations as the same as given in (4).

$$\begin{cases} \sum_{i=1}^{L-1} x_i = n-1, \\ g_j(\mathbf{x}) \leq 1, \quad for\ j = 1, \ldots, L-2. \\ 0 \leqslant x_i \leqslant 1, i = 1, \ldots, L-1. \end{cases} \tag{6}$$

Notice 3.1. *In this section we use the notation (a, b) to denote an open interval in the real axis where $a, b \in \mathbb{R}$. And don't be confused with the notation of a quaternion algebra in Section 2.*

The main objective of this section is to prove the following:

Theorem 3.2. *(6) does not have a real feasible solution.*

Sketch of the Proof. Let B be as above. The special case is that all $g_j(B) < 1$. We firstly work in this case. Suppose

$$\epsilon = \frac{\min_{1 \leq j \leq L-2}(1 - g_j(B))}{2(L-1)}.$$

Define a neighborhood of B as $U_\epsilon := \prod_{i=1}^{L-1}(t_i - \epsilon, t_i + \epsilon)$. It is easy to check that the intersection of U_ϵ and the hyperplane $P_0 : \sum_{i=1}^{L-1} x_i = n-1$ is lie in the feasible region of (6). However, by the density of $\mathbb{Q}^{L-1} \subseteq \mathbb{R}^{L-1}$, one can definitely find one rational point in $U_\epsilon \bigcap P_0$ and we will demonstrate why this fact is true in **Lemma**3.3. This leads to a contradiction with $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$.

In the general cases, let $I \subseteq \{1, 2, \ldots, L-2\}$ to denote the index set such that $g_n(B) = 1$ for each $n \in I$. And write $\bar{I}$ the complement of I in $\{1, \ldots, L-2\}$ .That is,

$$\begin{cases} g_n(B) = 1, \quad if\ n \in I, \\ g_n(B) < 1, \quad if\ n \in \bar{I}. \end{cases} \tag{7}$$

And suppose $\delta := \min_{j \in \bar{I}}(1 - g_j(B))/2(L-1)$. Define an open subset $U_\delta := \prod_{i \in I}(t_i - \delta, t_i) \times \prod_{i \in \bar{I}}(t_i - \delta, t_i + \delta)$. One can also find a rational point in $U_\delta \bigcap P_0$ which is also open in $\mathbb{R}^{L-1}$. This also contradicts the fact that $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$ (we will give a proof of this issue in **Lemma** 3.3 in the next subsection.

□

3.2 Filling the Gap in the Proof of $\mathcal{X}_{L-1}(\mathbb{R}) = \varnothing$

3.2.1 Shift the Irrational Point to a Rational Point

In this subsection, we prove some technical issues for the front subsection. The main goal is to construct a rational feasible point from the given hypothetical point B.

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Lemma 3.3. *Let U_ϵ and P_0 be the same as defined in the proof of Theorem 3.2. Then (1) when $g_j(\mathbf{x}) < 1$ for all $1 \le j \le L-1$, $U_\epsilon \cap P_0 \cap \mathbb{Q}^{L-1}$ is nonempty and contained in the feasible region of the system (6); (2) generally, let δ, U_δ be as stated in the last paragraph in the previous subsection, $U_\delta \cap P_0 \cap \mathbb{Q}^{L-1}$ is also nonempty.*

Proof. (1) Firstly, we need to show that U_ϵ lies in the feasible region of $g_j(\mathbf{x}) < 1$. Let $\mathbf{u} \in U_\epsilon$. Then

$$g_j(\mathbf{u}) - g_j(B) = u_j + \sum_{i=1}^{L-1-j} u_i u_{i+j} - t_j - \sum_{i=1}^{L-1-j} t_i t_{i+j} \tag{8}$$

$$\leqslant \epsilon + \sum_{i=1}^{L-1-j} (2\epsilon + \epsilon^2) \tag{9}$$

$$< 2(L-1)\epsilon \tag{10}$$

$$= \min_{1\le j\le L-2} (1 - g_j(B)) \tag{11}$$

Hence $g_j(\mathbf{u}) \leqslant 1$ for all $1 \leqslant j \leqslant L-2$. The above arguments justify that U_ϵ lies in the feasible region of $g_j(\mathbf{x}) < 1$. Then it remains to show that $U_\epsilon \cap P_0 \cap \mathbb{Q}^{L-1}$ is nonempty.

It is easy to construct such a rational point in $U_\epsilon \cap P_0 \cap \mathbb{Q}^{L-1}$. Not uniquely, choose $s_1 \in (t_1 - \frac{\epsilon}{L-1}, t_1 + \frac{\epsilon}{L-1}) \cap \mathbb{Q}$, $i = 1, \dots, L-2$, and choose $s_{L-1} = n - 1 - \sum_{i=1}^{L-2} s_i$. It is clear that

$$|s_{L-1} - t_{L-1}| \leqslant \sum_{i=1}^{L-2} |s_i - t_i| \tag{12}$$

$$< \epsilon \tag{13}$$

It implies that $(s_1, \dots, s_{L-1}) \in U_\epsilon \cap P_0 \cap \mathbb{Q}^{L-1}$.

(2) The proof of the general part is similar but requires a more delicate shift of each x_i. Set $\{\alpha_1, \alpha_2, \dots, \alpha_t\}$ an **irrational basis** of $\{t_1, \dots, t_{L-1}\}$ where $\alpha_1 = 1$, here by an irrational basis we mean that each t_i can be uniquely written as the summation from α_1 to α_t with rational coefficients. If $t = 1$, then all terms t_i are rational numbers that contradict $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$. We focus on the case where $t > 2$. Write

$$t_i = \sum_{p=1}^{t} a_{i,p}\alpha_p \ , \qquad i = 1, \dots, L-1.$$

For some computational reasons, we intend to properly choose the basis for each coefficient $a_{i,p} \geqslant 0$. Assume $1 = \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_t$, and set $\rho_p := \min_{1\leqslant i\leqslant L-1} a_{i,p}$ for each $p \in \{1, \dots, t\}$. And suppose that $\alpha'_t := \theta\alpha_t + \sum_{p=1}^{t-1} \rho_p\alpha_p$, where θ is the minimum integer such that $\theta\alpha_t + \sum_{p=1}^{t-1} \rho_p\alpha_p > 0$. It is clear that $\{\alpha_1, \dots, \alpha_{t-1}, \alpha'_t\}$ is also a transcendental basis for $\{t_1, \dots, t_{L-1}\}$. Furthermore, under this basis $\{\alpha_1, \dots, \alpha_{t-1}, \alpha'_t\}$, the coefficient $a'_{i,p}$ of α_p for t_i is always non-negative. We also use this basis in the following computations.

And we have

$$\sum_{i=1}^{L-1} t_i = \sum_{i=1}^{L-1} \sum_{p=1}^{t} a'_{i,p}\alpha'_p \tag{14}$$

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$$= \sum_{p=1}^{t} c_p \alpha'_p \tag{15}$$

$$= n - 1, \tag{16}$$

where $c_p = \sum_{i=1}^{L-1} a'_{i,p}$. It implies that $c_1 = n - 1$ and $c_p = 0$ for all $p \geqslant 2$ for $\{\alpha_1, \ldots, \alpha'_t\}$ are linearly independent over $\mathbb{Q}$. Moreover, rewriting other equations ($g_j(B) = 1$ when $j \in I$) under this basis we have

$$x_j + \sum_{i=1}^{L-1-j} x_i x_{i+j} = \sum_{p=1}^{t} a'_{i,p}\alpha_p + \sum_{i=1}^{L-1-j} (\sum_{p=1}^{t} a'_{i,p}\alpha_p) \cdot (\sum_{p=1}^{t} a'_{i+j,p}\alpha_p) \tag{17}$$

$$= \sum_{i=1}^{p} d_{i,p}\alpha'_p + \sum_{\substack{2 \leqslant p \leqslant t \\ 2 \leqslant q \leqslant t}} l_{jpq}\alpha'_p\alpha'_q \tag{18}$$

$$= 1, \quad j \in I \tag{19}$$

where $d_{i,p}$ is the sum of rational coefficients in (17) while l_{jpq} is the sum of coefficients of $\alpha_p\alpha_q$. Note that the coefficients $d_{i,p}$ and l_{jpq} in (18) are all nonnegative. If we substitute α'_p for a slightly smaller rational number, the value of (15) is still equal to $n - 1$ and the value of (18) will not exceed 1. Choose t positive rational numbers $\{\pi_p\}_{1 \leqslant p \leqslant t}$ where π_p is in the interval

$$(\max\{0, \alpha_p - \frac{\delta}{\max\{t, t \cdot \max_{i,p}|a'_{i,p}|\}}\}, \alpha_p),$$

and $\delta = \min_{j \in \bar{I}}(1 - g_j(B))/2(L-1)$ is as defined in the previous subsection. Then we substitute all α'_p with π_p and define

$$b_i := \sum_{p=1}^{t} a'_{i,p}\pi_p, \quad i = 1, \ldots, L-1.$$

Of course we have $0 \leqslant b_i \leqslant t_i \leqslant 1$. Moreover, we have

$$|b_i - t_i| \leqslant \sum_{p=1}^{t} a'_{i,p}|\pi_p - \alpha'_p| \tag{20}$$

$$< \sum_{p=1}^{t} \delta/t \tag{21}$$

$$= \delta. \tag{22}$$

Without a doubt, b_i is in $\mathbb{Q}$ for all $1 \leqslant i \leqslant L-1$. Denote $Q = (b_1, \ldots, b_{L-1}) \in \mathbb{Q}^{L-1}$. We also have:

$$\begin{cases} \sum_{i=1}^{L-1} b_i = n - 1, \\ g_j(Q) = b_j + \sum_{i=1}^{L-1-j} b_i b_{i+j} \leqslant 1, \quad j \in I. \end{cases} \tag{23}$$

And for $j \in \bar{I}$,

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$$
\begin{aligned}
|g_j(Q) - g_j(B)| &= |(b_j - t_j) + \sum_{i=1}^{L-1-j} (b_i b_{i+j} - t_i t_{i+j})| && (24)\\
&< \delta + \sum_{i=1}^{L-1-j} \left((t_i + t_{i+j})\delta + \delta^2\right) && (25)\\
&< 2(L-1-j)\delta + \delta + (L-1-j)\delta^2 && (26)\\
&\leqslant 2(L-j)\delta && (27)\\
&< \min_{j \in \bar{I}} (1 - g_j(B)). && (28)
\end{aligned}
$$

It follows that $g_j(Q) \leqslant 1$ for all $1 \leqslant j \leqslant L-1$. That is to say, Q lies in the the feasible region of 6 as a rational point.

□

3.2.2 Conclusion

Combining results in Section2 and Section 3 we give an entire proof of Conjecture 1.3. The method we use in the proof is purely algebraic and arithmetic including some elementary computations. In general, one can expect to reformulate an optimization problem as an algebraic geometric problem which is extremely powerful to deal with some special problems, for examples, the existence of integer feasible solutions of a given optimization model whose constraints are all rational polynomials. We also expect to find more optimization models whose discrete case and continuous case have the same optimal value.

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