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Article

Two-Parameter Exponentially-Fitted Taylor Method for Oscillatory/Periodic Problems

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Abstract: Classical numerical methods for solving ordinary differential equations often produce less accurate results when applied to problems with oscillatory or periodic behaviour. To adapt them for such problems, they are usually modified using the exponential fitting technique. This adaptation allows for the construction of new methods from their classical counterparts. The new methods are usually more accurate, efficient and suitable for handling the oscillatory or periodic behaviour of the problem. In this work, we construct a two-parameter exponentially-fitted Taylor method suitable for solving oscillatory or periodic problems that possess two frequencies. The construction algorithm is based on a proposed six-step flowchart discussed by authors in related literature. Two standard test problems were used to illustrate the accuracy and performance of the proposed method.

Keywords: Taylor; Exponentially-fitted; two-parameter; periodic; oscillatory; frequency

1. Introduction

One important and interesting class of initial value problems that often arise in practice consists of Ordinary Differential Equations (ODEs) whose solutions are periodic or oscillates with known frequency(ies). Usually, classical numerical methods produce less accurate results when applied to such problems but they can be adapted to efficiently give better accuracy [1–3]. This adaptation is the core of exponential fitting technique [3–7]. The adapted classical method is modified such that it is exact for problem whose solution is a linear combination of

$$1, t, \cdots, t^K, \exp(\pm\omega t), t\exp(\pm\omega t), \cdots, t^P\exp(\pm\omega t) \tag{1}$$

where K and P are integers. The adaptation of several classical methods for problems with the fitting space (1) has gained popularity in recent times [5,7–13]. However, very little has been done in the construction of exponentially fitted methods for fitting space with multiple frequencies [3,6,14]. Authors in [15–17], constructed Runge–Kutta type method for the fitting space

$$1, t, \cdots, t^K, \exp(\pm\omega t), \exp(\pm 2\omega t), \cdots, \exp(\pm(P+1)\omega t). \tag{2}$$

In [6], a multiparameter exponentially-fitted Numerov method for the space

$$1, t, \cdots, t^K, \exp(\pm\omega_0 t), \exp(\pm\omega_1 t), \cdots, \exp(\pm\omega_p t). \tag{3}$$

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for solving periodic problems with more than one frequency was proposed. This work is related to [6,12,18] and shall provide extension by the construction of a two-parameter exponentially fitted Taylor method for the fitting space

$$\left.\begin{array}{r} 1, t, \cdots, t^K, \exp(\pm\omega_i t), t\exp(\pm\omega_i t), \cdots, t^{P_i}\exp(\pm\omega_i t), \\ i = 1,2 \end{array}\right\} \tag{4}$$

2. Construction of Method

The general r-th order Taylor scheme is given as

$$u_{j+1} = u_j + hu'_j + \frac{1}{2}h^2 u''_j + \cdots + \frac{1}{r!}h^r u_j^{(r)}. \tag{5}$$

We set $r = 4$ in (5) to get the classical fourth-order Taylor method (6)

$$u_{j+1} = u_j + hu'_j + \frac{1}{2}h^2 u''_j + \cdots + \frac{1}{4!}h^4 u_j^{(4)}, \tag{6}$$

to be fitted exponentially in this work. Following the six-step algorithm proposed in [3], the base method (6) is first written in a more general form as

$$u_{j+1} = \alpha_0 u_j + \beta_1 h u'_j + \beta_2 h^2 u''_j + \beta_3 h^3 u_j^{(3)} + \beta_4 h^4 u_j^{(4)}. \tag{7}$$

and the associated linear difference operator $\mathcal{L}[h, \mathbf{a}]$ is given as

$$\mathcal{L}[h, \mathbf{a}]u(t) \quad = \quad u(t+h) - \alpha_0 u(t) - \beta_1 h u'(t) - \beta_2 h^2 u''(t) - \beta_3 h^3 u^{(3)}(t) - \beta_4 h^4 u^{(4)}(t)$$

where $\mathbf{a} := (\alpha_0, \beta_1, \beta_2, \beta_3, \beta_4)$. Next, we determine the maximum value of M that makes the system

$$L^*_m(\mathbf{a}) \quad = \quad h^{-m}\mathcal{L}[h, \mathbf{a}]t^m|_{t=0} = 0 | m = 0, 1, 2, \cdots M-1 \tag{8}$$

compatible. The system obtained from (8) is given as

$$L^*_0(\mathbf{a}) = 1 - \alpha_0 = 0 \tag{9}$$
$$L^*_1(\mathbf{a}) = 1 - \beta_1 = 0 \tag{10}$$
$$L^*_2(\mathbf{a}) = 1 - 2\beta_2 = 0 \tag{11}$$
$$L^*_3(\mathbf{a}) = 1 - 6\beta_3 = 0 \tag{12}$$
$$L^*_4(\mathbf{a}) = 1 - 24\beta_4 = 0 \tag{13}$$

and is compatible with $M = 5$. Solving (9) yields the coefficient of the well-known classical fourth-order Taylor method here referred to as ***S0***. In order to fit (6) exponentially, six-step algorithm requires that we obtain the expressions for $G^+(Z_i, \mathbf{a})$ and $G^-(Z_i, \mathbf{a})$ respectively defined as

$$G^+(Z_i, \mathbf{a}) = \frac{1}{2}(E^*_0(z_i, \mathbf{a}) + E^*_0(-z_i, \mathbf{a})) \tag{14}$$
$$G^-(Z_i, \mathbf{a}) = \frac{1}{2z_i}(E^*_0(z_i, \mathbf{a}) - E^*_0(-z_i, \mathbf{a})) \tag{15}$$

where

$$E^*_0(\pm z_i, \mathbf{a}) = e^{\mp\omega_i t}\mathcal{L}[h, \mathbf{a}]e^{\pm\omega_i t}$$

and

$$Z_i = z_i^2, \quad z_i = \omega_{h_i} = \omega_i h, \quad i = 1, 2.$$

Now, the respective expressions for $G^+(Z_i, \mathbf{a})$ and $G^-(Z_i, \mathbf{a})$ are obtained as

$$\begin{aligned} G^+(Z_i,\mathbf{a}) &= -\alpha_0 - \beta_4\omega_{h_i}^4 - \beta_2\omega_{h_i}^2 + \cosh(\omega_{h_i}) \\ G^-(Z_i,\mathbf{a}) &= -\beta_1 - \beta_3\omega_{h_i}^2 + \frac{\sinh(\omega_{h_i})}{\omega_{h_i}} \end{aligned}$$

where ω_i, the frequencies of oscillation are real or imaginary. Considering the general fitting space (4) with $M = 5$, application of step IV of the algorithm gave rise to a two-parameter exponentially fitted variant of (6) referred to as ***S1*** and characterized by:

$S1 : (K, P_1, P_2) = (0, 0, 0)$**:** The two-parameter exponentially fitted case with the set

$$\{1, \exp(\pm\omega_1 t), \exp(\pm\omega_2 t)\}.$$

By solving the nonlinear algebraic system (16), obtained via step V of the six-step flowchart, the coefficients of the two-parameter exponentially-fitted variant are given in (17)

$$\left.\begin{aligned} 1-\alpha_0 &= 0 \\ -\alpha_0 - \beta_4\omega_{h_1}^4 - \beta_2\omega_{h_1}^2 + \cosh(\omega_{h_1}) &= 0 \\ -\beta_1 - \beta_3\omega_{h_1}^2 + \frac{\sinh(\omega_{h_1})}{\omega_{h_1}} &= 0 \\ -\alpha_0 - \beta_4\omega_{h_2}^4 - \beta_2\omega_{h_2}^2 + \cosh(\omega_{h_2}) &= 0 \\ -\beta_1 - \beta_3\omega_{h_2}^2 + \frac{\sinh(\omega_{h_2})}{\omega_{h_2}} &= 0 \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} \alpha_0 &= 1 \\ \beta_1 &= \frac{Z_1^{3/2}\sinh\left(\sqrt{|Z_2|}\right) - Z_2^{3/2}\sinh\left(\sqrt{|Z_1|}\right)}{Z_1^{3/2}\sqrt{|Z_2|} - Z_2^{3/2}\sqrt{|Z_1|}} \\ \beta_2 &= \frac{Z_1^2\left(\cosh\left(\sqrt{|Z_2|}\right)-1\right) - Z_2^2\left(\cosh\left(\sqrt{|Z_1|}\right)-1\right)}{Z_1(Z_1-Z_2)Z_2} \\ \beta_3 &= \frac{\sqrt{|Z_2|}\sinh\left(\sqrt{|Z_1|}\right) - \sqrt{|Z_1|}\sinh\left(\sqrt{|Z_2|}\right)}{Z_1^{3/2}\sqrt{|Z_2|} - Z_2^{3/2}\sqrt{|Z_1|}} \\ \beta_4 &= \frac{Z_2\left(\cosh\left(\sqrt{|Z_1|}\right)-1\right) - Z_1\left(\cosh\left(\sqrt{|Z_2|}\right)-1\right)}{Z_1(Z_1-Z_2)Z_2} \end{aligned}\right\} \tag{17}$$

As expected, the exponentially-fitted variant reduce to the classical method as $Z \to 0$.

3. Error Analysis : Local Truncation Error (*LTE*)

The leading term of the local truncation error (*lte*) for an exponentially-fitted method with respect to the basis (4) is of the form

$$\begin{aligned} LTE(t) &= (-1)^{\sum_{i=1}^{I} P_i + I} h^M \frac{\mathcal{L}_{K+1}^*(\mathbf{a}(Z_i))}{(K+1)! Z_1^{P_1+1} \cdots Z_I^{P_I+1}} \times \\ & \prod_{i=1}^{I} (D^2 - \omega_i^2)^{P_i+1}, \quad D^m := \frac{d^m}{dt^m} \end{aligned} \tag{18}$$

with $K, P_1, \cdots, P_I$ and M satisfying the condition $K + 2(P_1 + \cdots + P_I) = M - 2I - 1$, ([19]). Using (18), local truncation error our exponentially-fitted method is obtained as:

$$LTE = h^5 \frac{(\beta_1 - 1)\left(u^{(5)}(t) - \omega_2^2 u^{(3)}(t) + \omega_1^2\left(\omega_2^2 u'(t) - u^{(3)}(t)\right)\right)}{Z_1 Z_2} \tag{19}$$

4. Numerical Results

A standard test problem was used to demonstrate the performance of our new method, referred to as *EF2PT*. We implement the constructed method on this test problem and the

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results obtained were compared with those of the classical fourth-order Taylor and Runge-Kutta (RK-4) methods. Consider the initial value problem given as

$$u'' = \frac{3}{4}u - \exp(t)\sin\left(\frac{t}{2}\right), \quad u(0) = 1, \; u'(0) = 1 \tag{20}$$

whose exact solution is

$$u(t) = \exp(t)\cos\left(\frac{t}{2}\right) \tag{21}$$

Problem (20) has two complex conjugate frequencies which are: $\omega_1 = 1 + \frac{1}{2}i$ and $\omega_2 = 1 - \frac{1}{2}i$ and has been studied by [6,12]. Here, we solve this problem using different steplength h, the maximum absolute error for each steplength is obtained and presented in Figure 1. Clearly, the two-parameter exponentially-fitted method gave better results compared with

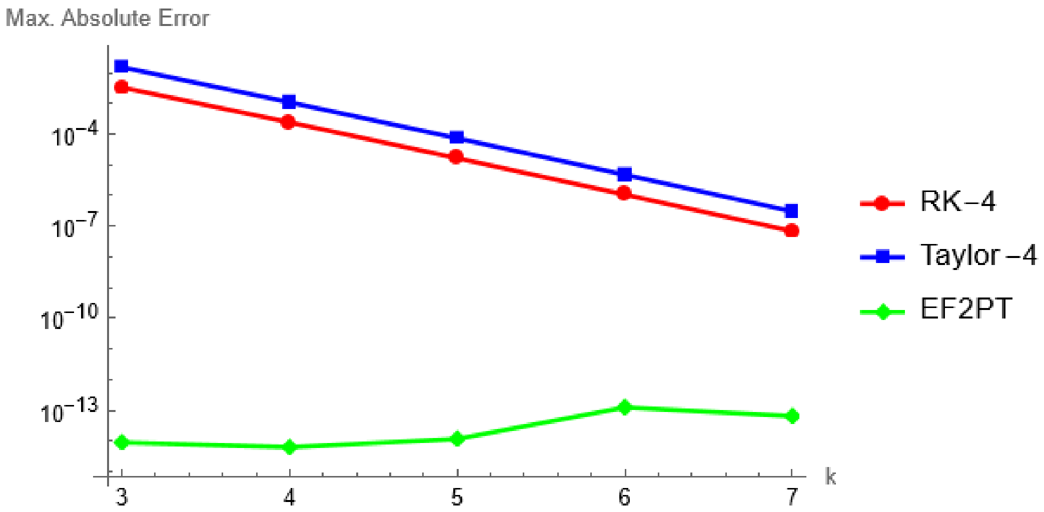

Figure 1. Maximum absolute errors for Problem 1 as a function of the step-size $h = 2^{-k}, k = 3(1)7$

its classical counterpart and the Runge–Kutta method as seen in Figure 1.

5. Conclusion

The two-parameter exponentially-fitted Taylor method constructed in this work is of algebraic order four and is self-starting. Its local truncation error is of order five. Compared with its classical counterparts, the results obtained from the numerical example showed that the new method is suitable for solving periodic/oscillatory problems.

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