The Proof of a Conjecture For a Continuos Golumb Ruler Model

Tianyu Liu

Department of Mathematics Science , University of Chinese Science Academy, China-100049
(e-mail: liutianyu201@mails.ucas.ac.cn)

Abstract

In this paper we study a conjecture proposed by P.Duxbury , C.laror , L.Leduino de Salles Neto in 2021[4] on the Golumb Ruler Problem which is a classical optimization model in discrete case . In [4] the authors constucted a continuous model for the Golumb Ruler Problem associated to the discrete case and conjectured that the optimal value of both cases are equal . We deal with this conjecture via algebraic geometrical methods .

Keywords: The Golumb Ruler Problem , Brauer Groups , Rational Approxmation

Contents

1 Introduction ii

2 The Rational Aspect of the Proof iii
2.1 Preliminary . . . iii
2.1.1 A First Review of Brauer Group . . . iii
2.1.2 Residue Homomorphism . . . iv
2.2 Proof of Rational Part . . . iv
2.2.1 Step I : Substitution . . . iv
2.2.2 Step II : Local Arguments on $\mathcal{X}_{L-1}$ . . . v

3 Approximating the Real Points viii
3.1 Outline of the Proof . . . viii
3.2 Filling the Gap in the Proof of $\mathcal{X}_{L-1}(\mathbb{R}) = \varnothing$ . . . ix
3.2.1 Reduce the Irrational Point to a Rational Point . . . ix
3.2.2 Conclusion . . . xi

4 Acknowledgement xi

1 Introduction

The Golumb Ruler Problem is a classical optimization problem and has various applications from astronomy [2] to information theory [1] , [14] . In [4] P.Duxbury and his cooperators constuct a discrete optimization model and a continuous one for the Golumb Ruler Problem . More surprisingly they proposed a conjecture which claims that the optimal value of the discrete model is equal to that of the continuos one . In this paper we study this conjecture via algebraic geometrical tools and some easy rational approximation .

Historically , a Golumb ruler is a finite set of marked points on a ruler at integer positions such that the distances of any two ponits are different. More explicitly , we say that a Golumb Ruler problem is to find the smallest length of the ruler such that the n marked points form a Golumb ruler . In [9] , a discrete model for the Golumb ruler problem was firstly presented . Given an positive inerger n and a upper bound L_n for the length of the ruler .

$$\min_{x_i \in \{0,1\}} t$$

$$\begin{cases} i x_i \leq t, \\ \sum_{i=1}^{L_n} x_i = n-1, \\ x_j + \sum_{i=1}^{L_n - j - 1} x_i x_{i+j} \leq 1, \quad j = 1, \ldots, L_n - 1. \end{cases} \tag{1}$$

In [4] a continuous model for the Golumb ruler is also established :

$$\min_{x_i \in [0,1]} t$$

$$\begin{cases} i x_i \leq t x_i, \\ \sum_{i=1}^{L_n} x_i = n-1, \\ x_j + \sum_{i=1}^{L_n - j - 1} x_i x_{i+j} \leq 1, \quad j = 1, \ldots, L_n - 1. \end{cases} \tag{2}$$

Moreover they propose a fascinating conjecture .

Conjecture 1.1 ([4])**.** *The optimal value of model (1) is identical with that of model (2) .*

We separate the study of this conjecure into three parts : (1)We regard the feasible regions of 2 and 1 as algebraic schemes after substituting $x_i = y_i^2$ and for n and L_n we write $\mathcal{X}_{L_n}$ to denote the associated schemes ; (2) We intend to prove that $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$ where L is the optimal value for the discrete model 1 ; (3) Lastly we need to verify that $\mathcal{X}_{L-1}(\mathbb{Q})$ is dense in $\mathcal{X}_{L-1}(\mathbb{R})$ and it follows immediately that $\mathcal{X}_{L-1}(\mathbb{R}) = \varnothing$ which suffices to prove conjeture 1.1 .

The most difficult and technical part turns out to be part (2) , in which we utilize algebraic-geometry-theoretical notions and tools . More specifically , it suffices to prove that $\mathcal{X}_{L-1}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \varnothing$ (we will explain this termination in Definition 2.16) which implies that $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$ (here $\mathbb{A}_{\mathbb{Q}}$ denotes the Adélic ring of $\mathbb{Q}$, for an entire definition of the Adélic ring of a local field , see A,Weil's textbook [19] or J.Neukirch's lecture [12]) . Hence we hope to find out some Azumaya algebras $\{A_s\}_{s \in S}$ indexed by a finite index S such that $\bigcap_{s \in S} \mathcal{X}_{L-1}(\mathbb{A}_{\mathbb{Q}})^{A_s} = \varnothing$ (this notation will be defined also in Definition 2.16) .

The idea to settle part (3) is quite simple . If an $\mathbb{R}$-points P exist in $\mathcal{X}_{L-1}$, we can pick out a $\mathbb{Q}$-points Q very close to P which also satisfies constraints in 2 . This leads to a contraction to the temporarily assumed fact that $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$.

Organization. This paper is organized as follow : In **Section 2** We recall some notions and properties in basic Brauer group theory and realize the goal of part 2 . In **Section 3** we finish part 3 .

2 The Rational Aspect of the Proof

2.1 Preliminary

Firstly let us recall some basic algebraic geometry . The main references of this subsection are Hartshorne's textbook [7] and Poonen's lecture note [13] . The readers can also check [11] , [17] [3] for more detailed illustrations and advanced topics on algebraic geometry and arithmetic theory . For more detailed and interesting descriptions of Brauer groups and Brauer-Manin obstruction , c.f. [18] , [15] , and Grothendieck's original research on Brauer groups [6] and [5] .

For the precise definition of schemes, the readers can check **Section I , II** in **Chapter 2** in [7] .

2.1.1 A First Review of Brauer Group

Definition 2.1. *Let X be a scheme , the* ***(cohomological) Brauer group*** *of X is defined as*

$$\mathrm{Br}X := \mathrm{H}^2_{et}(X, \mathbb{G}_m)$$

However , althugh the cohomological definition of Brauer group of schemes is elegant and general , it is not easy to be explicitly computed and applied . We will introduce another definition using **Azumaya algebra** .

Definition 2.2 (Azumaya Algebra)**.** *An Azumaya algebra on a scheme X is an $\mathscr{O}_X$-algebra $\mathcal{A}$ which is coherent as an $\mathscr{O}_X$-module with $\mathcal{A}_x \neq 0$ for all $x \in X$, that satisfies one of the following conditions :*

(i) There is an open covering $\{U_i \to X\}$ in the etale topology such taht for each i there exists $r_i \in \mathbb{Z}_{\geq o}$ such that $\mathcal{A} \otimes_{\mathscr{O}_X} \mathscr{O}_{U_i} \cong M_{r_i}(\mathscr{O}_{U_i})$.

(ii) $\mathcal{A}$ is locally free as an $\mathscr{O}_X$-module , and the fibre $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathscr{O}_X} k(x)$ is an Azumaya algebra over the residue field $k(x)$ for each $x \in X$.

Definition 2.3. *Two Azumaya algebra $\mathcal{A}$ and $\mathcal{A}'$ on X are* ***similar*** *if there exist locally free coherent $\mathscr{O}_X$-modules $\mathscr{E}$ and $\mathscr{E}'$ of positive rank at each $x \in X$ such that*

$$\mathcal{A} \otimes_{\mathscr{O}_X} \mathbf{End}_{\mathscr{O}_x}(\mathscr{E}) \simeq \mathcal{A}' \otimes_{\mathscr{O}_X} \mathbf{End}_{\mathscr{O}_x}(\mathscr{E}')$$

Definition 2.4. *Given a scheme X , the* ***Azumaya Brauer group*** $\mathrm{Br}_{\mathrm{Az}}X$ *is the set of similarity classes of Azumaya algebras on X . The multiplication is induced by $\mathcal{A}, \mathcal{B} \to \mathcal{A} \otimes_{\mathscr{O}_X} \mathcal{B}$, the inverse is induced by $\mathcal{A} \to \mathcal{A}^{\mathrm{opp}}$, and the identity is the class of $\mathscr{O}_X$.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 November 2022 doi:10.20944/preprints202211.0027.v1

Fact 1 (Theorem.6.6.17 in [13])**.** *a) For any scheme X , the following map*

$$\mathrm{Br}_{\mathrm{Az}}X \to \mathrm{Br}X$$

is an injective homomorphism .

b) An Azumaya algebra $\mathcal{A}$ on X is locally free of rank n^2 defines an element of $\mathrm{Br}_{\mathrm{Az}}X$ that is killed by n . In particular , if x has at most finitely many connected components , then $\mathrm{Br}_{\mathrm{Az}}X$ is torsion .

2.1.2 Residue Homomorphism

There are two common way to compute the Brauer group . The first is using the Hochschild-Serre spectral sequence in étale cohomology . The second one is apply the **residue homomorphism** . For practical reason our proof of rational emptiness is inspired by Iskovskikh's construction of Mannin-Brauer obstruction to his conic bundle with 4 singular fibres (c.f. [8]) , we only need to utilize some properties of residue homomorphism .

Let us recall some important facts for residue homomorphisms .

Proposition 2.5. *Let X be a regular integral notherian scheme . Then the sequence*

$$0 \to \mathrm{Br}X \to \mathrm{Br}K(X) \xrightarrow{res} \bigoplus_{x\in X^{(1)}} \mathrm{H}^1(K(X), \mathbb{Q}/\mathbb{Z})$$

is exact , where $K(X)$ is the function field of X , with the caveat that one must exclude the p-primary part of all the groups if X is of dimension$\leqslant 1$ and some $k(x)$ is imperfect of characteristic p , or X is of dimension$\geqslant 2$ and $k(x)$ is of characteristic p .

2.2 Proof of Rational Part

In this subsection we explain how to choose a substitution to translate the feasible region of models 1 and 2 and reformulate the conjecture 1.1 in the language of algebraic geometry .

2.2.1 Step I : Substitution

For an integer $n \in \mathbb{Z}_{>0}$, let L be the optimal value of the optimization model 1 , and let $x_i = \sum_{s=1}^4 z_{i,s}^2$, where x_i is with additional restriction $x_i \geq 0$, the same as given in models 2 and 1 , for $i = 1, \dots, L-1$ .And we allow

$$R := \mathbb{R}[z_{1,1}, z_{2,1}, z_{3,1}, z_{4,1}, \dots z_{1,L-1}, \dots, z_{4,L-1}, w_{1,1}, \dots, w_{4,1} \dots, w_{1,L-2}, \dots, w_{4,L-2}]$$

$$\mathcal{X}_{L-1} := \mathrm{Spec}R/(f_0, f_1, \dots, f_{L-2})$$

where the $L-1$ equations are given as :

$$\begin{cases} f_0 = \sum_{i=1}^{L-1}\sum_{s=1}^4 z_{i,s}^2 - n + 1, \\ f_j = \sum_{s=1}^4 z_{s,j}^2 + \sum_{s=1}^{L-1-j}(\sum_{s=1}^4 z_{s,i}^2)(\sum_{s=1}^4 z_{s,i+j}^2) + \sum_{s=1}^4 w_{s,j}^2 - 1, j = 1 \dots, L-2. \end{cases} \tag{3}$$

As convention we use $\mathcal{X}_{L-1}(M)$ to denote the M-valued points on $\mathcal{X}_{L-1}$ where M is any $\mathbb{Q}$-algebra . And we observe that :

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 November 2022 doi:10.20944/preprints202211.0027.v1

Observation 2.6. *If $\mathcal{X}_{L-1}(\mathbb{R}) = \varnothing$, then conjeture 1.1 holds .*

Proof. Let $\mathcal{F}_{L-1}$ to denote the feasible region determined by the following constraints :

$$\begin{cases} \sum_{i=1}^{L-1} x_i = n-1, \\ x_j + \sum_{i=1}^{L-1-j} x_i x_{i+j} \leqslant 1. \\ 0 \leqslant x_i \leqslant 1, i = 1, \dots, L-1. \end{cases} \tag{4}$$

On the one hand , since L is the optimal value of (1) , one can easily verify that 1 and 4 has no integer feasible solutions for $L_n = L-1$. We now give the explanation : if 4 has an integer solution , denoted as $\mathbf{a} = (a_1, \dots, a_{L-1})$, then $(a_1, \dots, a_{L-1}, 0, \dots, 0) \in \mathbb{R}^{L_n}$ is an feasible solution for 1 which implies that the optimal value of 1 is not bigger than $L-1$. This yields a contradiction . Similarly if 4 has a real solution then so do 2 . The above discussion tells us that to check whether L is the optimal value for the continuous model 2 we only need to know whether 4 has a real solution .

On the other hand , if $\mathcal{X}_{L-1}(\mathbb{R}) = \varnothing$ which is equivalent to say that the system 3 has no real solution , then 4 is also free of real solutions for the reason that every positive real number can be decomposed as the sum of four squares of real numbers (the same argument also holds for positive rational numbers , i.e. every positive rational number can be decomposed into the sum of four squares of rational numbers) . This implies the conjecture 1.1 .

□

The residue part of this section is devoted to prove that the **Rational Emptiness** of $\mathcal{X}_{L-1}$, i.e.

2.2.2 Step II : Local Arguments on $\mathcal{X}_{L-1}$

The following fact inspires us to study the $\mathbb{Q}$-points on $\mathcal{X}_{L-1}$.

Proposition 2.7. *If $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$, then (4) has no rational solutions .*

Proof. This is a straightforward result by Lagrange's four-square theorem (c.f. Landau's classical textbook [10] for a detailed proof) . If there exists a rational solution of (4) , then we can definitely find four rational numbers $z_{1,i}, z_{2,i}, z_{3,i}, z_{4,i} \in \mathbb{Q}$ such that $x_i = \sum_{s=1}^{4} z_{s,i}^2$ for each $i \in \{1, \dots, L-1\}$ which means that $\mathcal{X}_{L-1}(\mathbb{Q}) \neq \varnothing$. This leads to a contradiction .

□

According to the above proposition , we only need to work on $\mathcal{X}_{L-1}$.

Theorem 2.8. *Let $\mathcal{X}_{L-1}$ be as above . Then $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$.*

Before proving this theorem , let us recall some basic results of invariant map of Brauer groups .

Lemma 2.9. *Let k be a local field . Then we have the following properties :*

(i) There is an injection $\mathrm{inv} : \mathrm{Br}k \to \mathbb{Q}/\mathbb{Z}$ *, whose image is :*

$$\begin{cases} \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } k = \mathbb{R}, \\ 0 & \text{if } k = \mathbb{C}, \\ \mathbb{Q}/\mathbb{Z} & \text{if } k \text{ is nonarchimedean.} \end{cases} \tag{5}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 November 2022 doi:10.20944/preprints202211.0027.v1

(ii) *Every element of* Brk *has period equal to index . Especially, if* $A \in \mathrm{Br}k$ *is a quaternion algebra , then* $\mathrm{inv}_p A$ *is equal to* $\frac{1}{2}$ *if* A *is not split or* 0 *if* A *is split .*

Proof. One can find the proof of (i) in p.130 of [16] and (ii) in p.25 of [13] . □

Lemma 2.10 (Proposition 1.5.23. in [13])**.** *Given a field* k *. Let* $L \supseteq k$ *be an cyclic extension of* k *of degree* n *and* $\chi : \mathrm{Gal}(\bar{k}/k) \to \mathbb{Z}/n\mathbb{Z}$ *be a continuos homomorphism . Then for* $a \in k^\times$ *, the* k*-algebra* (a,χ) *is split if and only if* $a \in N_{L/k}(L^\times)$ *.*

Proof. The proof can be found in [13] . □

Remark 2.11. *Notice that a cyclic algebra* (a,χ) *given above is an Azumaya algebra in* Brk *.* (a,χ) *being split over* k *means that it is an identity element in* Brk *, hence* $\mathrm{inv}_v(a,\chi) = 0$ *for every place* v *of* k *.*

Lemma 2.12. *Given a field* k *. And let* $a,b,c,d \in k^\times$ *. We have*

$$(a,b) \simeq (ac^2, bd^2)$$

Proof. See Exercise 4.(2) in **Chapter 2** of [18] . □

Next we intend to introduce an important notion : the **Brauer-Manin obsruction** for a k-variety where k is a given field .

Definition 2.13 (**Evaluation of the Brauer group**)**.** *Let* $A \in \mathrm{Br}X$ *. If* L *is a* k*-algebra and* $x \in X(L)$ *, then* $\mathrm{Spec}L \xrightarrow{x} X$ *induces a homomorphism* $\mathrm{Br}X \to \mathrm{Br}L$ *, which maps* A *to an element of* BrL *tht we call* $A(x)$ *. For more detailed explanations , c.f. Section 8.1.1. in [13] .*

Now assume that k is a global field and let $A \in \mathrm{Br}X$. Then we have :

Proposition 2.14. *If* $(x_v) \in X(\mathbb{A}_k)$ *, then we have* $A(x_v) = 0$ *for almost all places* v *of* k *.*

Proof. See Proposition 8.2.1 in [13] . □

Proposition 2.15. *If* $x \in X(k) \subseteq X(\mathbb{A}_k)$ *, then we have* $(A,x) = 0$ *.*

Proof. See Proposition 8.2.2 in [13] . □

Definition 2.16. *For* $A \in \mathrm{Br}X$ *, define that*

$$X(\mathbb{A}_k)^A := \{(x_v) \in X(\mathbb{A}_k) : (A,(x_v)) = 0\} \quad .$$

Furthermore we define

$$X(\mathbb{A}_k)^{\mathrm{Br}} := \bigcap_{A \in \mathrm{Br}X} X(\mathbb{A}_k)^A \quad .$$

Proposition 2.17. *It is clear that* $X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}}$ *.*

Proof. c.f. Corollary 8.2.6 in [13] . □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 November 2022 doi:10.20944/preprints202211.0027.v1

Proof of Theorem2.8. As stated in the Introduction section , we want to find finitely many quaternion algebras to obstruct $\mathcal{X}_{L-1}$. Let $A^1_{i,j} := (z^2_{i,j} - 2, -1), A^2_{i,j} := (z^2_{i,j} - 3, -1), A^3_{i,j} := (z^2_{i,j} - 4, -1), A^4_{i,j} := (z^2_{i,j} - 5, -1)$, to denote the quaternion algebra generated by $(z^2_{i,j} - a)$ for $a = 2, 3, 4, 5$ and $-1 \in K(\mathcal{X}_{L-1})^\times$ for some i, j where $1 \le i \le 4$ and $1 \le j \le L - 1$.

First we need to show that $A^k_{i,j}$ are all in $\mathrm{Br}(\mathcal{X}_{L-1})$ for $k = 1, 2, 3, 4$. According to the construction of $\mathcal{X}_{L-1}$, it is clear that $\mathcal{X}_{L-1}(\mathbb{Z}) = \varnothing$. Hence $z^2_{i,j} - k$ has no zeroes in $\mathcal{X}_{L-1}$. That is to say $A^k_{i,j}$ has no residue along $\mathcal{X}_{L-1}$. By **Proposition** 2.5 , we know that $A^k_{i,j}$ are all in $\mathrm{Br}(\mathcal{X}_{L-1})$ for $k = 1, 2, 3, 4$.

Then we need to compute $\mathrm{inv}_p A^k_{i,j}(P)$ for all places p of $\mathbb{Q}$ and each $P \in \mathcal{X}_{L-1}(\mathbb{Q}_p)$.

(1) If $p \notin \{2, \infty\}$, we will show that $\mathrm{inv}_p A^k_{i.j}(P) = 0$. If $v_p(z_{i,j}) < 0$, then $v_p(1 - k/z^2_{i,j}) = 0$ which means that $1 - (k+1)/z^2_{i,j} \in \mathbb{Z}^\times_p$. If $v_p(z_{i,j}) \ge 0$ and $z^2_{i,j}(P) - k - 1 \ne 0$, we have $z^2_{i,j} - k - 1 \in \mathbb{Z}^\times$. Therefore $A^k_{i,j} \in \mathrm{Br}\mathbb{Z}_p$ applying the assumption $p \ne 2$. However for any local field K , its Brauer group is equal to zero . Hence $\mathrm{Br}\mathbb{Z}_p = 0$. It follows that $\mathrm{inv}_p A^k_{i,j}(P) = 0$ when $p \ne 2, \infty$.

(2) If $p = \infty$, it is obvious that $z^2_{i,j}(P) - k - 1 < 0$ that is imposible to be a norm form in $N_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}(\mathbb{Q}(\sqrt{-1})^\times)$. It implies that $\mathrm{inv}_\infty A^k_{i,j}(P) = \frac{1}{2}$.

(3) If $p = 2$, the case is different from those of $p \ne 2$.

(a) We firstly study this case when $v_2(z_{i,j}) > 0$. One immediately obtains that $z^2_{i,j} - k - 1 \equiv -k - 1 (\bmod 4)$ for $k = 1, 2, 3, 4$. Therefore when $k = 4$, $z^2_{i,j} - 5 \equiv -1(\pmod 4)$ which means that $z^2_{i,j} - 5 \notin N_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2}(\mathbb{Q}_2(\sqrt{-1})^\times)$ and $\mathrm{inv}_2 A^4_{i.j} = 1/2$. Except when $k = 4$, $z^2_{i,j} - k - 1$ can always be written as a norm form in $N_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2}(\mathbb{Q}_2(\sqrt{-1})^\times)$ and it is direct to see that $\mathrm{inv}_2 A^k_{i,j} = 0$.

(b) If $v_2(z_{i,j}) = 0$, then $z^2_{i,j} - 2$, $z^2_{i,j} - 3$, $z^2_{i,j} - 4$, $z^2_{i,j} - 5$ also range over the residue class modulo 4 , similarly we have exactly one $b \in \{1, 2, 3, 4\}$ such that $\mathrm{inv} A^b_{i,j} = 1/2$ and $\mathrm{inv} A^k_{i,j} = 0$ for $k \in \{1, 2, 3, 4\} \setminus \{b\}$.

(c) If $v_2(z_{i,j}) < 0$, we have $v_2(1 - \frac{k+1}{z^2_{i.j}}) = 0$ and then $1 - \frac{k+1}{z^2_{i.j}} \equiv 1 \pmod 4$ for all $k = 1, 2, 3, 4$. That is to say , $1 - \frac{k+1}{z^2_{i.j}}$ is always a norm form in $N_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2}(\mathbb{Q}_2(\sqrt{-1})^\times)$. Moreover , the basic property of quaternion algebras , i.e. **Lemma** 2.12 and **Lemma**2.10 , tell us that $A^k_{i,j} - (1 - \frac{k+1}{z^2_{i.j}}, -1) = (z^2_{i,j}, -1) = 0$. Hence $\mathrm{inv}_2 A^k_{i,j} = 0$ for $k = 1, 2, 3, 4$.

Combining the computation results above in (1) , (2) and (3) , we achieve that if $P \in \mathcal{X}_{L-1}(\mathbb{A}_\mathbb{Q})$ then $(A^k_{i,j}, P) := \sum_p \mathrm{inv}_p A^k_{i,j}(P) = 0$ or $1/2$, where $k = 1, 2, 3, 4$. What's more , by the result of part (3) we know that there are at least two $k, k' \in \{1, 2, 3, 4\}$ such that $(A^k_{i,j}, P) \ne (A^{k'}_{i,j}, P)$. Therefore , we can conclude that $\bigcap_{k=1}^4 \mathcal{X}_{L-1}(\mathbb{A}_\mathbb{Q})^{A^k_{i.j}} = \varnothing$. It suffice to justify that $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$.

□

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 November 2022 doi:10.20944/preprints202211.0027.v1

3 Approximating the Real Points

3.1 Outline of the Proof

The proof of emptiness of real-valued points only depends on the original form of 1.1 and will not involve any algebraic-geometry-theorectical notions and techniques . If a real point exists in $\mathcal{X}_{L-1}$, it is equivalent to say that 2 has a real-valued feasible point B . Hence our strategy is to turn back to model 2 and find a rational feasible point close enough to B that also lies in the feasible region of 2 . This will contradict with $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$ we have proven in **Section 2** .

We let $\mathbf{x} = (x_1, \ldots, x_{L-1})$, $g_j(\mathbf{x}) = x_j + \sum_{i=1}^{L-1-j} x_i x_{i+j}$ for $j = 1, \ldots, L-2$. And explicitly write $B = (t_1, \ldots, t_{L-1}) \in \mathbb{R}^{L-1}$ which is the hypothetical real-valued feasible solution of the following equations which has been given in (4) .

$$\begin{cases} \sum_{i=1}^{L-1} x_i = n-1, \\ g_j(\mathbf{x}) \leq 1, \quad for\ j = 1, \ldots, L-2. \\ 0 \leqslant x_i \leqslant 1, i = 1, \ldots, L-1. \end{cases} \tag{6}$$

Notice 3.1. *In this section we use notation (a, b) to denote an open interval in the real axis where $a, b \in \mathbb{R}$. And don't be confused with the notation of a quaternion algebra in Section 2 .*

The main target of this section is to prove :

Theorem 3.2. *(6) has no real feasible solutions .*

Sketch of the Proof. Let B be as above . The special case is that all $g_j(B) < 1$. We firstly work in this case . Suppose

$$\epsilon = \frac{\min_{1 \leq j \leq L-2}(1 - g_j(B))}{2(L-1)}.$$

Define a neighborhood of B as $U_\epsilon := \prod_{i=1}^{L-1}(t_i - \epsilon, t_i + \epsilon)$. It is easy to check that the intersection of U_ϵ and the hyperplane $P_0 : \sum_{i=1}^{L-1} x_i = n-1$ is lie in the feasible region of (6) . However , by the density of $\mathbb{Q}^{L-1} \subseteq \mathbb{R}^{L-1}$, one can definitely find one rational point in $U_\epsilon \bigcap P_0$ and we will demonstrate why this fact is true in **Lemma**3.3 . This leads to a contradiction with $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$.

In the general cases , let $I \subseteq \{1, 2, \ldots, L-2\}$ to denote the index set such that $g_n(B) = 1$ for each $n \in I$. And write $\bar{I}$ the complement of I in $\{1, \ldots, L-2\}$ .That is ,

$$\begin{cases} g_n(B) = 1, & if\ n \in I, \\ g_n(B) < 1, & if\ n \in \bar{I}. \end{cases} \tag{7}$$

And suppose $\delta := \min_{j \in \bar{I}}(1 - g_j(B))/2(L-1)$. Define an open subset $U_\delta := \prod_{i \in I}(t_i - \delta, t_i) \times \prod_{i \in \bar{I}}(t_i - \delta, t_i + \delta)$. One can also find a rational point in $U_\delta \bigcap P_0$ which is also open in $\mathbb{R}^{L-1}$. This also contradicts with the fact that $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$ (we will give a proof of this issue in **Lemma** 3.3 in the next subsection) .

□

3.2 Filling the Gap in the Proof of $\mathcal{X}_{L-1}(\mathbb{R}) = \varnothing$

3.2.1 Reduce the Irrational Point to a Rational Point

In this subsection we prove some technical issues for the front subsection . The main goat is to construct a rational feasible from the given hypothetical point B .

Lemma 3.3. *Let U_ϵ and P_0 be the same as defined in the proof of Theorem 3.2 . Then (1) when $g_j(\mathbf{x}) < 1$ for all $1 \le j \le L-1$, $U_\epsilon \cap P_0 \cap \mathbb{Q}^{L-1}$ is nonempty and contained in the feasible region of the system (6) ; (2) generally , let δ , U_δ be as stated in the last paragraph in the previous subsection , $U_\delta \cap P_0 \cap \mathbb{Q}^{L-1}$ is also nonempty .*

Proof. (1) Firstly we need to show that U_ϵ lies in the feasible region of $g_j(\mathbf{x}) < 1$. Let $\mathbf{u} \in U_\epsilon$. Then

$$\begin{aligned} g_j(\mathbf{u}) - g_j(B) &= u_j + \sum_{i=1}^{L-1-j} u_i u_{i+j} - t_j - \sum_{i=1}^{L-1-j} t_i t_{i+j} && (8)\\ &\leqslant \epsilon + \sum_{i=1}^{L-1-j} (2\epsilon + \epsilon^2) && (9)\\ &< 2(L-1)\epsilon && (10)\\ &= \min_{1 \le j \le L-2} (1 - g_j(B)) && (11) \end{aligned}$$

Hence $g_j(\mathbf{u}) \leqslant 1$ for all $1 \leqslant j \leqslant L-2$. This justifies that U_ϵ lies in the feasible region of $g_j(\mathbf{x}) < 1$. Then it remains to show that $U_\epsilon \cap P_0 \cap \mathbb{Q}^{L-1}$ is nonempty .

It is easy to construct such a rational point in $U_\epsilon \cap P_0 \cap \mathbb{Q}^{L-1}$. Not uniquely , choose $s_1 \in (t_1 - \frac{\epsilon}{L-1}, t_1 + \frac{\epsilon}{L-1}) \cap \mathbb{Q}$, $i = 1, \ldots, L-2$, and choose $s_{L-1} = n - 1 - \sum_{i=1}^{L-2} s_i$. It is clear that :

$$\begin{aligned} |s_{L-1} - t_{L-1}| &\leqslant \sum_{i=1}^{L-2} |s_i - t_i| && (12)\\ &< \epsilon && (13) \end{aligned}$$

It implies that $(s_1, \ldots, s_{L-1}) \in U_\epsilon \cap P_0 \cap \mathbb{Q}^{L-1}$.

(2) The proof of the general part is similar but requires more delicate shifting of each x_i . Set $\{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ an **irrational basis** of $\{t_1, \ldots, t_{L-1}\}$ where $\alpha_1 = 1$, here by an irrational basis we mean that each t_i can be uniquely written as the summation from α_1 to α_t with rational coefficients . If $t = 1$, then all the terms t_i are rational numbers that contradicts wuth $\mathcal{X}_{L-1}(\mathbb{Q}) = \varnothing$. We focus on the case when $t > 2$. Write

$$t_i = \sum_{p=1}^{t} a_{i,p}\alpha_p \ , \qquad i = 1, \ldots, L-1.$$

For some computational reasons we intend to choose the basis properly to make each coefficient $a_{i,p} \geqslant 0$. Assume $1 = \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_t$, and set $\rho_p := \min_{1 \leqslant i \leqslant L-1} a_{i,p}$ for each $p \in \{1, \ldots, t\}$. And suppose that $\alpha_t' := \theta\alpha_t + \sum_{p=1}^{t-1} \rho_p\alpha_p$, where θ is the minimum integer such that $\theta\alpha_t + \sum_{p=1}^{t-1} \rho_p\alpha_p > 0$. It is clear that $\{\alpha_1, \ldots, \alpha_{t-1}, \alpha_t'\}$ is also a transcendental basis for

$\{t_1,\ldots,t_{L-1}\}$. Furthermore , under this basis $\{\alpha_1,\ldots,\alpha_{t-1},\alpha'_t\}$, the coefficient $a'_{i,p}$ of α_p for t_i is always non-negative . And we use this basis in the following computations .

And we have

$$\sum_{i=1}^{L-1} t_i = \sum_{i=1}^{L-1}\sum_{p=1}^{t} a'_{i,p}\alpha'_p \tag{14}$$
$$= \sum_{p=1}^{t} c_p\alpha'_p \tag{15}$$
$$= n-1, \tag{16}$$

where $c_p = \sum_{i=1}^{L-1} a'_{i,p}$. It implies that $c_1 = n-1$ and $c_p = 0$ for all $p \geqslant 2$ for $\{\alpha_1,\ldots,\alpha'_t\}$ are linearly independent over $\mathbb{Q}$. Moreover , rewriting other equations ($g_j(B) = 1$ when $j \in I$) under this basis we have

$$x_j + \sum_{i=1}^{L-1-j} x_i x_{i+j} = \sum_{p=1}^{t} a'_{i,p}\alpha_p + \sum_{i=1}^{L-1-j}(\sum_{p=1}^{t} a'_{i,p}\alpha_p)\cdot(\sum_{p=1}^{t} a'_{i+j,p}\alpha_p) \tag{17}$$
$$= \sum_{i=1}^{p} d_{i,p}\alpha'_p + \sum_{\substack{2\leqslant p\leqslant t\\ 2\leqslant q\leqslant t}} l_{jpq}\alpha'_p\alpha'_q \tag{18}$$
$$= 1, \quad j \in I \tag{19}$$

where $d_{i,p}$ is the sum of rational coefficients in (17) whereas l_{jpq} is the sum of coefficients of $\alpha_p\alpha_q$. Notice that the coefficients $d_{i,p}$ and l_{jpq} in (18) are all non-negative . If we substitute α'_p by a slightly smaller rational number , the value of (15) is still equal to $n-1$ and the value of (18) will not exceed 1 . Choose t positive rational numbers $\{\pi_p\}_{1\leqslant p\leqslant t}$ where π_p is in the interval

$$(\max\Big\{0, \alpha_p - \frac{\delta}{\max\{t, t\cdot \max_{i,p}|a'_{i,p}|\}}\Big\}, \alpha_p),$$

and $\delta = \min_{j\in\bar{I}}(1-g_j(B))/2(L-1)$ as defined in the previous subsection . And substitute all α'_p by π_p and define

$$b_i := \sum_{p=1}^{t} a'_{i,p}\pi_p, \quad i = 1,\ldots,L-1.$$

Of course we have $0 \leqslant b_i \leqslant t_i \leqslant 1$. Moreover we have

$$|b_i - t_i| \leqslant \sum_{p=1}^{t} a'_{i,p}|\pi_p - \alpha'_p| \tag{20}$$
$$< \sum_{p=1}^{t} \delta/t \tag{21}$$
$$= \delta. \tag{22}$$

Without doubt b_i is in $\mathbb{Q}$ for all $1 \leqslant i \leqslant L-1$. Denote $Q = (b_1,\ldots,b_{L-1}) \in \mathbb{Q}^{L-1}$. What's more , we have

$$\begin{cases} \sum_{i=1}^{L-1} b_i = n-1, \\ g_j(Q) = b_j + \sum_{i=1}^{L-1-j} b_i b_{i+j} \leqslant 1, \quad j \in I. \end{cases} \tag{23}$$

And for $j \in \bar{I}$,

$$|g_j(Q) - g_j(B)| = |(b_j - t_j) + \sum_{i=1}^{L-1-j} (b_i b_{i+j} - t_i t_{i+j})| \tag{24}$$

$$< \delta + \sum_{i=1}^{L-1-j} \big((t_i + t_{i+j})\delta + \delta^2\big) \tag{25}$$

$$< 2(L-1-j)\delta + \delta + (L-1-j)\delta^2 \tag{26}$$

$$\leqslant 2(L-j)\delta \tag{27}$$

$$< \min_{j \in \bar{I}} (1 - g_j(B)). \tag{28}$$

This immediately forces that $g_j(Q) \leqslant 1$ for all $1 \leqslant j \leqslant L-1$. That is to say , Q lies in the the feasible region of 6 as a rational points .

□

3.2.2 Conclusion

Combining results in Section2 and Section 3 we give an entire proof of conjecture 1.1 . The method we use in the proof is purely algebraic and arithmetic including some elementary computations . In general one can expect to reformulate a optimization problem as an algebraic geometric problem which is extremely powerful to deal with some special problems , for examples , the existence of integer feasible solutions of a given optimization model whose constraints are all rational polynomials . It is also expected to find out more optimization models whose discrete case and continuous case have the same optimal value .

4 Acknowledgement

Firstly the author wants to thank his supervisor LUO CUICUI for her patient guidance and encouragements during the time when writing this paper . Secondly the author wants to thank Wang Yidan for her help on typing the LaTeX format of this paper . Lastly the author wants to sincerely thank all the referees .

References

[1] W. C. Babcock. Intermodulation interference in radio systems frequency of occurrence and control by channel selection. *The Bell System Technical Journal*, 32(1):63–73, 1953.

[2] E. Blum, J. Ribes, and F. Biraud. Some new possibilities of optimum synthetic linear arrays for radioastronomy. *Astronomy and Astrophysics*, 41:409–411, 1975.

[3] B. D. Conrad. *Arithmetic algebraic geometry*, volume 9. American Mathematical Soc., 2001.

[4] P. Duxbury, C. Lavor, and L. l. de Salles-Neto. A conjecture on a continuous optimization model for the golumb ruler problem. *RAIRO–Operations Research*, 2021.

[5] A. Grothendieck. Le groupe de brauer. i. algèbres d'azumaya et interprétations diverses. *Dix exposés sur la cohomologie des schémas*, 3(46-66):15, 1968.

[6] A. Grothendieck. Le groupe de brauer. ii. théorie cohomologique. *Dix exposés sur la cohomologie des schémas*, 3:67–87, 1968.

[7] R. Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[8] V. A. Iskovskikh. Minimal models of rational surfaces over arbitrary fields. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 43(1):19–43, 1979.

[9] B. Kocuk and W.-J. v. Hoeve. A computational comparision of optimization methods for the golumb ruler problem. In *International Conference on Intergration of Constraint Programming , Artificial Intelligence and Operation Research*, pages 409–425. Springer, 2019.

[10] E. Landau. *Elementary number theory*, volume 125. American Mathematical Society, 2021.

[11] Q. Liu and R. Erne. *Algebraic Geometry and Arithmetic Curves*, volume 6. Oxford University Press, 2006.

[12] J. Neukirch. *Algebraic number theory*, volume 322. Springer Science & Business Media, 2013.

[13] B. Poonen. *Rational points on varieties*, volume 186. American Mathematical Soc., 2017.

[14] J. Robinson and A. Bernstein. A class of binary recurrent codes with limited error propagation. *IEEE Transactions on Information Theory*, 13(1):106–113, 1967.

[15] V. Scharaschkin. *Local-global problems and the Brauer-Manin obstruction*. University of Michigan, 1999.

[16] J.-P. Serre. *Local Fields*, volume 67. Springer Science & Business Media, 2013.

[17] I. R. Shafarevich and M. Reid. *Basic algebraic geometry*, volume 2. Springer, 1994.

[18] J. Voight. *Quaternion algebras*. Springer Nature, 2021.

[19] A. Weil. *Basic number theory.*, volume 144. Springer Science & Business Media, 2013.