doi:10.20944/preprints202210.0473.v1

*Article*

# Parameter Estimation for Some Probability Distributions Used in Hydrology

**Cristian Gabriel Anghel and Cornel Ilinca** *

Faculty of Hydrotechnics, Technical University of Civil Engineering Bucharest, Lacul Tei, nr.122-124, Bucharest 020396, Romania;
* Correspondence: cornel@utcb.ro; Tel.: +40-723-071-247

**Abstract:** Estimating the parameters of statistical distributions generally involves solving a system of nonlinear equations or a nonlinear equation, being a technical difficulty in their usual application in hydrology. This article presents improved approximations and, in some cases, new approximations for the estimation with the method of ordinary moments and the method of linear moments, which are useful for the direct calculation of the parameters, because the errors in the approximate estimation are similar to the use of iterative numerical methods. Thirteen statistical distributions of two and three parameters frequently used in hydrology are presented, for which parameter estimation was laborious. Thus, the approximate estimation of the parameters by the two methods is simple but also precise and easily applicable by hydrology researchers. The approximate forms characterized by very small errors are the result of the research conducted within the Faculty of Hydrotechnics to update the romanian normative standards in the hydrotechnical field.

**Keywords:** probability distributions; estimation parameters; approximate form; method of ordinary moments; method of L-moments.

## 1. Introduction

Statistical distributions are used in hydrology to determine the probability of occurrence of extreme events: maximum flows, maximum volumes, maximum precipitation.

A criticism of the probabilistic estimation of extreme events is that it does not take climate changes into account, this being generally solved with hydrological modeling using learning machines [1] of the neural network type and others. However, statistical modeling is complementary and necessary to other types of black-box and gray-box modeling, because the input and output data from these models must be presented probabilistically, especially for extreme events with low probability, necessary for the design of hydrotechnical constructions.

In the case of determining the maximum flows, three or more parameter distributions are recommended, while for the determination of precipitation and maximum volumes, two parameter distributions are recommended [2-4].

In general, finding approximate relations to express the parameters is very important because many of these probability distributions, for exact determination, require solving with the Levenberg-Marquardt algorithm [5] for nonlinear least squares curve-fitting problems, which can sometimes represent a difficulty requiring the use of specific calculation programs.

Another important aspect is the expression of the quantile function (inverse of the distribution function) of these distributions, both of which facilitate the use of these distributions in determination of the analyzed extreme values.

This article presents approximate forms of the expressions of some parameters of some frequently used probability distributions in hydrology, such as: Pearson III (PE3), Generalized extreme value (GEV), Weibull (W3), Weibull (W2), Fréchet (F3), Fréchet (F2),

 doi:10.20944/preprints202210.0473.v1



Generalized Pareto (P3), Log-Logistics (LG3), Kappa (K3-Generalized Gumbel), Kappa (K3-Park), Pearson V (PV3), Pearson V (PV2), Generalized Exponential (EG2).

The approximate forms are for determining the parameters using the method of ordinary moments (MOM) and the method of linear moments (L-moments). These methods are some of the most frequently used methods for estimating parameters in hydrology [2, 3, 4, 6, 8]. The approximations use rational and polynomial functions in which the coefficients were obtained by least squares nonlinear regression on the domain of definition considered.

In some cases the approach was similar by using the same type of function but with more terms, and in other cases the approach was different by using other types of functions.

For the method of ordinary moments, the approximation range most often used for skewness is 0-4, but in certain regions of the world they meet up to a skewness equal to 9, [3]. The definition range of approximations 0-8 was chosen in the case of MOM, and the definition range for the L-moments method is the default 0-1. Considering that the approximation relations must be as accurate as possible, the relative error graphs are also presented. The relative error is the ratio between the difference between the exact and approximate value of the skewness, and the exact value. In this article, the new approximation relations are noted as "a better approximation". In Romania, the skewness is adopted by multiplying the coefficient of variation with a chosen positive coefficient [9], so the skewness is always positive, even if that of the given sample may have negative skewness values. For the L-skewness of the sample, it can have negative values, generally greater than -0.4.

Thus, all of these novelty elements  such as s better approximation for PE3 shape parameter for L-moments, a better and a new approximation for GEV shape parameter for MOM and L-moments, a new approximation for W3 shape parameters for MOM and L-moments, a new approximation for W2 shape parameters for MOM, a new approximation for F3 shape parameters for MOM and L-moments, a new approximation for F2 shape parameters for MOM, a new approximation for P3 shape parameters for MOM, a new approximation for K3 shape parameters for MOM and L-moments, a new approximation for PV3 shape parameters for MOM and L-moments, a new approximation for PV2 shape parameters for L-moments, a new approximations for EG2 parameter for MOM and L-moments, will help hydrology researchers to use these distributions easily. The research also had as its object the use of mathematics by the large mass of engineers in the field of hydrotechnics, because the use of dedicated software without knowledge of mathematical methods is not beneficial. In this way, functions from Mathcad were presented comparatively, which can be equated by researchers with functions from their favorite programs (Matlab, Excel, Python, R, etc.).

## 2. Probability distributions

In this section the analyzed probability distributions are presented, the approximate estimation of the parameters for MOM and L-moments, using various approximation functions; the variation graphs of approximately estimated parameters, as well as the graphs of their relative estimation errors.

For some statistical distributions, the same type of functions, rational or polynomial, were used, and in some cases both functions were used on some distinct defining intervals.

The new approximations compared to the existing ones are characterized by smaller relative errors generally due to the increase in the number of terms of the approximation function, i.e. by a higher degree of the approximation polynomials.

This section presents the general description of each distribution and the functions of their parameters, thus highlighting the influence of the estimated parameters in their

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



structure, which are important aspects in the use of the estimated value in the direct determination of the other parameters of the distribution.

The estimated parameters for the analyzed distribution, can only be obtained exactly by numerical methods, because the moments equations are non-linear, being presented in Appendix B. Mathematical notations for the built-in functions in the Mathcad program used in this article are shown in appendix A.

*2.1. Pearson III Distribution (PE3)*

The Pearson III distribution belongs to the Gamma distribution family, being a generalized form of the two-parameter Gamma distribution [10], with shifted x, and a particular case of the four-parameter gamma distribution [5, 11].

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and quantile function, $x(p)$, of the Pearson III distribution are [12]:

$$f(x)=\frac{(x-\gamma)^{\alpha-1}}{\beta^{\alpha}\cdot\Gamma(\alpha)}\cdot\exp\left(-\frac{x-\gamma}{\beta}\right)=\frac{1}{\beta}\cdot\text{dgamma}\left(\frac{x-\gamma}{\beta},\alpha\right) \tag{1}$$

$$F(x)=1-\int_{\gamma}^{x}f(x)dx=1-\frac{1}{\beta\cdot\Gamma(\alpha)}\cdot\int_{\gamma}^{x}\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}\cdot\exp\left(-\frac{x-\gamma}{\beta}\right)dx=\frac{\Gamma\left(\alpha,\frac{x-\gamma}{\beta}\right)}{\Gamma(\alpha)} \tag{2}$$

$$F(x)=1-\text{pgamma}\left(\frac{x-\gamma}{\beta},\alpha\right)=1-\text{pchisq}\left(2\cdot\frac{x-\gamma}{\beta},2\cdot\alpha\right) \tag{3}$$

$$\begin{aligned}
x(p)&=F^{-1}(x)=\gamma+\beta\cdot\Gamma^{-1}(1-p;\alpha)=\gamma+\beta\cdot\text{qgamma}(1-p,\alpha)\\
x(p)&=\mu+\sigma\cdot\frac{\Gamma^{-1}(1-p;\alpha)-\alpha}{\sqrt{\alpha}}=\mu+\sigma\cdot\frac{\text{qgamma}(1-p,\alpha)-\alpha}{\sqrt{\alpha}}\\
x(p)&=\gamma+\beta\cdot\text{qchisq}(1-p,2\cdot\alpha)=\mu+\sigma\cdot\left(\frac{\text{qchisq}(1-p,2\cdot\alpha)}{2\cdot\sqrt{\alpha}}-\sqrt{\alpha}\right)
\end{aligned} \tag{4}$$

where $\alpha,\beta,\gamma$ are the shape, the scale and the position parameters and $x$ can take any values of range $\gamma<x<\infty$ if $\beta>0$ or $-\infty<x<\gamma$ if $\beta<0$ and $\alpha>0$; $\mu,\sigma$ represent the mean (expected value) and standard deviation. If $\beta<0$ (negative skewness) then the first argument of the inverse of the distribution function Gamma, $\Gamma^{-1}(1-p;\alpha)$ becomes $\Gamma^{-1}(p;\alpha)$.

The expressions of the inverse function $x(p)=F^{-1}(x)$, using the mean and standard deviation, are valid only for the method of ordinary moments.

For estimation with MOM, the distribution parameters have the following expressions [10]:

$$\alpha=\left(\frac{2}{C_s}\right)^2 \tag{5}$$

$$\beta=\frac{\sigma}{2}\cdot C_s \tag{6}$$

$$\gamma=\mu-\alpha\cdot\beta \tag{7}$$

where $C_s$ represents the skewness coefficient.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



The parameter $\alpha$ can be estimated using the rational function presented by Hosking in 1997, named in this section the "Hosking approximation", or based on an approximation made up of two polynomial functions and one rational, depending on the definition domain of the estimated parameter, named here " a better approximation"

Thus, for the estimation with the L-moments, the shape parameter $\alpha$ can be evaluated numerically with the following approximate forms, depending on L-skewness ( $\tau_3$ ):

the Hosking approximation, rational function [10, 13]:

if $0<\left|\tau_3\right|<\frac{1}{3}$:

$$\alpha=\frac{1+2.73884\cdot\tau_3^2}{9.424778\cdot\tau_3^2+16.7171359\cdot\tau_3^4+37.0028906\cdot\tau_3^6} \tag{8}$$

if $\frac{1}{3}\le\left|\tau_3\right|<1$:

$$\alpha=\frac{0.36067\cdot\left(1-\left|\tau_3\right|\right)-0.59567\cdot\left(1-\left|\tau_3\right|\right)^2+0.25361\cdot\left(1-\left|\tau_3\right|\right)^3}{1-2.78861\cdot\left(1-\left|\tau_3\right|\right)+2.56096\cdot\left(1-\left|\tau_3\right|\right)^2-0.77045\cdot\left(1-\left|\tau_3\right|\right)^3} \tag{9}$$

a better approximation, polynomial and rational functions:

if $0<\left|\tau_3\right|\le\frac{1}{3}$:

$$\alpha=\exp\begin{pmatrix}-3.164791927-5.108735285\cdot\ln\left(\left|\tau_3\right|\right)-4.116014079\cdot\ln\left(\left|\tau_3\right|\right)^2-\\ 2.985250105\cdot\ln\left(\left|\tau_3\right|\right)^3-1.327399577\cdot\ln\left(\left|\tau_3\right|\right)^4-0.373944875\cdot\ln\left(\left|\tau_3\right|\right)^5-\\ 0.065421611\cdot\ln\left(\left|\tau_3\right|\right)^6-0.006508037\cdot\ln\left(\left|\tau_3\right|\right)^7-0.000281969\cdot\ln\left(\left|\tau_3\right|\right)^8\end{pmatrix} \tag{10}$$

if $\frac{1}{3}<\left|\tau_3\right|\le\frac{2}{3}$:

$$\alpha=\exp\begin{pmatrix}-3.9918551-10.781466\cdot\ln\left(\left|\tau_3\right|\right)-21.557807\cdot\ln\left(\left|\tau_3\right|\right)^2-\\ 33.8752604\cdot\ln\left(\left|\tau_3\right|\right)^3-35.0641585\cdot\ln\left(\left|\tau_3\right|\right)^4-22.921163\cdot\ln\left(\left|\tau_3\right|\right)^5-\\ 8.5491823\cdot\ln\left(\left|\tau_3\right|\right)^6-1.3855653\cdot\ln\left(\left|\tau_3\right|\right)^7\end{pmatrix} \tag{11}$$

if $\frac{2}{3}<\left|\tau_3\right|<1$:

$$\alpha=\frac{\begin{array}{l}5.17817436-26.209448756\cdot\left|\tau_3\right|+62.12494027\cdot\tau_3^2-84.39423264\cdot\left|\tau_3\right|^3+\\ 67.08589624\cdot\tau_3^4-29.150288079\cdot\left|\tau_3\right|^5+5.364968945\cdot\tau_3^6\end{array}}{1+0.0005134\cdot\left|\tau_3\right|+0.00063644\cdot\tau_3^2} \tag{12}$$

The scale parameter $\beta$ and the position parameter $\gamma$ are determined with the following expressions [13]:

$$\beta=L_2\cdot\sqrt{\pi}\cdot\frac{\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)} \tag{13}$$

$$\gamma=L_1-\alpha\cdot\beta \tag{14}$$

where $L_1, L_2, \tau_3$ are the sample L-moments and L-skewness, [6,10,13].

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



The variation graph of the parameter $\alpha$ depending on skewness and L-skewness is presented in Figure 1.

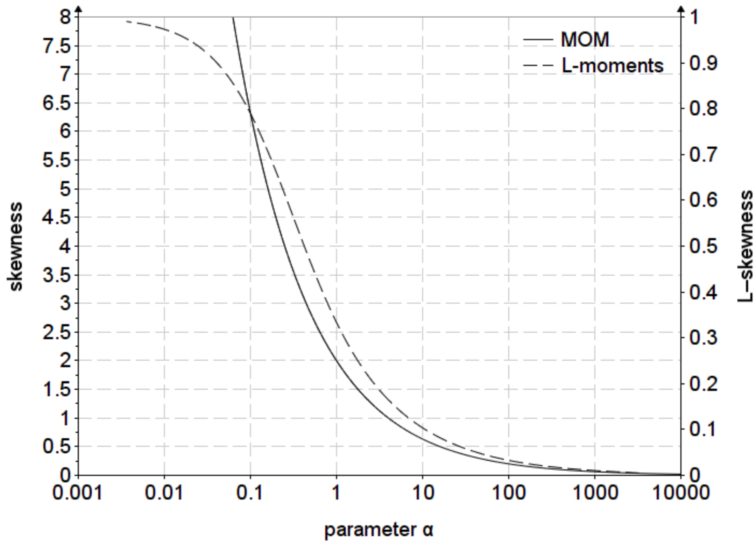



**Figure 1.** The variation of parameter $\alpha$

The relative errors for the approximations are presented in Figure 2. It can be observed that the new approximation is generally better than the Hosking approximation, in the interval of L-skewness $0.1 < \tau_3 < 0.85$.

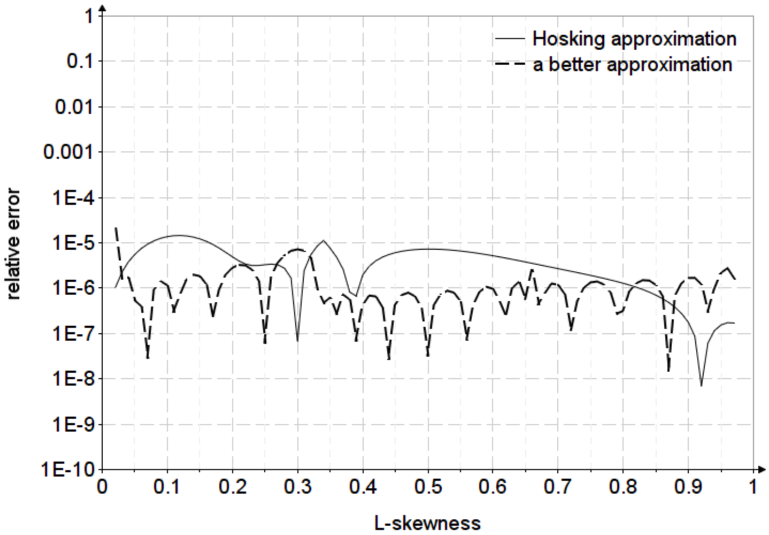



**Figure 2.** The variation of errors depending on $\tau_3$

*2.2. Generalized Extreme Value Distribution (GEV)*

The GEV distribution was introduced for the first time by Jenkins in 1955 [10] for the analysis of extreme values. It is also known as the Fisher–Tippett distribution.

Depending on the sign of the shape parameter, this can be transformed into the Weibull ($\alpha < 0$), Fréchet ($\alpha > 0$) or Gumbel ($\alpha = 0$) distributions.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the GEV distribution are [10,12,13]:

$$f(x)=\left(1-\frac{\alpha}{\beta}\cdot(x-\gamma)\right)^{\frac{1}{\alpha}-1}\cdot\frac{1}{\beta}\cdot\exp\left(-\left(1-\frac{\alpha}{\beta}\cdot(x-\gamma)\right)^{\frac{1}{\alpha}}\right) \tag{15}$$

$$F(x)=1-\exp\left(-\left(1-\frac{\alpha}{\beta}\cdot(x-\gamma)\right)^{\frac{1}{\alpha}}\right) \tag{16}$$

$$x(p)=\gamma+\frac{\beta}{\alpha}\cdot\left(1-\left(-\ln(1-p)\right)^{\alpha}\right) \tag{17}$$

where $\alpha,\beta,\gamma$ are the shape, the scale and the position parameters; $x$ can take any values of range $x<\left(\gamma+\frac{\beta}{\alpha}\right)$ if $\alpha>0$, and $x>\left(\gamma+\frac{\beta}{\alpha}\right)$ if $\alpha<0$.

The parameter $\alpha$ can be estimated using the ''approximation 1'' which represents a polynomial function presented by Rao et.al [10] or using two new approximations of the polynomial or rational type, hereafter named "approximation 2" and "approximation 3".

Thus, the parameter $\alpha$ has the following approximate forms depending on $C_s$:

approximation 1, polynomial function [10]:

if $C_s \le 1.14$:

$$\alpha=0.277648-0.322016\cdot C_s+0.060278\cdot C_s^2+0.016759\cdot C_s^3-0.005873\cdot C_s^4-0.00244\cdot C_s^5-0.000050\cdot C_s^6 \tag{18}$$

if $C_s > 1.14$:

$$\alpha=0.2858221-0.357983\cdot C_s+0.116659\cdot C_s^2-0.022725\cdot C_s^3+0.002604\cdot C_s^4-0.000161\cdot C_s^5+0.000004\cdot C_s^6 \tag{19}$$

a better approximations:

approximation 2, polynomial form:

if $C_s \le 1.14$:

$$\alpha=0.2775961-0.3217629\cdot C_s+0.0608352\cdot C_s^2+0.0170626\cdot C_s^3-0.0100647\cdot C_s^4+0.0004883\cdot C_s^5+0.0003895\cdot C_s^6 \tag{20}$$

If $C_s > 1.14$:

$$\alpha=0.2786297-0.3425507\cdot C_s+0.1041188\cdot C_s^2-0.0177578\cdot C_s^3+0.001584\cdot C_s^4-0.0000572\cdot C_s^5 \tag{21}$$

approximation 3, rational form:

$$\alpha=\frac{0.277593723-0.116805709\cdot C_s-0.071360121\cdot C_s^2-0.035014727\cdot C_s^3}{1+0.73874734\cdot C_s+0.37485038\cdot C_s^2+0.103516119\cdot C_s^3} \tag{22}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



$$\beta = \frac{\alpha}{|\alpha|} \cdot \frac{\sigma \cdot \alpha}{\sqrt{\Gamma(1+2\cdot\alpha) - \Gamma(1+\alpha)^2}} \tag{23}$$

$$\gamma = \mu - \frac{\alpha}{|\alpha|} \cdot \frac{\sigma \cdot \left[1 - \Gamma(1+\alpha)\right]}{\sqrt{\Gamma(1+2\cdot\alpha) - \Gamma(1+\alpha)^2}}$$

The variation graph of the parameter $\alpha$ depending on skewness and L-skewness is presented in Figure 3.

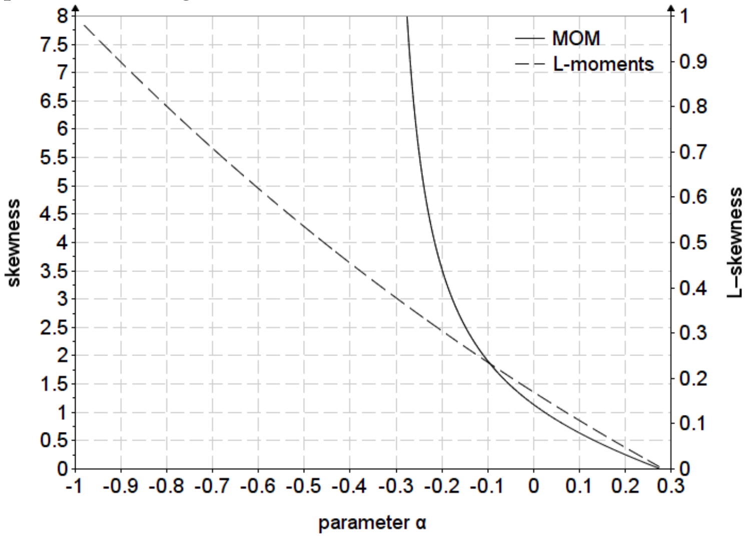



**Figure 3.** The variation of parameter $\alpha$

The relative errors for the approximations are presented in Figure 4. The two new approximations are characterized by smaller relative errors, in the range of 0-1 the polynomial approximation being more accurate, but on the whole domain, the rational approximation is more accurate.

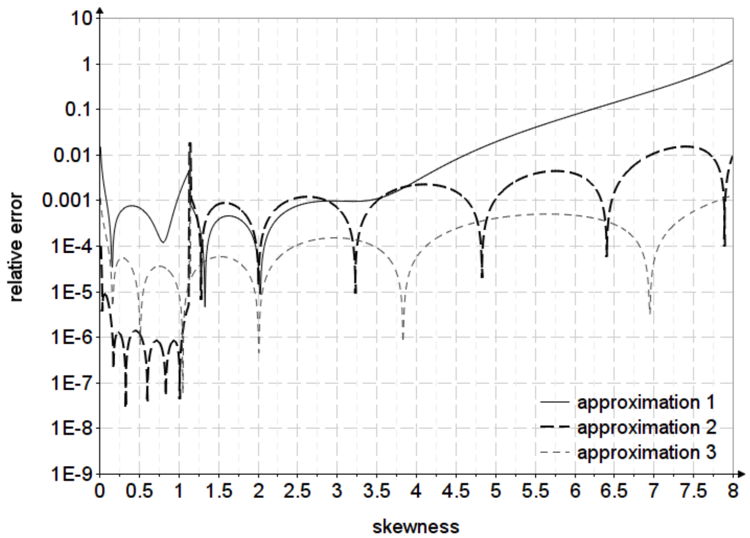



**Figure 4.** The variation of errors depending on $C_s$

For estimation with the L-moments, parameter $\alpha$ has the following approximate forms depending on $\tau_3$:

approximation 1, polynomial form, presented in Rao et.al [10] and Hosking [13]:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



$$\alpha = 7.85890 \cdot \left( \frac{2}{3+\tau_3} - \frac{\ln(2)}{\ln(3)} \right) + 2.9554 \cdot \left( \frac{2}{3+\tau_3} - \frac{\ln(2)}{\ln(3)} \right)^2 \tag{24}$$

approximation 2, also a polynomial function but more accurately, presented in Rao et.al [10]:

$$\alpha = 7.817740 \cdot \left( \frac{2}{3+\tau_3} - \frac{\ln(2)}{\ln(3)} \right) + 2.930462 \cdot \left( \frac{2}{3+\tau_3} - \frac{\ln(2)}{\ln(3)} \right)^2 + 13.641492 \cdot \left( \frac{2}{3+\tau_3} - \frac{\ln(2)}{\ln(3)} \right)^3 + 17.206675 \cdot \left( \frac{2}{3+\tau_3} - \frac{\ln(2)}{\ln(3)} \right)^4 \tag{25}$$

a better approximation, named in this section approximation 3, being a rational function:

$$\alpha = \frac{0.283759107 - 1.669931462 \cdot |\tau_3|}{1 + 0.441588375 \cdot |\tau_3| - 0.071007671 \cdot \tau_3^2 + 0.015634368 \cdot |\tau_3|^3} \tag{26}$$

$$\beta = \frac{L_2}{\Gamma(\alpha) \cdot (1 - 2^{-\alpha})} \tag{27}$$

$$\gamma = L_1 + \frac{\beta}{\alpha} \cdot [\Gamma(1+\alpha) - 1] \tag{28}$$

The relative errors for the approximations are presented in Figure 5. It can be observed, that the new approximation with the rational function is characterized by smaller relative errors over the entire domain of L-skewness.

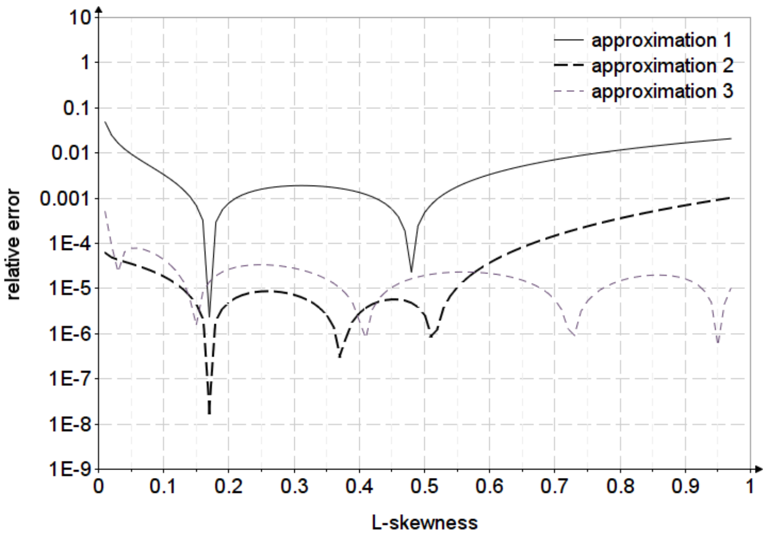



**Figure 5.** The variation of errors depending on $\tau_3$

*2.3. Weibull Distribution (W3)*

The distribution represents a particular case of the GEV distribution when the shape parameter is negative. It is also known as the Type II extreme value distribution, [10].

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the three parameters Weibull distribution are [10,14,15]:

$$f(x)=\frac{\alpha}{\beta}\cdot\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}\cdot e^{-\left(\frac{x-\gamma}{\beta}\right)^{\alpha}}=\frac{\alpha}{\beta}\cdot \text{dweibull}\left(\frac{x-\gamma}{\beta},\alpha\right) \tag{29}$$

$$F(x)=e^{-\left(\frac{x-\gamma}{\beta}\right)^{\alpha}}=1-\text{pweibull}\left(\frac{x-\gamma}{\beta},\alpha\right) \tag{30}$$

$$x(p)=\gamma+\beta\cdot\left(-\ln(p)\right)^{1/\alpha}=\gamma+\beta\cdot \text{qweibull}(1-p,\alpha) \tag{31}$$

where $\alpha,\beta,\gamma$ are the shape, the scale and the position parameters; $\alpha,\beta>0, x\geq\gamma$.

The variation graph of the parameter $\alpha$ depending on skewness and L-skewness is presented in Figure 6.

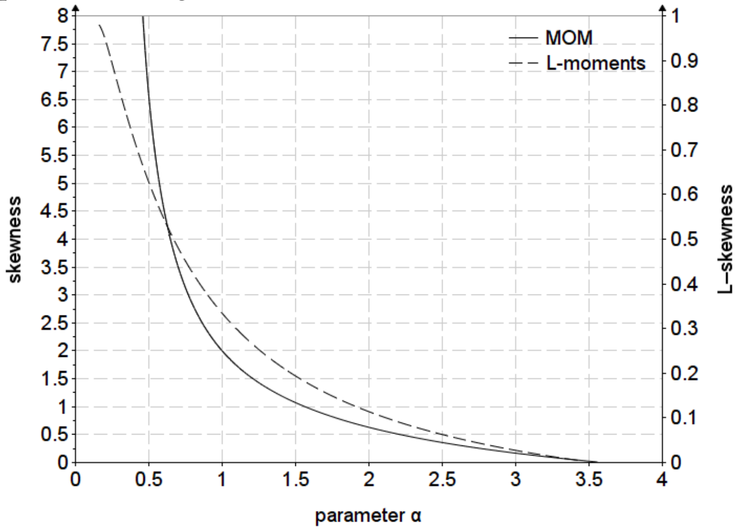



**Figure 6.** The variation of parameter $\alpha$

For estimation with MOM, parameter $\alpha$ can be approximated, depending on the skewness, using the "approximation 1" of the rational function type presented by Rao, valid for a maximum skewness of 2, or using the new approximation called here ‘’a better aproximation’’ valid for a skewness in the range of 0-8:

approximation 1, for $C_s \leq 2$, [10]:

$$\alpha=\frac{1}{0.2777561+0.3219\cdot C_s+0.061566\cdot C_s^2-0.017376\cdot C_s^3-0.00771\cdot C_s^4+0.00398\cdot C_s^5-0.00051\cdot C_s^6} \tag{32}$$

a better approximation, called here approximation 2, valid for $0<C_s\leq 8$:

$$\alpha=\frac{3.5947875-3.7374529\cdot C_s+2.7023151\cdot C_s^2-1.1835102\cdot C_s^3+0.4318107\cdot C_s^4-0.0042306\cdot C_s^5+0.0045545\cdot C_s^6-0.0046984\cdot C_s^7}{1+0.0931316\cdot C_s-0.12783\cdot C_s^2+0.0671082\cdot C_s^3+0.1113667\cdot C_s^4-0.0425475\cdot C_s^5+0.0673629\cdot C_s^6-0.0157628\cdot C_s^7} \tag{33}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



$$\beta = \frac{\sigma}{\sqrt{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}} \tag{34}$$

$$\gamma = \mu - \beta\cdot\Gamma\left(1+\frac{1}{\alpha}\right) \tag{35}$$

The relative errors for the approximations are presented in Figure 7. It can be seen that the new rational aproximation is characterized by much smaller relative errors, especially for skewness greater than 3.

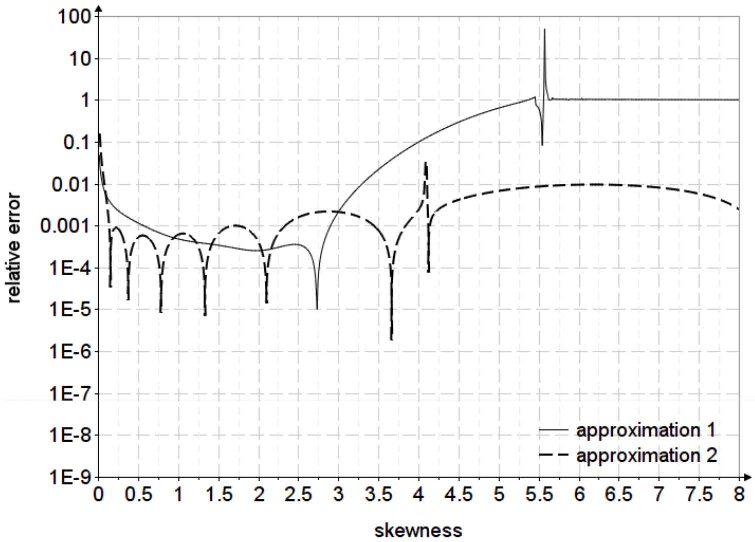



**Figure 7.** The variation of errors depending on $C_s$

For estimation with the L-moments, parameter $\alpha$ has the following approximate forms depending on $\tau_3$:

approximation 1, a rational function prezented by Rao et.al. [10]:

$$\alpha = \frac{1}{7.85890\cdot\left(\frac{2}{3-\tau_3}-\frac{\log(2)}{\log(3)}\right)+2.9554\cdot\left(\frac{2}{3-\tau_3}-\frac{\log(2)}{\log(3)}\right)^2} \tag{36}$$

approximation 2, polynomial form, for positive $\tau_3$, prezented by Y.Goda [14]:

$$\alpha = 285.3\cdot\tau_3^6 - 658.6\cdot\tau_3^5 + 622.8\cdot\tau_3^4 - 317.2\cdot\tau_3^3 + 98.52\cdot\tau_3^2 - 21.256\cdot\tau_3 + 3.516 \tag{37}$$

a better approximation, named in the Figure 8, approximation 3, witch is a rational function, valid for negative and positive values of L-skewness:

$$\alpha = \frac{3.528107902 - 6.294082546\cdot|\tau_3| + 2.767652838\cdot\tau_3^2}{1+4.599024923\cdot|\tau_3| - 7.993601572\cdot\tau_3^2 + 2.423742593\cdot|\tau_3|^3} \tag{38}$$

$$\beta = \frac{L_2}{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\left(1-2^{-\frac{1}{\alpha}}\right)} \tag{39}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



$$\gamma = L_1 - \frac{\beta}{\alpha} \cdot \Gamma\left(\frac{1}{\alpha}\right) \tag{40}$$

The relative errors for the approximations are presented in Figure 8. Comparing the three approximations, it can be observed that the approximation 3 ,using the rational function, is characterized by smaller errors, especially in the range of 0.6-1 of the L-skewness.

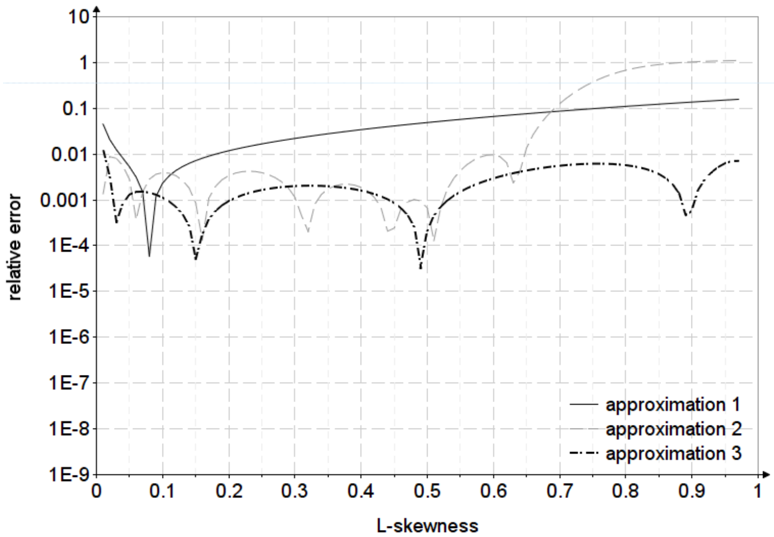



**Figure 8.** The variation of errors depending on $\tau_3$

*2.4. Weibull Distribution (W2)*

The distribution represents a particular case of the W3 distribution when the position parameter is 0.

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the two parameters Weibull distribution are [10,15]:

$$f(x) = \frac{\alpha}{\beta} \cdot \left(\frac{x}{\beta}\right)^{\alpha-1} \cdot \exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right) = \frac{1}{\beta} \cdot \text{dweibull}\left(\frac{x}{\beta}, \alpha\right) \tag{41}$$

$$F(x) = \exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right) = 1 - \text{pweibull}\left(\frac{x}{\beta}, \alpha\right) \tag{42}$$

$$x(p) = \beta \cdot (-\ln(p))^{1/\alpha} = \beta \cdot \text{qweibull}(1-p, \alpha) \tag{43}$$

where $\alpha, \beta$ are the shape and the scale parameters; $\alpha, \beta, x > 0$.

The variation graph of the parameter $\alpha$ depending on coefficient of variation ($C_v$) and L-coefficient of variation ($\tau_2$), is presented in Figure 9.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



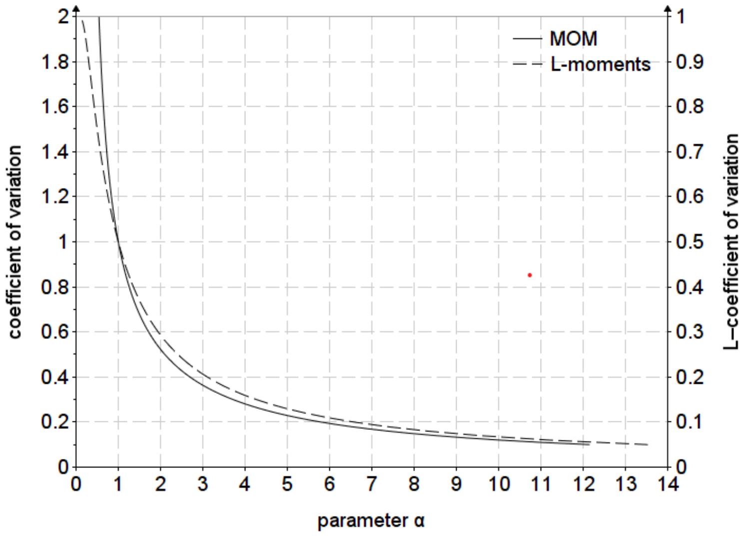



**Figure 9.** The variation of parameter $\alpha$

For estimation with MOM, parameter $\alpha$ has the following approximate forms depending on the coefficient of variation $C_v \geq 0.1$:

$$\alpha = \exp\left(\begin{array}{l} 0.000006199\text{-}1.000228322 \cdot \ln(C_v) + 0.144564211 \cdot \ln(C_v)^2 + \\ 0.055543308 \cdot \ln(C_v)^3 - 0.016265526 \cdot \ln(C_v)^4 - 0.013691787 \cdot \ln(C_v)^5 - \\ 0.000586501 \cdot \ln(C_v)^6 + 0.001239068 \cdot \ln(C_v)^7 + 0.000240162 \cdot \ln(C_v)^8 \end{array}\right) \tag{44}$$

$$\beta = \frac{\mu}{\Gamma\left(1 + \frac{1}{\alpha}\right)} \tag{45}$$

The relative errors for the approximation are presented in Figure 10.

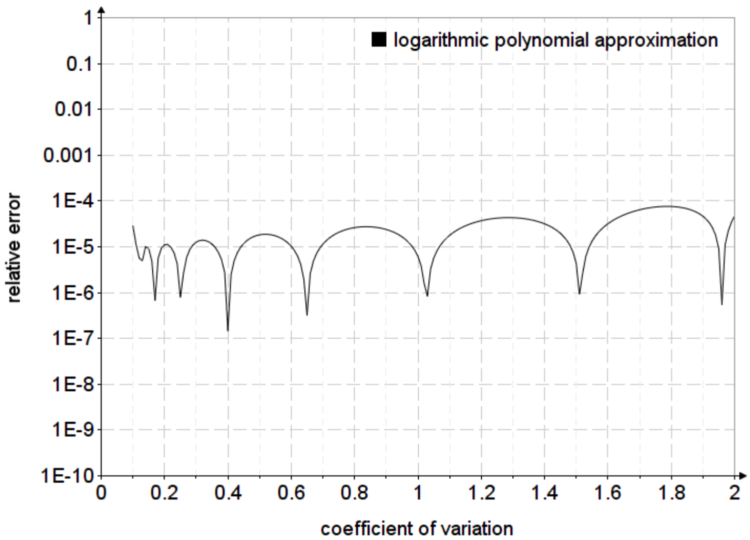



**Figure 10.** The variation of errors depending on $C_v$

For estimation with the L-moments, parameters have the following expressions [10, 15, 16]:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



$$\alpha = \frac{-\ln(2)}{\ln\left(1-\frac{L_2}{L_1}\right)} \tag{46}$$

$$\beta = \frac{L_1}{\Gamma\left(1+\frac{1}{\alpha}\right)} \tag{47}$$

*2.5. Fréchet Distribution (F3)*

The distribution represents a particular case of the GEV distribution when the shape parameter is positive. It is also known as the Type III extreme value distribution, [10].

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the three parameters Fréchet distribution are [17]:

$$f(x) = \frac{\alpha}{\beta}\cdot\left(\frac{x-\gamma}{\beta}\right)^{-\alpha-1}\cdot\exp\left(-\left(\frac{x-\gamma}{\beta}\right)^{-\alpha}\right) \tag{48}$$

$$F(x) = 1-\exp\left(-\left(\frac{x-\gamma}{\beta}\right)^{-\alpha}\right) \tag{49}$$

$$x(p) = \gamma+\beta\cdot(-\ln(1-p))^{-1/\alpha} \tag{50}$$

where $\alpha,\beta,\gamma$ are the shape, the scale and the position parameters; $\alpha,\beta > 0, -\infty < \gamma < \infty, x > \gamma$.

The variation graph of the parameter $\alpha$ depending on skewness and L-skewness is presented in the Figure 11.

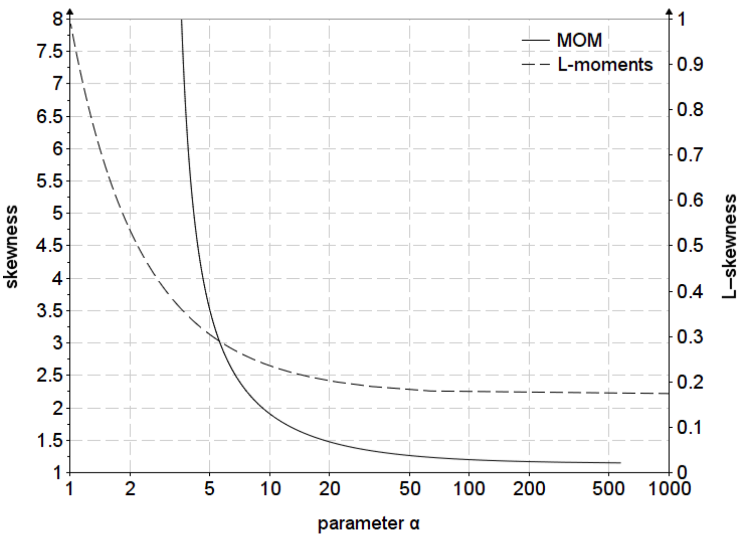



**Figure 11.** The variation of parameter $\alpha$

For estimation with MOM, parameter $\alpha$ has the following approximate forms depending on $C_s$:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



rational approximation form, for $C_s > 1.14$:

$$\alpha = \frac{1 + 0.73874734 \cdot C_s + 0.37485038 \cdot C_s^2 + 0.103516119 \cdot C_s^3}{-0.277593723 + 0.116805709 \cdot C_s + 0.071360121 \cdot C_s^2 + 0.035014727 \cdot C_s^3} \quad (51)$$

$$\beta = \frac{\sigma}{\sqrt{\Gamma\left(1 - \frac{2}{\alpha}\right) - \Gamma\left(1 - \frac{1}{\alpha}\right)^2}} \quad (52)$$

$$\gamma = \mu - \beta \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \quad (53)$$

The relative errors for the approximation are presented in Figure 12.

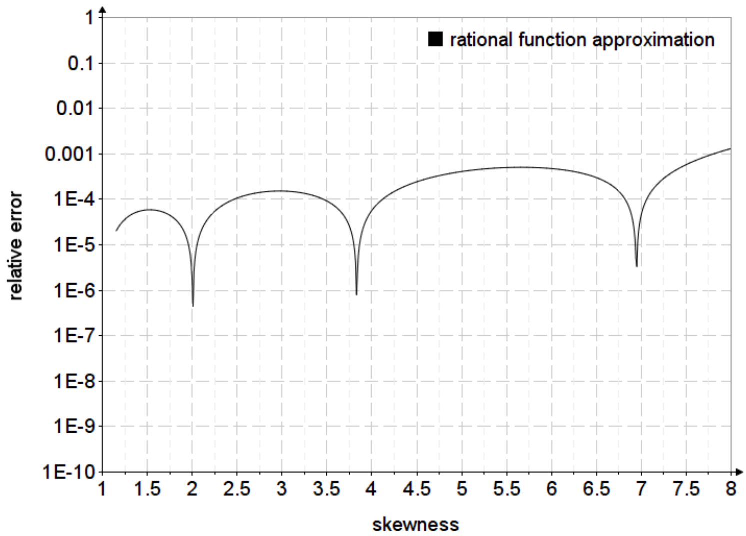



**Figure 12.** The variation of errors depending on $C_s$

For estimation with the L-moments, parameter $\alpha$ has the following approximate forms depending on L-skewness, $|\tau_3| \geq 0.17$:

approximation 1, a rational function, adopted from GEV approximation [10]:

$$\alpha = \frac{1}{\left(-7.817740 \cdot \left(\frac{2}{3+\tau_3} - \frac{\ln(2)}{\ln(3)}\right) - 2.930462 \cdot \left(\frac{2}{3+\tau_3} - \frac{\ln(2)}{\ln(3)}\right)^2 - 13.641492 \cdot \left(\frac{2}{3+\tau_3} - \frac{\ln(2)}{\ln(3)}\right)^3 - 17.206675 \cdot \left(\frac{2}{3+\tau_3} - \frac{\ln(2)}{\ln(3)}\right)^4\right)} \quad (54)$$

a better approximation, named here approximation 2, also a rational function form, but characterized by smaller relative errors, in the range of 0.6-1 of the L-skewness:

$$\alpha = \frac{1 + 0.441588375 \cdot |\tau_3| - 0.071007671 \cdot \tau_3^2 + 0.015634368 \cdot |\tau_3|^3}{-0.283759107 + 1.669931462 \cdot |\tau_3|}$$

$$\beta = \frac{L_2}{\Gamma\left(-\frac{1}{\alpha}\right) \cdot \left(1 - 2^{-\frac{1}{\alpha}}\right)} \quad (5$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



$$\gamma = L_1 - \beta \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) \quad (56)$$

The relative errors for the approximations are presented in Figure 13.

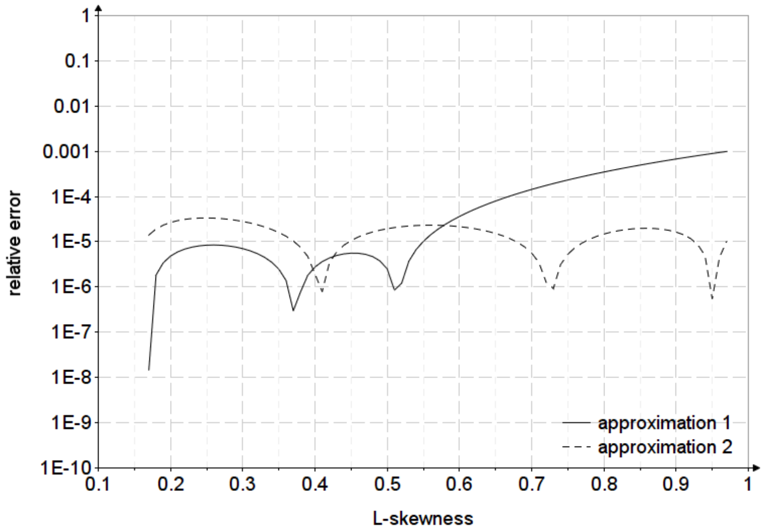



**Figure 13.** The variation of errors depending on $\tau_3$

*2.6. Fréchet Distribution (F2)*

The distribution represents a particular case of the F3 distribution when the position parameter is 0.

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the two parameters Fréchet distribution are [15]:

$$f(x) = \frac{\alpha}{\beta} \cdot \left(\frac{\beta}{x}\right)^{\alpha+1} \cdot \exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right) \quad (57)$$

$$F(x) = 1 - \exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right) \quad (58)$$

$$x(p) = \beta \cdot \left(-\ln(1-p)\right)^{-1/\alpha} \quad (59)$$

where $\alpha, \beta$ are the shape and the scale parameters; $\alpha, \beta, x > 0$.

The variation graph of the parameter $\alpha$ depending on coefficient of variation and L- coefficient of variation is presented in Figure 14.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



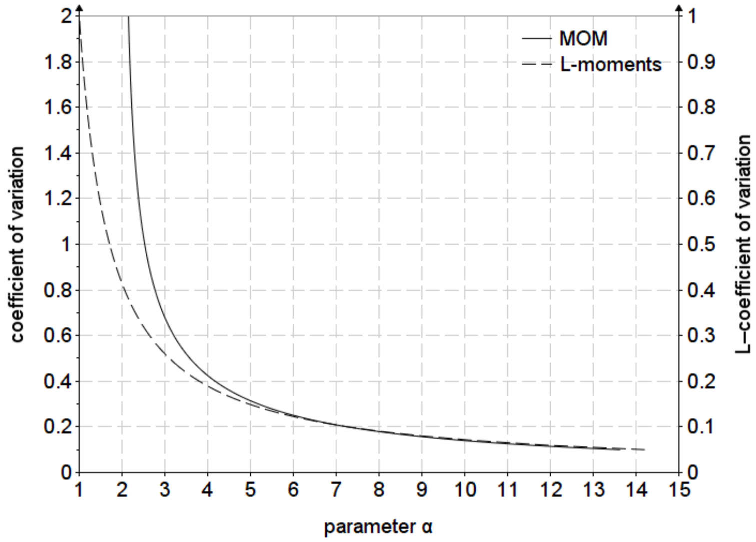



**Figure 14.** The variation of parameter $\alpha$

For estimation with MOM, parameter $\alpha$ has the following approximate forms depending on the coefficient of variation $C_v$:

$$\alpha = \exp\begin{pmatrix} 0.928192087\text{-}0.358563901\cdot\ln(C_v)+0.206039026\cdot\ln(C_v)^2- \\ 0.023717668\cdot\ln(C_v)^3-0.029286344\cdot\ln(C_v)^4+0.003741887\cdot\ln(C_v)^5+ \\ 0.008353766\cdot\ln(C_v)^6+0.002758305\cdot\ln(C_v)^7+0.000306859\cdot\ln(C_v)^8 \end{pmatrix} \quad (60)$$

$$\beta = \frac{\mu}{\Gamma\left(1-\frac{1}{\alpha}\right)} \quad (61)$$

The relative errors for the approximation are presented in Figure 15.

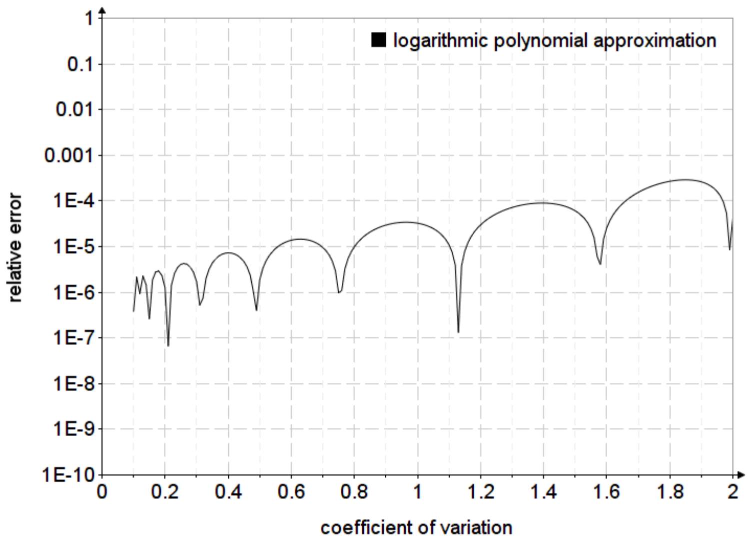



**Figure 15.** The variation of errors depending on $C_v$

For estimation with the L-moments, parameters have the following expressions [15]:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



$$\alpha = \frac{\ln(2)}{\ln\left(\frac{L_2 + L_1}{L_1}\right)} \tag{62}$$

$$\beta = \frac{L_1}{\Gamma\left(1 - \frac{1}{\alpha}\right)} \tag{63}$$

*2.7. Generalized Pareto Distribution (P3)*

The Pareto distribution was introduced by Pickands in 1975 [16]. The distribution represents a special case of the five-parameter Wakeby distribution [10, 13], respectively of the four-parameter Wakeby distribution [12].

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the three parameters Pareto distribution are [16]:

$$f(x) = \frac{1}{\beta}\cdot\left(1 - \frac{\alpha}{\beta}\cdot(x-\gamma)\right)^{\frac{1}{\alpha}-1} \tag{64}$$

$$F(x) = \left(1 - \frac{\alpha}{\beta}\cdot(x-\gamma)\right)^{\frac{1}{\alpha}} \tag{65}$$

$$x(p) = \gamma + \frac{\beta}{\alpha}\cdot\left(1 - p^{\alpha}\right) \tag{66}$$

where $\alpha, \beta, \gamma$ are the shape, the scale and the position parameters; $x \geq \gamma$ if $\alpha < 0$ or $\gamma \leq x \leq \gamma + \frac{\beta}{\alpha}$ if $\alpha > 0$.

The variation graph of the parameter $\alpha$ depending on skewness and L-skewness is presented in Figure 16.

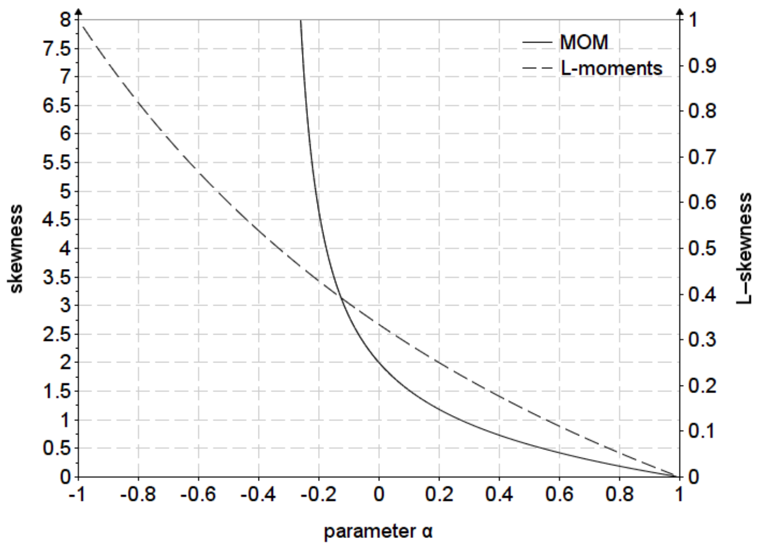



**Figure 16.** The variation of parameter $\alpha$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



For estimation with MOM, parameter $\alpha$ has the following approximate polynomial or rational forms depending on $C_s$:

for $C_s \le 2$:

$$\alpha = 999.998156889\cdot 10^{-3} - 1.154659839\cdot C_s + 555.214061169\cdot 10^{-3}\cdot C_s^2 - 137.553895649\cdot 10^{-3}\cdot C_s^3 - 2.365065448\cdot 10^{-3}\cdot C_s^4 + 13.308919064\cdot 10^{-3}\cdot C_s^5 - 3.931108819\cdot 10^{-3}\cdot C_s^6 + 409.717304137\cdot 10^{-6}\cdot C_s^7 \quad (67)$$

for $C_s > 2$:

$$\alpha = \frac{880.965365848\cdot 10^{-3} - 926.199345052\cdot 10^{-3}\cdot C_s + 394.547875439\cdot 10^{-3}\cdot C_s^2 - 103.261733369\cdot 10^{-3}\cdot C_s^3 + 16.938774757\cdot 10^{-3}\cdot C_s^4 - 1.812785982\cdot 10^{-3}\cdot C_s^5 + 103.169053863\cdot 10^{-6}\cdot C_s^6 - 2.562841773\cdot 10^{-6}\cdot C_s^7}{1 + 449.853791343\cdot 10^{-6}\cdot C_s + 1.18940502\cdot 10^{-3}\cdot C_s^2 + 353.672107629\cdot 10^{-6}\cdot C_s^3 + 1.35505086\cdot 10^{-3}\cdot C_s^4 + 84.527491397\cdot 10^{-6}\cdot C_s^5} \quad (68)$$

$$\beta = \sigma\cdot(\alpha+1)\cdot\sqrt{2\cdot\alpha+1} \quad (69)$$

$$\gamma = \mu - \frac{\beta}{\alpha+1} \quad (70)$$

The relative errors for the approximation are presented in Figure 17.

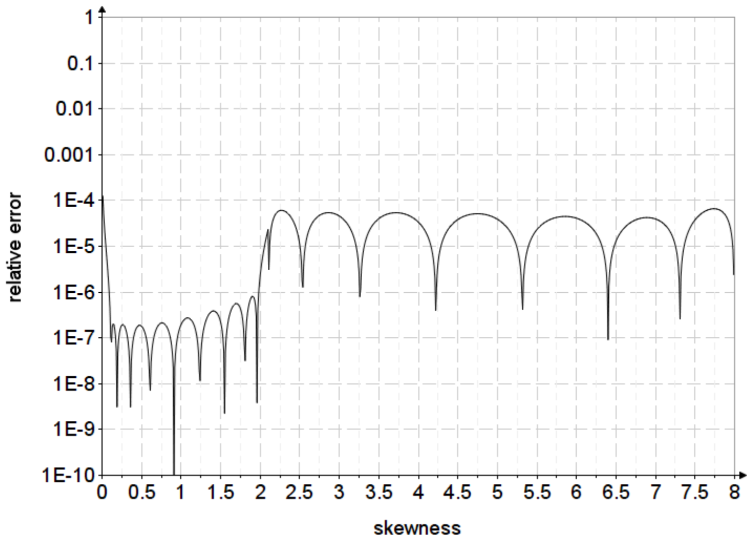



**Figure 17.** The variation of errors depending on $C_s$

For estimation with the L-moments, parameters have the following expressions [10, 13, 15, 16]:

$$\alpha = \frac{1 - 3\cdot\tau_3}{\tau_3 + 1} \quad (71)$$

$$\beta = L_2\cdot(1+\alpha)\cdot(2+\alpha) \quad (72)$$

$$\gamma = L_1 - L_2\cdot(2+\alpha) \quad (73)$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



### *2.8. Log-Logistic Distribution (LL3)*

The distribution was popularized in hydrology by Ahmad, et al. in 1988 and represents a generalized form of the two-parameter Log-Logistic distribution [16].

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the three parameters Log-Logistic distribution are [10,16]:

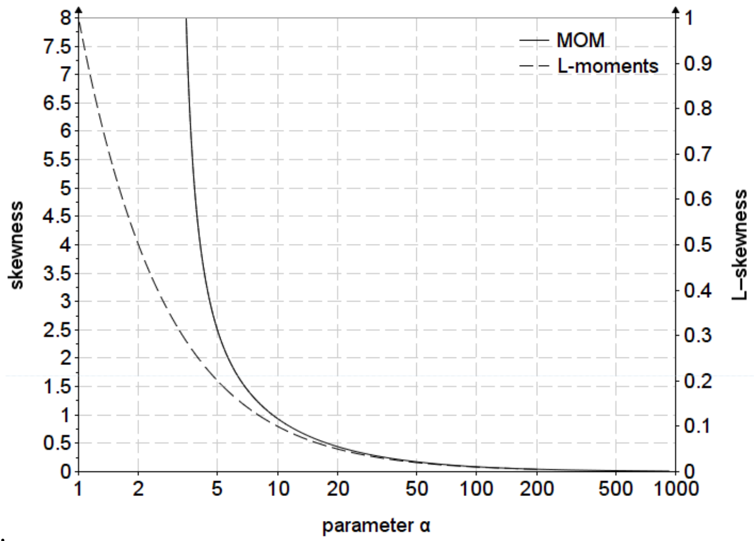
$$f(x)=\frac{\alpha}{\beta}\cdot\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}\cdot\left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha}+1\right)^{-2} \quad (74)$$

$$F(x)=\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-1} \quad (75)$$

$$x(p)=\gamma+\beta\cdot\left(\frac{1}{p}-1\right)^{\frac{1}{\alpha}} \quad (76)$$

where $\alpha,\beta,\gamma$ are the shape, the scale and the position parameters; $\alpha,\beta>0$, $\gamma<x_{min}$.

The variation graph of the parameter $\alpha$ depending on skewness and L-skewness is presented in Figure 18.




.

**Figure 18.** The variation of parameter $\alpha$

For estimation with MOM, parameter $\alpha$ has the following approximate forms depending on $C_s>0$:

approximation 1, adopted from the expression for generalized Logistic [10, 18]:

for $C_s\le 2.5$, the approximation function is polynomial:

$$\alpha=\exp\left(2.246-0.848\cdot\ln(C_s)+0.1272\cdot\ln(C_s)^2-0.04008\cdot\ln(C_s)^3\right) \quad (77)$$

for $C_s>2.5$, the approximation function has a rational form:

$$\alpha=\frac{4.007+3.411\cdot C_s+2.985\cdot C_s^2}{C_s^2} \quad (78)$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



approximation 2, adopted from the expression for generalized Logistic [10]:

$$\alpha = \frac{-3\cdot\pi}{2\cdot a\tan\left(-0.59484\cdot C_s\right)} \tag{79}$$

a better approximation referred here as approximation 3 which is a polynomial function:

for $C_s \le 2.5$:

$$\alpha = \exp\begin{pmatrix} 2.22464301 - 850.7876728\cdot10^{-3}\cdot\ln(C_s) + 121.776022\cdot10^{-3}\cdot\ln(C_s)^2 + \\ 51.4528279\cdot10^{-3}\cdot\ln(C_s)^3 + 5.3737403\cdot10^{-3}\cdot\ln(C_s)^4 - \\ 2.8910802\cdot10^{-3}\cdot\ln(C_s)^5 - 3.2030341\cdot10^{-3}\cdot\ln(C_s)^6 - \\ 671.4922523\cdot10^{-6}\cdot\ln(C_s)^7 - 51.670024\cdot10^{-6}\cdot\ln(C_s)^8 \end{pmatrix} \tag{80}$$

for $C_s > 2.5$:

$$\begin{aligned} \alpha = {} & 15.786327481 - 11.37147504\cdot C_s + 5.201363981\cdot C_s^2 - \\ & 1.418144703\cdot C_s^3 + 0.239556022\cdot C_s^4 - 0.02462292\cdot C_s^5 + \\ & 0.001412092\cdot C_s^6 - 0.00003466\cdot C_s^7 \end{aligned} \tag{81}$$

$$\beta = \frac{\sigma}{\sqrt{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right) - \Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^2}} \tag{82}$$

$$\gamma = \mu - \frac{\beta}{\alpha}\cdot\Gamma\left(\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right) \tag{83}$$

The relative errors for the approximations are presented in Figure 19.

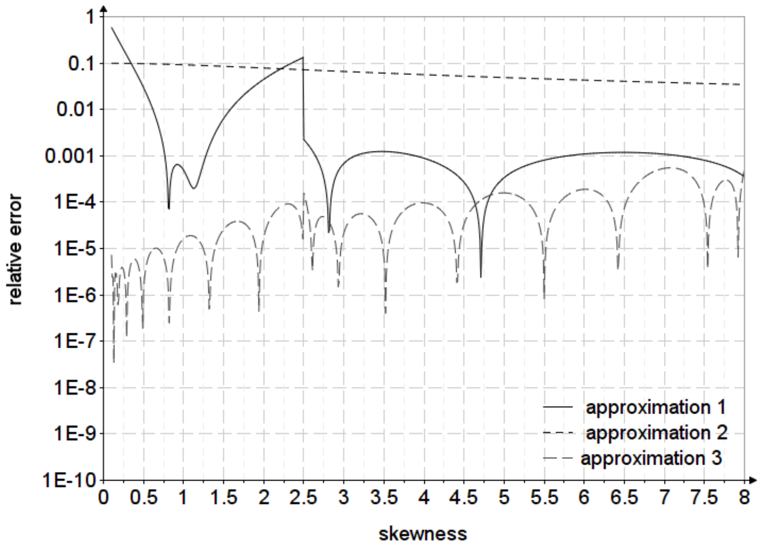



**Figure 19.** The variation of errors depending on $C_s$

For estimation with the L-moments, parameters have the following expressions [10]:

$$\alpha = \frac{L_2}{L_3} \tag{84}$$

 doi:10.20944/preprints202210.0473.v1



$$\beta = \frac{L_2}{\Gamma\left(\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)} \quad (85)$$

$$\gamma = L_1 - \frac{\beta}{\alpha}\cdot\Gamma\left(\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right) \quad (86)$$

*2.9. Kappa Distribution (K3-Generalized Gumbel)*

The distribution represents a particular case of the four-parameter Kappa distribution. It is also known as the generalized Gumbel distribution [19].

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the K3-Generalized Gumbel distribution are [19]:

$$f(x) = \frac{1}{\beta}\cdot\exp\left(-\frac{x-\gamma}{\beta}\right)\cdot\left(1-\alpha\cdot\exp\left(-\frac{x-\gamma}{\beta}\right)\right)^{\frac{1}{\alpha}-1} \quad (87)$$

$$F(x) = 1-\left(1-\alpha\cdot\exp\left(-\frac{x-\gamma}{\beta}\right)\right)^{\frac{1}{\alpha}} \quad (88)$$

$$x(p) = \gamma+\beta\cdot\ln\left(\frac{\alpha}{1-(1-p)^{\alpha}}\right) \quad (89)$$

The variation graph of the parameter $\alpha$ depending on skewness and L-skewness is presented in Figure 20.

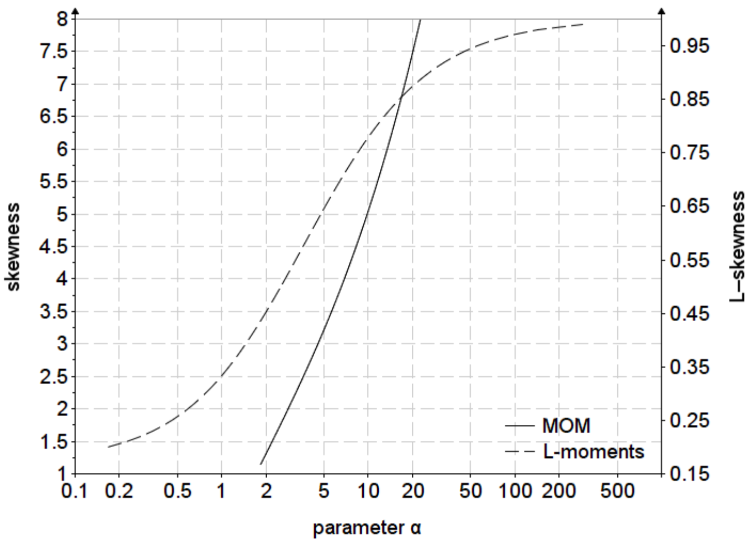



**Figure 20.** The variation of parameter $\alpha$

For estimation with MOM, parameter $\alpha$ has the following approximate polynomial form, for $C_s > 1.14$:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



$$\alpha = -918.79032552\cdot10^{-3} + 768.82077754\cdot10^{-3}\cdot C_s - 91.79270267\cdot10^{-3}\cdot C_s^2 + 136.3152042\cdot10^{-3}\cdot C_s^3 - 27.00071596\cdot10^{-3}\cdot C_s^4 + 3.20994481\cdot10^{-3}\cdot C_s^5 - 210.04317387\cdot10^{-6}\cdot C_s^6 + 5.80948381\cdot10^{-6}\cdot C_s^7 \quad (90)$$

$$\beta = \frac{\sigma}{\left(\frac{7.7175\cdot\alpha^8 - 326.34\cdot\alpha^7 - 3711.7\cdot\alpha^6 + 67724.37\cdot\alpha^5 + 321482.3241\cdot\alpha^4 - 7529351.76\cdot\alpha^3 + 38043509.47\cdot\alpha^2 - 19935867.29\cdot\alpha + 219479848.41}{115221657.6}\left(\psi\left(\frac{1}{\alpha}+1\right)+\ln(\alpha)+\gamma_e\right)^2\right)^{0.5}} \quad (91)$$

$$\gamma = \mu - \beta\cdot\left(\psi\left(\frac{1}{\alpha}+1\right)+\ln(\alpha)+\gamma_e\right) \quad (92)$$

where $\gamma_e$ is the Euler-Macheroni constant and $\psi(..)$ represents the digamma function which has the following approximate form [10]:

$$\psi(\alpha) = \frac{d}{d\alpha}\left(\ln\left(\Gamma(\alpha)\right)\right) = \mathrm{Psi}(\alpha)$$
$$\psi(\alpha) \approx \ln(\alpha+2) - \frac{1}{\alpha} - \frac{1}{\alpha+1} - \frac{1}{2\cdot(\alpha+2)} - \frac{1}{12\cdot(\alpha+2)^2} + \frac{1}{120\cdot(\alpha+2)^4} \quad (93)$$

The relative errors for the approximation are presented in Figure 21.

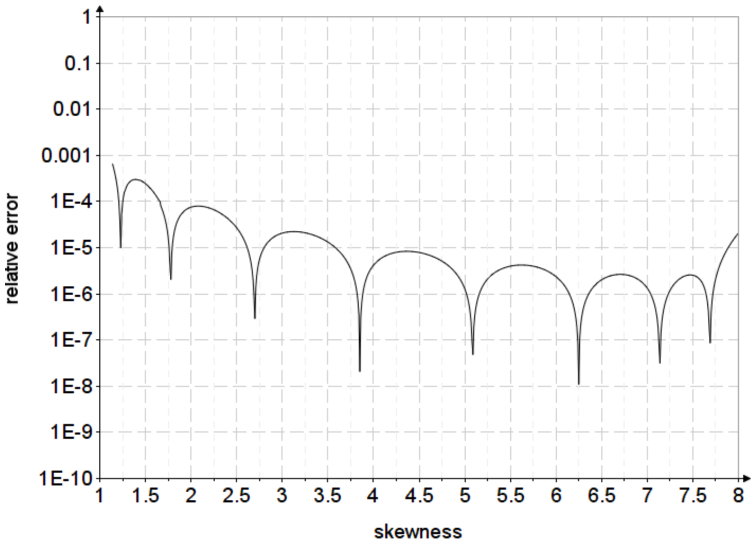



**Figure 21.** The variation of errors depending on $C_s$

For estimation with the L-moments, parameter $\alpha$ has the following approximate forms depending on $\tau_3$:

for $0.17 \le |\tau_3| \le \frac{1}{3}$:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



$$\alpha = -971.10886182 \cdot 10^{-3} + 6.3606944 \cdot |\tau_3| - 6.35405524 \cdot \tau_3^2 + 15.03572719 \cdot |\tau_3|^3 \quad (94)$$

for $\frac{1}{3} < |\tau_3| < 1$:

$$\alpha = \frac{-661.76303922 \cdot 10^{-3} + 4.30832382 \cdot |\tau_3| - 1.09504916 \cdot \tau_3^2 + 373.12384033 \cdot 10^{-3} \cdot |\tau_3|^3}{1 - 999.98952576 \cdot 10^{-3} \cdot |\tau_3|} \quad (95)$$

$$\beta = \frac{L_2}{\psi\left(\frac{2}{\alpha}+1\right) - \psi\left(\frac{1}{\alpha}+1\right)} \quad (96)$$

$$\gamma = L_1 - \beta \cdot \left[\psi\left(\frac{1}{\alpha}+1\right) + \gamma_e + \ln(\alpha)\right] \quad (97)$$

The relative errors for the approximation are presented in Figure 22.

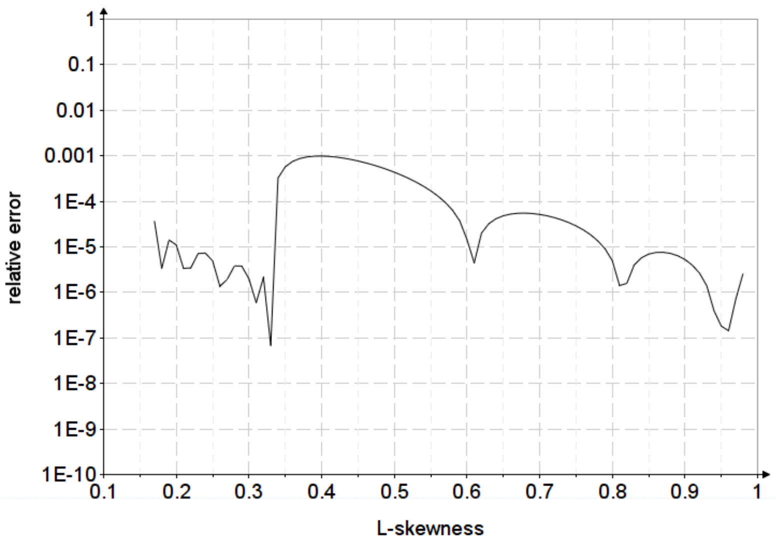



**Figure 22.** The variation of errors depending on $\tau_3$

*2.10. Kappa Distribution (K3-Park)*

The distribution represents a particular case of the four-parameter kappa distribution and a generalized form of the two-parameter Kappa distribution by adding a location parameter (shifted x), being presented for the first time in 2009 by Park et.al [19].

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the K3-Park distribution are [19]:

$$f(x) = \frac{\alpha}{\beta} \cdot \left(\alpha + \left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-\frac{\alpha+1}{\alpha}} \quad (98)$$

$$F(x) = 1 - \frac{x-\gamma}{\beta} \cdot \left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha} + \alpha\right)^{-\frac{1}{\alpha}} \quad (99)$$

 doi:10.20944/preprints202210.0473.v1



$$x(p)=\gamma+\beta\cdot\left(\frac{\alpha\cdot(1-p)^{\alpha}}{1-(1-p)^{\alpha}}\right)^{\frac{1}{\alpha}} \tag{100}$$

The variation graph of the parameter $\alpha$ depending on skewness and L-skewness is presented in Figure 23.

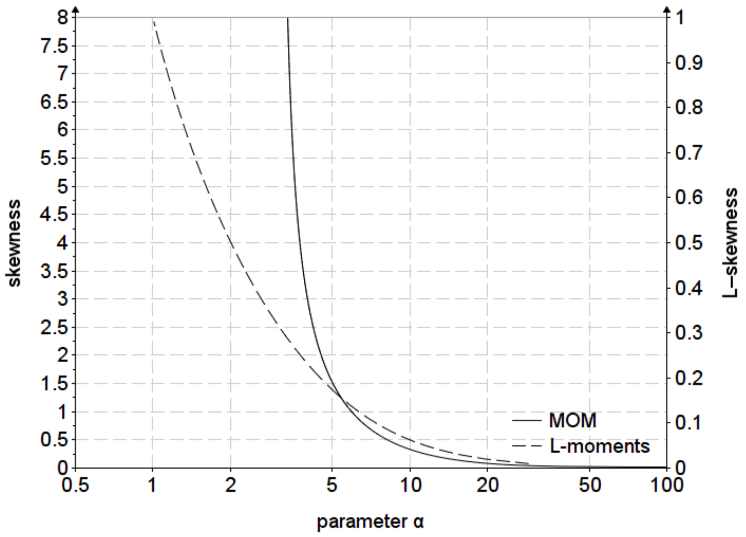



**Figure 23.** The variation of parameter $\alpha$

For estimation with MOM, parameter $\alpha$ has the following approximate logarithmic polynomial form, depending on $C_s>0$:

$$\alpha=\exp\begin{pmatrix}1.7822619-0.437806883\cdot\ln(C_s)+0.049586114\cdot\ln(C_s)^2+\\0.019798571\cdot\ln(C_s)^3+0.002017836\cdot\ln(C_s)^4-\\0.001937961\cdot\ln(C_s)^5-0.000620839\cdot\ln(C_s)^6+\\0.000097814\cdot\ln(C_s)^7+0.000046042\cdot\ln(C_s)^8\end{pmatrix} \tag{101}$$

$$\beta=\sigma\cdot\sqrt{\frac{\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma\left(\frac{3}{\alpha}\right)\cdot\alpha^{\frac{2}{\alpha}-1}\cdot\Gamma\left(1-\frac{2}{\alpha}\right)-\frac{\Gamma\left(\frac{2}{\alpha}\right)\cdot\alpha^{\frac{2}{\alpha}-2}\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^{2}}{\Gamma\left(\frac{1}{\alpha}+1\right)}}} \tag{102}$$

$$\gamma=\mu-\frac{\beta\cdot\Gamma\left(\frac{2}{\alpha}\right)\cdot\alpha^{\frac{1}{\alpha}}\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)} \tag{103}$$

The relative errors for the approximation are presented in Figure 24.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



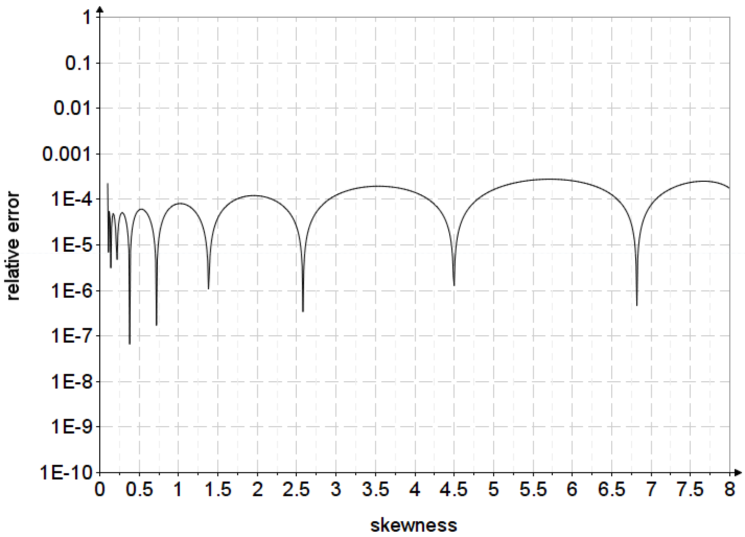



**Figure 24.** The variation of errors depending on $C_s$

For estimation with the L-moments, parameter $\alpha$ has the following approximate forms depending on L-skewness:

$$\alpha = \exp\left(\begin{array}{l} 0.000753505 - 0.983447042\cdot\ln(|\tau_3|) + 0.162877024\cdot\ln(|\tau_3|)^2 + 0.247492727\cdot\ln(|\tau_3|)^3 + \\ 0.116264397\cdot\ln(|\tau_3|)^4 + 0.029183504\cdot\ln(|\tau_3|)^5 + 0.004031622\cdot\ln(|\tau_3|)^6 + \\ 0.000270699\cdot\ln(|\tau_3|)^7 + 0.000005628\cdot\ln(|\tau_3|)^8 \end{array}\right) \quad (104)$$

$$\beta = \frac{L_2}{\alpha^{\frac{1}{\alpha}}\cdot\Gamma\left(1-\frac{1}{\alpha}\right)\cdot\left(\frac{\Gamma\left(\frac{3}{\alpha}\right)}{\Gamma\left(\frac{2}{\alpha}\right)} - \frac{\Gamma\left(\frac{2}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)}\right)} \quad (105)$$

$$\gamma = L_1 - \beta\cdot\alpha^{\frac{1}{\alpha}}\cdot\frac{\Gamma\left(\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)} \quad (106)$$

The relative errors for the approximation are presented in Figure 25.

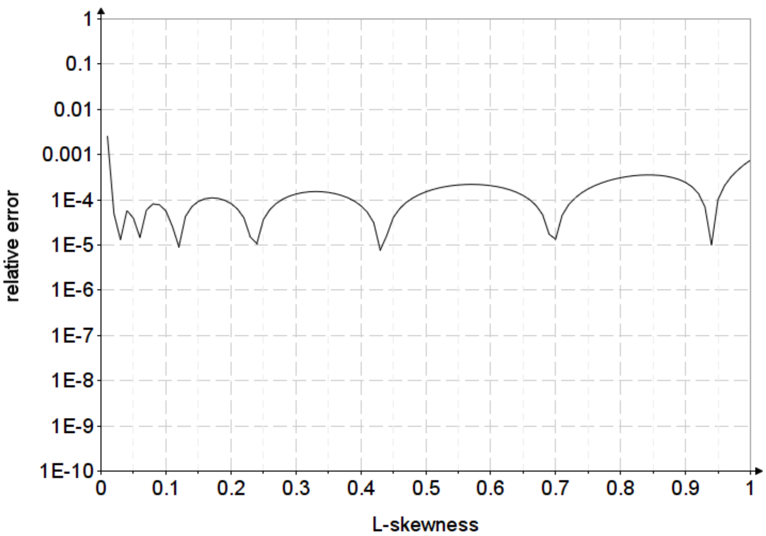


**Figure 25.** The variation of errors depending on $\tau_3$

*2.11. Pearson V Distribution (PV3)*

The distribution represents the inverse of the Pearson III distribution.

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the three parameters Pearson V distribution are [20]:

$$f(x)=\frac{\exp\left(-\frac{\beta}{x-\gamma}\right)}{\beta\cdot\Gamma(\alpha-1)}\cdot\left(\frac{x-\gamma}{\beta}\right)^{-\alpha} \tag{107}$$

$$F(x)=\frac{\Gamma\left(\alpha-1,\frac{\beta}{x-\gamma}\right)}{\Gamma(\alpha-1)}=1-\text{pgamma}\left(\frac{\beta}{x-\gamma},\alpha-1\right) \tag{108}$$

$$x(p)=\gamma+\frac{\beta}{\Gamma^{-1}(p;\alpha-1)}=\gamma+\frac{\beta}{\text{qgamma}(p,\alpha-1)} \tag{109}$$

where $\alpha,\beta,\gamma$ are the shape, the scale and the position parameters; $\alpha,\beta>0;\gamma<x_{min}$ and $x$ can take any values of range $\gamma<x<\infty$.

The variation graph of the parameter $\alpha$ depending on skewness and L-skewness is presented in Figure 26.

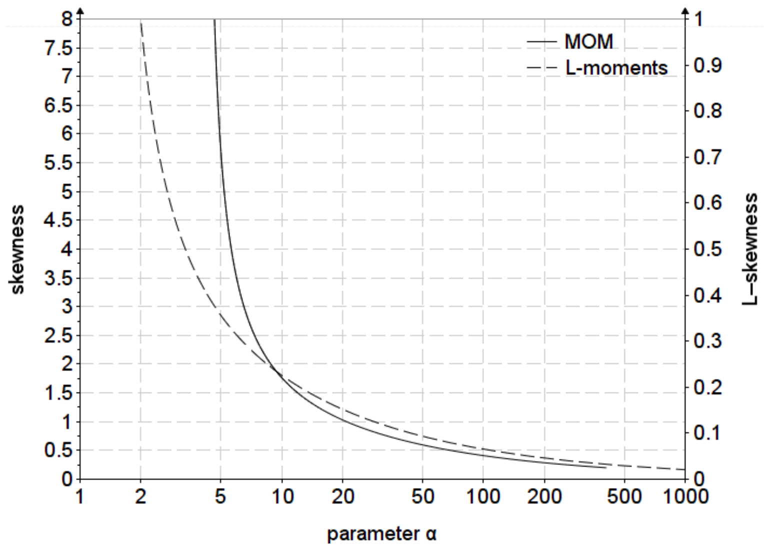



**Figure 26.** The variation of parameter $\alpha$

For estimation with MOM, parameter $\alpha$ can be evaluated numerically with the following approximate, depending on $C_s \geq 0.2$:

$$\alpha=\exp\begin{pmatrix}3.04183273-1.532335271\cdot\ln(C_s)+0.350524466\cdot\ln(C_s)^2+\\0.114266418\cdot\ln(C_s)^3-0.018372078\cdot\ln(C_s)^4-0.026493586\cdot\ln(C_s)^5-\\0.000919818\cdot\ln(C_s)^6+0.004375959\cdot\ln(C_s)^7+\\0.000310213\cdot\ln(C_s)^8-0.000357638\cdot\ln(C_s)^9\end{pmatrix} \tag{110}$$

$$\beta = \sigma \cdot (\alpha - 2) \cdot \sqrt{\alpha - 3} \tag{111}$$

$$\gamma = \mu - \frac{\beta}{\alpha - 2} \tag{112}$$

The relative errors for the approximation are presented in Figure 27.

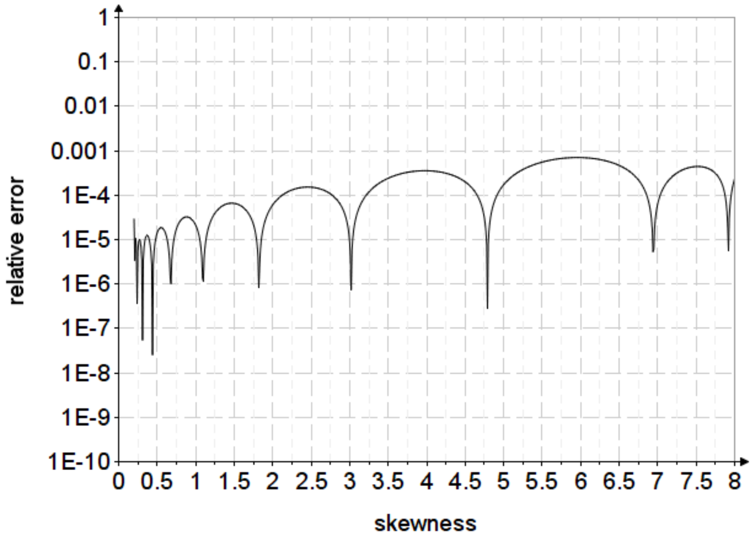



**Figure 27.** The variation of errors depending on $C_S$

For estimation with the L-moments, parameter $\alpha$ has the following approximate forms depending on L-skewness:

$$\alpha = \exp\begin{pmatrix} 0.692740135 - 0.466055586 \cdot \ln(|\tau_3|) + 0.460508385 \cdot \ln(|\tau_3|)^2 + \\ 0.470172127 \cdot \ln(|\tau_3|)^3 + 0.938027763 \cdot \ln(|\tau_3|)^4 + \\ 0.805363294 \cdot \ln(|\tau_3|)^5 + 0.359846034 \cdot \ln(|\tau_3|)^6 + \\ 0.08945247 \cdot \ln(|\tau_3|)^7 + 0.011775528 \cdot \ln(|\tau_3|)^8 + 0.000641664 \cdot \ln(|\tau_3|)^9 \end{pmatrix} \tag{113}$$

$$\beta = \frac{L_2}{\frac{1}{\alpha - 2} - 2 \cdot z} \tag{114}$$

$$\gamma = L_1 - \frac{\beta}{\alpha - 2} \tag{115}$$

where $z$ has the following expression:

$$z = \exp\begin{pmatrix} 4.083852739 - 11.442833473 \cdot \ln(\alpha) + 10.701793287 \cdot \ln(\alpha)^2 - \\ 6.513639901 \cdot \ln(\alpha)^3 + 2.527142191 \cdot \ln(\alpha)^4 - \\ 0.640389591 \cdot \ln(\alpha)^5 + 0.105546499 \cdot \ln(\alpha)^6 - \\ 0.010903032 \cdot \ln(\alpha)^7 + 0.000641094 \cdot \ln(\alpha)^8 - 0.000016371 \cdot \ln(\alpha)^9 \end{pmatrix} \tag{116}$$

The relative errors for the $\alpha$ approximation are presented in Figure 28.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



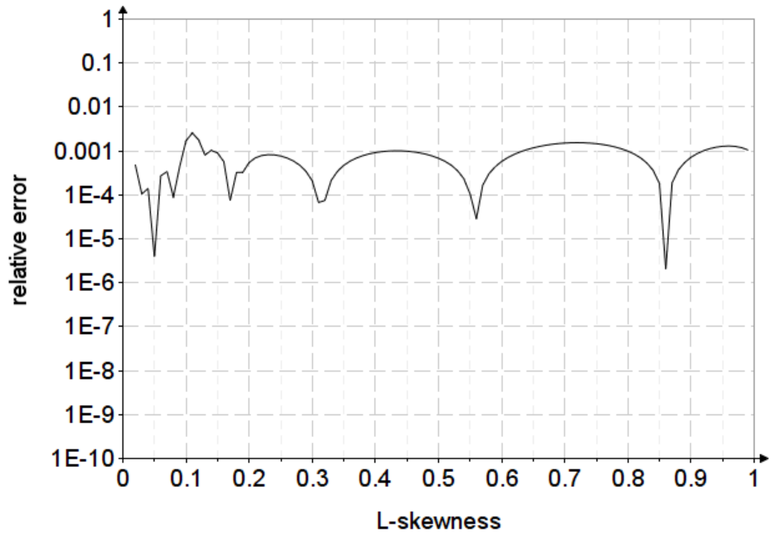



**Figure 28.** The variation of errors depending on $\tau_3$

*2.12. Pearson V Distribution (PV2)*

Two parameters Pearson V distribution, represent a particular form of PV5 when position parameter is 0.

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the two parameters Pearson V distribution are:

$$f(x)=\frac{\exp\left(-\frac{\beta}{x}\right)}{\beta\cdot\Gamma(\alpha-1)}\cdot\left(\frac{x}{\beta}\right)^{-\alpha} \tag{117}$$

$$F(x)=\frac{\Gamma\left(\alpha-1,\frac{\beta}{x}\right)}{\Gamma(\alpha-1)}=1-\text{pgamma}\left(\frac{\beta}{x},\alpha-1\right) \tag{118}$$

$$x(p)=\frac{\beta}{\Gamma^{-1}(p;\alpha-1)}=\frac{\beta}{\text{qgamma}(p,\alpha-1)} \tag{119}$$

where $\alpha,\beta>0$ are the shape and the scale parameters; $x>0$

For estimation with the MOM, the distribution parameters have the following expressions:

$$\alpha=2+\frac{1}{C_v^2} \tag{120}$$

$$\beta=\mu\cdot\left(1+\frac{1}{C_v^2}\right) \tag{121}$$

The variation graph of the parameter $\alpha$ depending on $C_v$ and $\tau_2$ is presented in Figure 29.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



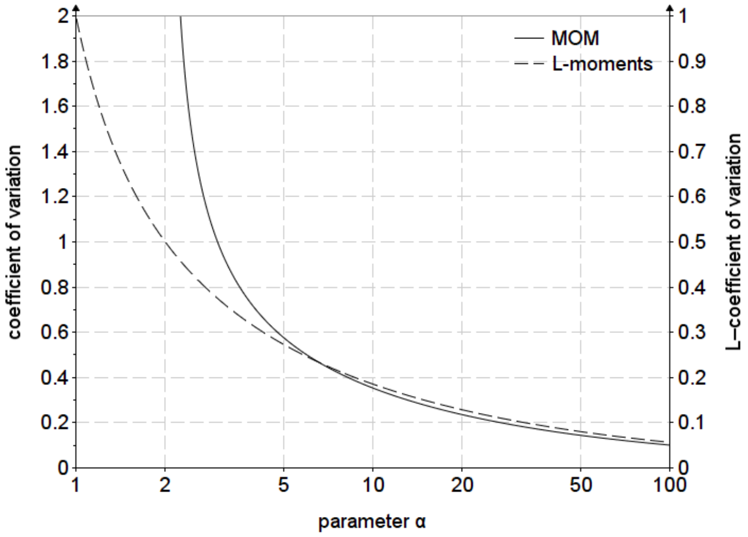



**Figure 29.** The variation of parameter $\alpha$

For estimation with the L-moments, parameter $\alpha$ has the following approximate forms depending on $\tau_2$:

$$\alpha = \exp\begin{pmatrix} 0.000105303 - 0.719316463 \cdot \ln(\tau_2) + 0.365000334 \cdot \ln(\tau_2)^2 - \\ 0.063429562 \cdot \ln(\tau_2)^3 + 0.043630525 \cdot \ln(\tau_2)^4 + \\ 0.110250588 \cdot \ln(\tau_2)^5 + 0.060323921 \cdot \ln(\tau_2)^6 + \\ 0.015685798 \cdot \ln(\tau_2)^7 + 0.002043393 \cdot \ln(\tau_2)^8 + 0.000107581 \cdot \ln(\tau_2)^9 \end{pmatrix} \quad (122)$$

$$\beta = \frac{L_1}{\exp\begin{pmatrix} 3.524909041 - 11.725123564 \cdot \ln(\alpha) + 17.230914474 \cdot \ln(\alpha)^2 - \\ 16.342259941 \cdot \ln(\alpha)^3 + 9.681973166 \cdot \ln(\tau_2)^4 - \\ 3.719303884 \cdot \ln(\alpha)^5 + 0.941443582 \cdot \ln(\alpha)^6 - \\ 0.15590484 \cdot \ln(\alpha)^7 + 0.016253258 \cdot \ln(\alpha)^8 - \\ 0.000967866 \cdot \ln(\alpha)^9 + 0.000025104 \cdot \ln(\alpha)^{10} \end{pmatrix}} \quad (123)$$

The relative errors for the $\alpha$ approximation are presented in Figure 30.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



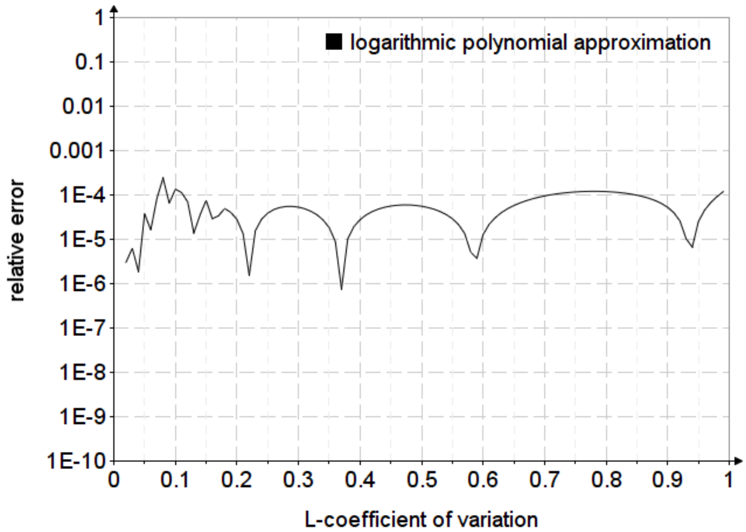



**Figure 30.** The variation of errors depending on $\tau_2$

*2.13. Generalized Exponential Distribution (EG2-Gupta)*

The Generalized Exponential Distribution is an alternative to the two-parameter Gamma and Weibul distributions. It was introduced by Gupta and Kundu in 1999 [21].

The probability density function, $f(x)$; the complementary cumulative distribution function, $F(x)$, and the quantile function, $x(p)$, of the EG distribution are [21]:

$$f(x)=\alpha\cdot\beta\cdot\exp(-\beta\cdot x)\cdot(1-\exp(-\beta\cdot x))^{\alpha-1} \tag{124}$$

$$F(x)=1-(1-\exp(-\beta\cdot x))^{\alpha} \tag{125}$$

$$x(p)=-\frac{\ln\left(1-(1-p)^{1/\alpha}\right)}{\beta} \tag{126}$$

where $\alpha,\beta>0$ are the shape and the scale parameters; $x>0$

The parameter $\alpha$ can only be obtained accurately by numerical methods, because the coefficient of variation and the L-coefficient of variation are non-linear equations presented in Appendix B.

For estimation with MOM, parameter $\alpha$ has the following approximate forms depending on $C_v$:

$$\alpha=\exp\begin{pmatrix}-0.00031096-2.262833217\cdot\ln(C_v)+0.260607141\cdot\ln(C_v)^2- \\ 0.211003998\cdot\ln(C_v)^3+0.094017376\cdot\ln(C_v)^4+ \\ 0.045622834\cdot\ln(C_v)^5+0.049036309\cdot\ln(C_v)^6+ \\ 0.016045195\cdot\ln(C_v)^7+0.002009549\cdot\ln(C_v)^8\end{pmatrix} \tag{127}$$

$$\beta=\frac{\psi(\alpha+1)+\gamma_e}{\mu} \tag{128}$$

where $\gamma_e$ represent the Euler-Mascheroni's constant and $\psi(\alpha)$ represents the digamma function.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



The variation graph of the parameter $\alpha$ depending on $C_v$ and $\tau_2$ presented in Figure 31.

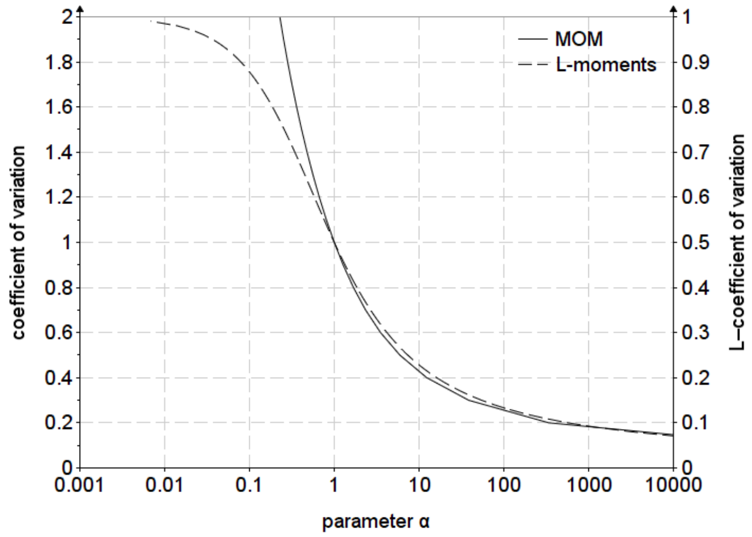



**Figure 31.** The variation of parameter $\alpha$

The relative errors for the $\alpha$ approximation for MOM are presented in Figure 32.

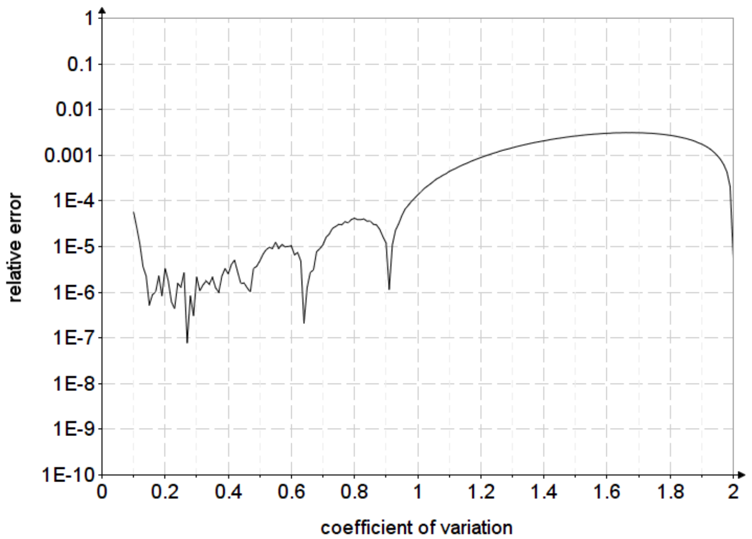



**Figure 32.** The variation of errors depending on $C_v$

For estimation with the L-moments, parameter $\alpha$ has the following approximate forms depending on $\tau_2$:

if $0.05 \le \tau_2 \le 0.5$

$$\alpha = \exp\begin{pmatrix} -2.705734288 - 6.147348286 \cdot \ln(\tau_2) - 5.63695924 \cdot \ln(\tau_2)^2 - \\ 4.843168776 \cdot \ln(\tau_2)^3 - 2.533007093 \cdot \ln(\tau_2)^4 - \\ 0.9053571 \cdot \ln(\tau_2)^5 - 0.19830547 \cdot \ln(\tau_2)^6 - \\ 0.026613323 \cdot \ln(\tau_2)^7 - 0.001663193 \cdot \ln(\tau_2)^8 \end{pmatrix} \quad (129)$$

if $0.5 < \tau_2 < 1$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



$$\alpha = \frac{\begin{array}{l}34.8898543 - 252.3059668 \cdot \tau_2 + 838.0635788 \cdot \tau_2^2 - \\ 1605.8220859 \cdot \tau_2^3 + 1885.2871002 \cdot \tau_2^4 - \\ 1344.7635811 \cdot \tau_2^5 + 536.9323004 \cdot \tau_2^6 - 92.2812201 \cdot \tau_2^7\end{array}}{1 + 0.0001378 \cdot \tau_2 + 0.0005466 \cdot \tau_2^2 + 0.0005326 \cdot \tau_2^3} \quad (130)$$

$$\beta = \frac{\psi(\alpha+1) + \gamma_e}{L} \quad (131)$$

The relative errors for the $\alpha$ approximation are presented in Figure 33.

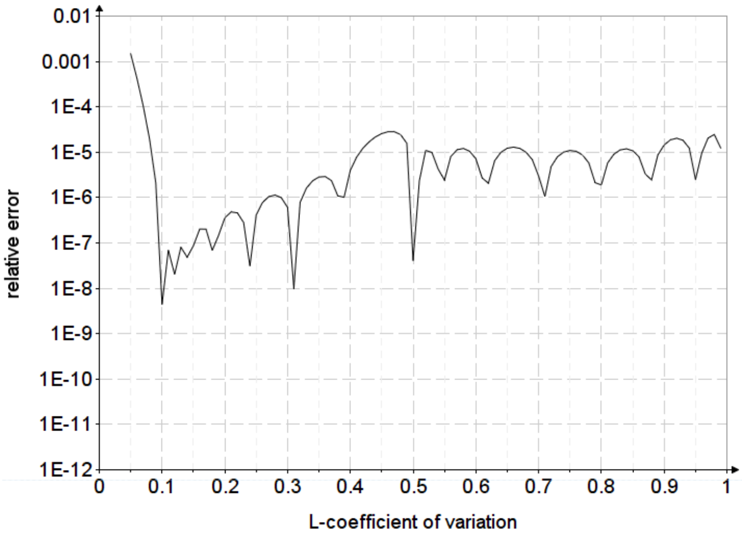



**Figure 33.** The variation of errors depending on $\tau_2$

## 3. Conclusions

The statistical distributions presented here are frequently used in hydrology to calculate maximum rainfall, maximum and minimum flows and volumes of synthetic floods.

The need to approximate the parameters is given by obtaining some initial values for the numerical calculation of the parameters, reducing the number of iterations, thus the calculation time.

The approximate values calculated with the formulas presented here can be used directly to estimate the parameters of the statistical distributions due to very small errors.

The relative errors of the parameter estimates are generally well below 1‰, which has a much smaller implication on the relative errors of the inverse function values, and are independent of the length of the observed data string. The first-order derivatives of the parameters determined from the approximations show negligible errors, especially for "better approximations".

The functions used in the approximation relations are of the rational and polynomial type, sometimes with a logarithmic argument.

The comparative presentation of the variation of the estimated parameter, for the two methods of estimating the parameters of the statistical distributions, is useful in choosing the asymmetry coefficient in the case of MOM, considering that the linear moments are closer to reality, an important aspect in Romania where the asymmetry it is chosen according to the genesis of the debts, a legacy from the USSR normative standards. [22].

In general, in Romania tabular calculation is used for a small number of distributions (Pearson III and Kritsky-Menkel) using linear interpolation, so the approximations presented in this article prove to be extremely useful, facilitating the use of a larger number

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



of distributions, a necessity regarding the updating of Romanian normative standards at the international level.

All research was carried out by authors in the Faculty of Hydrotechnics with hydrological data provided by the National Institute of Hydrology and Water Management and National Administration "Romanian Waters".

The presentation of some approximate forms of the parameters, especially for the estimation with the L-moments method, represents a step forward in the implementation stage of a transition from MOM to a regionalization based on L-moments in Romania.

**Author Contributions:** Conceptualization, C.I. and C.G.A.; methodology, C.I. and C.G.A.; software, C.I. and C.G.A.; validation, C.I. and C.G.A.; formal analysis, C.I. and C.G.A.; investigation, C.I. and C.G.A.; resources, C.I. and C.G.A.; data curation, C.I. and C.G.A.; writing—original draft preparation, C.I. and C.G.A.; writing—review and editing, C.I. and C.G.A.; visualization, C.I. and C.G.A.; supervision, C.I. and C.G.A.; project administration, C.I. and C.G.A.; funding acquisition, C.I. and C.G.A. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable

**Informed Consent Statement:** Not applicable

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflicts of interest.

**Abbreviations**

| | |
|---|---|
| MOM | method of ordinary moments |
| L-moments | method of liniar moments |
| $\mu$ | expected value; arithmetic mean |
| $\sigma$ | standard deviation |
| $C_V$ | coefficient of variation |
| $C_S$ | coefficient of skewness; skewness |
| $L_2$, $L_2$, $L_3$ | liniar moments |
| $\tau_2$ | coefficient of variation based on the L-moments method |
| $\tau_3$ | coefficient of skewness based on the L-moments method |
| PE3 | Pearson III distribution |
| GEV | generalized extreme value distribution |
| W3 | three parameters Weibull distribution |
| W2 | two parameters Weibull distribution |
| F3 | three parameters Fréchet distribution |
| F2 | two parameters Fréchet distribution |
| P3 | three parameters Pareto distribution |
| LL3 | three parameters Log-Logistic distribution |
| K3 | three parameters Kappa distribution |
| PV3 | three parameters Pearson V distribution |
| PV2 | two parameters Pearson V distribution |
| EG2 | Generalized exponential distribution |

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2022 doi:10.20944/preprints202210.0473.v1



**Appendix A. Built-In Function in Mathcad**

$\Gamma(\alpha)=\int_0^{\infty} t^{\alpha-1}\cdot e^{-t}dt$ the complete gamma function;

$\Gamma(\alpha,x)=\int_x^{\infty} t^{\alpha-1}\cdot e^{-t}dt$ returns the value of the upper incomplete gamma function of x with parameter $\alpha$;

$dgamma(x,\alpha)=\frac{x^{\alpha-1}\cdot e^{-x}}{\Gamma(\alpha)}$ returns the probability density for value x, for the Gamma distribution;

$pgamma(x,\alpha)=\int_0^{x}\frac{x^{\alpha-1}\cdot e^{-x}}{\Gamma(\alpha)}dx=\frac{\gamma(\alpha,x)}{\Gamma(\alpha)}=1-\frac{\Gamma(\alpha,x)}{\Gamma(\alpha)}$ returns the cumulative probability distribution for value x, for the Gamma distribution;

$qgamma(p,\alpha)=\Gamma^{-1}(p;\alpha)$ returns the inverse cumulative probability distribution for probability p, for the Gamma distribution.

$pchisq(x,d)=\int_0^{x}\frac{x^{\frac{d}{2}-1}\cdot e^{-x}}{2^{\frac{d}{2}}\cdot\Gamma\left(\frac{d}{2}\right)}dx=1-\Gamma\left(\frac{d}{2},\frac{x}{2}\right)\Big/\Gamma\left(\frac{d}{2}\right)$ returns the cumulative probability distribution for value x, for the Chi-Squared distribution;

$qchisq(p,d)$ returns the inverse cumulative probability distribution for probability p, for the Chi-Squared distribution;

$dweibull(x,s)=s\cdot x^{s-1}\cdot e^{-x^{s}}$ returns the probability density for value x, for the Weibull distribution;

$pweibull(x,s)=1-e^{-x^{s}}$ returns the cumulative probability distribution for value x, for the Weibull distribution;

$qweibull(p,s)=(-\ln(1-p))^{\frac{1}{s}}$ returns the inverse cumulative probability distribution for probability p, for the Weibull distribution;

$ibeta(a,x,y)=B(a;x,y)=\frac{\Gamma(x+y)}{\Gamma(x)\cdot\Gamma(y)}\cdot\int_0^{a}t^{x-1}\cdot(1-t)^{y-1}dt$, the incomplete Beta function, returns the value of the incomplete beta function of x and y with parameter a;

$Psi(z)=\psi(z)=\frac{d}{dz}\ln(\Gamma(z))=\frac{\frac{d}{dz}\Gamma(z)}{\Gamma(z)}$, the digamma function.

$Psi(n,z)=\frac{d^{n}}{dz^{n}}\psi(z)=\frac{d^{n+1}}{dz^{n+1}}\ln(\Gamma(z))$, the polygamma function of order n.

**Appendix B. The exact formulas of the parameters expressed with approximate expressions**

| | |
|---|---|
| PE3 | $\tau_3 = 3\cdot\left(2\cdot\frac{\Gamma(3\cdot\alpha)}{\Gamma(\alpha)\cdot\Gamma(2\cdot\alpha)}\cdot\int_0^{1/3} x^{\alpha-1}\cdot(1-x)^{2\cdot\alpha-1}dx-1\right)= 3\cdot\left(2\cdot B\left(\frac{1}{3};\alpha,2\cdot\alpha\right)-1\right)=3\cdot\left(2\cdot \text{ibeta}\left(\frac{1}{3},\alpha,2\cdot\alpha\right)-1\right)$ |
| GEV | $C_s=\text{sign}(\alpha)\cdot\frac{3\cdot\Gamma(2\cdot\alpha+1)\cdot\Gamma(\alpha+1)-\Gamma(3\cdot\alpha+1)-2\cdot\Gamma(\alpha+1)^3}{\sqrt{\left(\Gamma(2\cdot\alpha+1)-\Gamma(\alpha+1)^2\right)^3}}$ <br> $\tau_3=\frac{2\cdot(1-3^{-\alpha})}{(1-2^{-\alpha})}-3$ |
| W3 | $C_s=\frac{\Gamma\left(\frac{3}{\alpha}+1\right)+2\cdot\Gamma\left(\frac{1}{\alpha}+1\right)^3-3\cdot\Gamma\left(\frac{2}{\alpha}+1\right)\cdot\Gamma\left(\frac{1}{\alpha}+1\right)}{\sqrt{\left(\Gamma\left(\frac{2}{\alpha}+1\right)-\Gamma\left(\frac{1}{\alpha}+1\right)^2\right)^3}}$ <br> $\tau_3=2\cdot\frac{2^{-\frac{1}{\alpha}}-3^{-\frac{1}{\alpha}}}{2^{-\frac{1}{\alpha}}-1}+1$ |
| W2 | $C_v=\sqrt{\frac{\Gamma\left(1+\frac{2}{\alpha}\right)}{\Gamma\left(1+\frac{1}{\alpha}\right)^2}-1}$ |
| F3 | $C_s=\frac{\Gamma\left(1-\frac{3}{\alpha}\right)+2\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^3-3\cdot\Gamma\left(1-\frac{2}{\alpha}+\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\sqrt{\left(\Gamma\left(1-\frac{2}{\alpha}\right)-\Gamma\left(1-\frac{1}{\alpha}\right)^2\right)^3}}$ <br> $\tau_3=\frac{2\cdot 3^{\frac{1}{\alpha}}+1-3\cdot 2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}-1}-1$ |
| F2 | $C_v=\sqrt{\frac{\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}\right)^2}-1}$ |
| P3 | $C_s=\text{sign}(1-\alpha)\cdot\frac{2\cdot(1-\alpha)\cdot\sqrt{2\cdot\alpha+1}}{3\cdot\alpha+1}$ |
| LL3 | $C_s=\frac{\Gamma\left(1+\frac{3}{\alpha}\right)\cdot\Gamma\left(1-\frac{3}{\alpha}\right)+2\cdot\Gamma\left(1+\frac{1}{\alpha}\right)^3\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^3-3\cdot\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right)\cdot\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\left[\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^2\right]^{1.5}}$ |

| | |
|---|---|
| K3 – generalized Gumbel | $C_s=\int_0^1 \frac{\left[\ln\left(\frac{\alpha}{1-(1-p)^{\alpha}}\right)-\psi\left(\frac{1}{\alpha}+1\right)-\ln(\alpha)-\gamma_e\right]^3}{\left[\int_0^1 \ln\left(\frac{\alpha}{1-(1-p)^{\alpha}}\right)^2 dp-\left(\psi\left(\frac{1}{\alpha}+1\right)+\ln(\alpha)+\gamma_e\right)^2\right]^{1.5}}dp$ <br> $\tau_3=2\cdot\frac{\psi\left(1+\frac{3}{\alpha}\right)-\psi\left(1+\frac{1}{\alpha}\right)}{\psi\left(1+\frac{2}{\alpha}\right)-\psi\left(1+\frac{1}{\alpha}\right)}-3$ |
| K3 – Park | $C_s=\frac{2\cdot\Gamma\left(\frac{2}{\alpha}\right)^3\cdot\alpha^{\frac{3}{\alpha}}\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^3+\Gamma\left(\frac{1}{\alpha}\right)^2\cdot\Gamma\left(\frac{4}{\alpha}\right)\cdot\alpha^{\frac{3}{\alpha}}\cdot\Gamma\left(1-\frac{3}{\alpha}\right)-3\cdot\Gamma\left(\frac{1}{\alpha}\right)\cdot\Gamma\left(\frac{2}{\alpha}\right)\cdot\Gamma\left(\frac{3}{\alpha}\right)\cdot\Gamma\left(1-\frac{2}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)\cdot\alpha^{\frac{3}{\alpha}}}{\Gamma\left(\frac{1}{\alpha}\right)^3\cdot\left[\frac{\Gamma\left(\frac{3}{\alpha}\right)\cdot\alpha^{\frac{2}{\alpha}}\cdot\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)}-\frac{\Gamma\left(\frac{2}{\alpha}\right)^2\cdot\alpha^{\frac{2}{\alpha}}\cdot\Gamma\left(1-\frac{1}{\alpha}\right)^2}{\Gamma\left(\frac{1}{\alpha}\right)^2}\right]^{1.5}}$ <br> $\tau_3=\frac{2\cdot\frac{\Gamma\left(\frac{4}{\alpha}\right)}{\Gamma\left(\frac{3}{\alpha}\right)}-3\cdot\frac{\Gamma\left(\frac{3}{\alpha}\right)}{\Gamma\left(\frac{2}{\alpha}\right)}+\frac{\Gamma\left(\frac{2}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)}}{\frac{\Gamma\left(\frac{3}{\alpha}\right)}{\Gamma\left(\frac{2}{\alpha}\right)}-\frac{\Gamma\left(\frac{2}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)}}$ |
| PV3 | $C_s=\frac{4\cdot\sqrt{\alpha-3}}{\alpha-4}$ <br> $\tau_3=\frac{\int_0^1 \frac{1}{\Gamma^{-1}(p;\alpha-1)}\cdot\left(1-6\cdot p+6\cdot p^2\right)dp}{\int_0^1 \frac{1}{\Gamma^{-1}(p;\alpha-1)}\cdot(1-2\cdot p)dp}=\frac{\int_0^1 \frac{1-6\cdot p+6\cdot p^2}{\mathrm{qgamma}(p,\alpha-1)}dp}{\int_0^1 \frac{1-2\cdot p}{\mathrm{qgamma}(p,\alpha-1)}dp}$ |
| PV2 | $\tau_2=\frac{\int_0^1 \frac{1}{\Gamma^{-1}(p;\alpha)}\cdot(1-2\cdot p)dp}{\int_0^1 \frac{1}{\Gamma^{-1}(p;\alpha)}dp}=\frac{\int_0^1 \frac{1-2\cdot p}{\mathrm{qgamma}(p,\alpha)}dp}{\int_0^1 \frac{1}{\mathrm{qgamma}(p,\alpha)}dp}$ |
| EG2 | $C_v=\sqrt{\frac{\frac{\pi^2}{6}-\frac{d}{d\alpha}\psi(\alpha+1)}{\left(\psi(\alpha+1)+\gamma_e\right)^2}}$ <br> $\tau_2=\frac{\psi(2\cdot\alpha+1)-\psi(\alpha+1)}{\psi(\alpha+1)+\gamma_e}$ |

## References


1. Adnan, R.M., Petroselli, A., Heddam, S. et al. Comparison of different methodologies for rainfall–runoff modeling: machine learning vs conceptual approach. Nat Hazards 105, 2987–3011 (2021). https://doi.org/10.1007/s11069-020-04438-2
2. *EM 1110-2-1415 Hydrologic Frequency Analysis, Engineering and Design*; Department of the Army U.S. Army Corps of Engineers: 1993.
3. *Bulletin 17B Guidelines for determining Flood Flow Frequency*; Hydrology Subcommittee, Interagency Advisory Committee on Water Data, U.S. Departament of the Interior, U.S. Geological Survey, office of Water Data Coordination, Reston, Virginia: 1981.

 doi:10.20944/preprints202210.0473.v1




4. *Bulletin 17C Guidelines for determining Flood Flow Frequency*; U.S. Department of the Interior, U.S. Geological Survey, Reston, Virginia: 2017.
5. Song, S.; Kang, Y.; Song, X.; Singh, V.P. MLE-Based Parameter Estimation for Four-Parameter Exponential Gamma Distribution and Asymptotic Variance of Its Quantiles. Water 2021, 13, 2092. https://doi.org/10.3390/w13152092
6. Hosking, J.R.M. L-moments: Analysis and Estimation of Distributions using Linear, Combinations of Order Statistics. *J. R. Statist. Soc.* **1990**, *52*, 105–124.
7. Gubareva, T.S.; Gartsman, B.I. Estimating Distribution Parameters of Extreme Hydrometeorological Characteristics by L-Moment Method. *Water Resour.* **2010**, *37*, 437–445.
8. World Meteorological Organization. *(WMO-No.718) 1989 Statistical Distributions for Flood Frequency Analysis*; Operational Hydrology Report no. 33; WHO: Geneva, Switzerland, 1989.
9. Constantinescu, M.; Golstein, M.; Haram, V.; Solomon, S. *Hydrology*; Technical Publishing House: Bucharest, Romania, 1956.
10. Rao, A.R.; Hamed, K.H. *Flood Frequency Analysis*; CRC Press LLC: Boca Raton, FL, USA, 2000.
11. Song, S.; Song, X.; Kang, Y. Entropy-Based Parameter Estimation for the Four-Parameter Exponential Gamma Distribution. *Entropy* **2017**, *19*, 189.
12. Ilinca, C.; Anghel, C.G. Flood-Frequency Analysis for Dams in Romania. *Water* **2022**, *14*, 2884. https://doi.org/10.3390/w14182884
13. Hosking, J.R.M; Wallis, J.R. Regional Frequency Analysis, An Approach based on L-moments. Cambridge University Press, the Edinburgh Building, Cambridge CB2 2RU, UK: 1997; ISBN 13 978-0-521-43045-6.
14. Goda, Y.; Kudaka, M.; Kawai, H. Incorporation of Weibull distribution in *L*-moments method for regional frequency analysis of peaks-over-threshold wave heights. *Coast. Eng., Proceedings of 32nd international conference on coastal engineering, ASCE, Shanghai, China,* **2010.**
15. Grimaldi, S.; Kao, S.-C.; Castellarin, A.; Papalexiou, S.-M.; Viglione, A.; Laio, F.; Aksoy, H.; Gedikli, A. Statistical Hydrology. In *Treatise on Water Science*; Academic Press; Elsevier B.V: Oxford, UK, 2011; Volume 2, pp. 479–517.
16. Singh, V.P. *Entropy-Based Parameter Estimation in Hydrology*; Springer Science + Business Media: Dordrecht, The Netherlands, 1998; ISBN 978-90-481-5089-2, ISBN 978-94-017-1431-0 (eBook). https://doi.org/10.1007/978-94-017-1431-0.
17. Y. Wu, S. Younas, K. Abbas, A. Ali and S. A. Khan, "Monitoring Reliability for Three-Parameter Frechet Distribution Using Control Charts," in *IEEE Access*, vol. 8, pp. 71245-71253, 2020, doi: 10.1109/ACCESS.2020.2987422.
18. TEFARUK HAKTANIR (1991) Statistical modelling of annual maximum flows in Turkish rivers, Hydrological Sciences Journal, 36:4, 367-389, DOI: 10.1080/02626669109492520.
19. Murshed, Md. Sharwar, Byung-Jun Park, Bo-Yoon Jeong, and Jeong-Soo Park. "LH-Moments of Some Distributions Useful in Hydrology." *Communications for Statistical Applications and Methods*. The Korean Statistical Society, July 31, 2009. doi:10.5351/ckss.2009.16.4.647.
20. Matalas N.C. Probability Distribution of Low Flows. Statistical Studies in Hydrology. Geological Survey, United States Covernment Printing Office, USA, Washington:1963.
21. Rameshwar D. Gupta & Debasis. Kundu (2001) Generalized exponential distribution: different method of estimations, Journal of Statistical Computation and Simulation, 69:4, 315-337, DOI: 10.1080/00949650108812098.
22. *STAS 4068/1-82*; Maximum Water Discharges and Volumes, Determination of maximum Water Discharges and Volumes of Watercourses, the Romanian standardization institute, Romania.