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Article

Newton's Like Normal S-iteration under Weak Conditions

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Abstract: In the present paper, we introduced a quadratically convergent Newton's like normal S-iteration method free from the second derivative for the solution of nonlinear equations permitting $f'(x) = 0$ at some points in the neighborhood of the root. Our proposed method works well when the Newton method fails. Numerically it has been verified that the Newton's like normal S-iteration method converges faster than Fang et al. method [A cubically convergent Newton-type method under weak conditions, *J. Compute. and Appl. Math.*, **220** (2008), 409-412]. We studied different aspects of normal S-iteration method. Lastly, fractal patterns support the numerical results and explain the convergence, divergence, and stability of method.

Keywords: Newton's method; normal S-iteration; weak condition; simple root; order of convergence

1. Introduction

In this work, we have proposed a Newton's like normal S-iteration method for solving nonlinear algebraic and transcendental equations of the form ([17], [18], [19])

$$f(x) = 0. \tag{1}$$

Newton's method [13] is a basic method for solving (1), which converges to the root quadratically under some conditions. Newton's method is defined as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots . \tag{2}$$

Some weakness of Newton's method are as follows ([1]-[25]):
(i) It is only of order two.
(ii) The initial approximation should be near to the root.
(iii) The denominator term of Newton's method must not be zero, at the root or near to the root.

To remove these weakness, Wu [25] developed a quadratic convergent method in 2000 as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{\lambda_n f(x_n) + f'(x_n)}, \quad n = 0, 1, 2, \cdots , \tag{3}$$

where $|\lambda_n| \in (0, \infty)$.

Fang et al. [15] studied a method in 2008 as follows:

$$\begin{cases} y_n = x_n + \frac{f(x_n)}{\lambda_n f(x_n) + f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{\lambda_n f(x_n) + f'(x_n)}, \quad n = 0, 1, 2, \cdots , \end{cases} \tag{4}$$

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where $|\lambda_n| \leq 1$. They claimed that their method (4) is of cubic convergence. More precisely,

Theorem 1. *[15] Let $f : I \subseteq \Re \to \Re$ be a function and assume that*
(L1) $x^* \in I$ is a simple zero of f,
(L2) f is three times differentiable on I,
(L3) $\lambda_n f(x) + f'(x) \neq 0$, for all $x \in N(x^*)$, where $N(x^*)$ is neighborhood of x^*. Then the method (4) converges cubically to x^*.

Recently, Wang and Liu [10] identified that the Fang et al. method given by (4) is only of order two. Wang and Liu [10] revised Theorem 1 as follows:

Theorem 2. *Let $f : I \subseteq \Re \to \Re$ be a function and assume that*
(i) $x^* \in I$ is a simple zero of f,
(ii) f is three times differentiable on I,
(iii) $\lambda_n f(x) + f'(x) \neq 0$, for all $x \in N(x^*)$, where $N(x^*)$ is neighborhood of x^*. Then method (4) converges quadratically to x^*.

Contemporary, Wang and Liu [10] modified method (4) for third-order convergence as follows:

$$\begin{cases} y_n = x_n + \frac{f(x_n)}{\lambda_n f(x_n) - f'(x_n)}, \\ x_{n+1} = y_n + \frac{f(y_n)}{\lambda_n f(x_n) - f'(x_n)}, \quad n = 0, 1, 2, \cdots, \end{cases} \tag{5}$$

where $|\lambda_n| \leq 1$ and it is equal to -$sign(f(x_n)f'(x_n))min\{1, |f(x_n)|\}$. Under above modification, Wang and Liu [10] settled third-order convergence Theorem as follows:

Theorem 3. *Let $f : I \subseteq \Re \to \Re$ be a function and assume that*
(W1) $x^* \in I$ is a simple zero of f,
(W2) f is three times differentiable on I,
(W3) $\lambda_n f(x) - f'(x) \neq 0$, for all $x \in N(x^*)$, where $N(x^*)$ is neighborhood of x^*. Then, the iterative method (5) is cubically convergent.

It is clear from condition *(W2)* of Theorem 3, that the sufficient condition for the convergence of method (5) to the zero of the function f is that the third derivative of f must exist. But, we often come across the situation, when the third order differential of the function does not exist, while f has a zero in the interval I. Consider the function f_1 defined by

$$f_1(x) = x^{5/2} - \exp(x) + 1.$$

Here $x^* = 0.0$. Note that $f_1(x^*) = 0$ and $f_1'''(x^*)$ does not exist. Hence, we observe that

(i) Newton's method (2) can not be used.

(ii) Wang and Liu method (5) doesn't satisfy the condition *(W2)* of Theorem 3. At this stage, following natural **question** arises: Is it possible to propose an iterative method for finding solution of (1), when f is not three times differentiable on I.

The objective of this work is to introduce Newton's like normal S-iteration method for solving nonlinear equation (1). Taking into account, we describe the new method in which second derivative of the function f is sufficient for the convergence and is comparable to the third order methods. Thus, our method provides not only an affirmative answer of the question, but also behaves well in comparison of third order Wang and Liu method [10].

Rest of the paper is arranged as follows: Section 2 is Preliminary. In section 3, we have proposed the new Newton's like normal S-iteration method and established its convergence analysis. In section 4, numerical examples are given to check the theoretical results . Lastly, dynamical analysis supports the numerical and theoretical results in section 5.

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2. Preliminary

Let x^* be a root of non-linear equation (1) and f be a sufficiently differentiable function and $x_n \in N(x^*)$, where $N(x^*)$ is neighborhood of x^*. Then, the numerical solution of (1) can be written as

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt. \tag{6}$$

Approximating the integral by $(x - x_n)f'(x_n)$ with $x = x^*$ in (6), we get

$$0 \approx f(x_n) + (x^* - x_n)f'(x_n).$$

Therefore, a new approximation x_{n+1} to x^* can be written as (2). Newton's method (2) fails when derivative of the f becomes zero in the neighbourhood of the root. On replacing $f'(x_n)$ in (2) by $f'(x_n) + \lambda_n f(x_n)$, we obtain an approximation x_{n+1} as given in (3), which is quadratically convergent method given by Wu [25].

3. New Newton's like method and its Convergence Analysis

In this section, we introduce new Newton like normal S-iteration method and study its convergence analysis.

In [5], Sahu introduced normal S-iteration process as follows:

Definition 1. Let D be a nonempty convex subset of a normed space X and $T : D \to D$ be an operator. Then for arbitrary $x_0 \in D$, the normal S-iteration process is defined by

$$x_{n+1} = T((1 - \beta_n)x_n + \beta_n T(x_n)), \quad n = 0, 1, 2, \cdots,$$

where the sequence $\beta_n \in (0, 1)$.

There are many papers dealing with S-iteration process and normal S-iteration process in the literature. In [6], Sahu introduced Newton's like method based on normal S-iteration process as follows:

$$\begin{cases} x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}, \\ y_n = (1 - \beta_n)x_n + \beta_n u_n, \\ u_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots, \end{cases}$$

where the sequence $\beta_n \in (0, 1)$ and $f'(x)$ is the derivative of f at point x.

We now introduce our new Newton's like normal S-iteration method for solving nonlinear equation (1), when f' may be zero in the neighbourhood of the root, as

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n G(x_n), \\ x_{n+1} = G(y_n), \quad n = 0, 1, 2, \cdots, \end{cases} \tag{7}$$

where

$$G(x_n) = x_n + \frac{f(x_n)}{\lambda_n f(x_n) - f'(x_n)}, \tag{8}$$

$\beta_n \in (0, 1)$ and λ_n is a sequence in $\Re$, such that $|\lambda_n| \leq 1$. The parameter λ_n is chosen in such a manner that both $\lambda_n f(x_n)$ and $-f'(x_n)$ have same sign and hence denominator is non zero in equation (8). For this purpose we use signum function as follows:

$$sign(x) = \begin{cases} 1, & if\ x \geq 0, \\ -1, & if\ x < 0. \end{cases}$$

We are ready to establish main result of this paper, which is as follows:

Theorem 4. *Let $f : I \subseteq \Re \to \Re$ be a function and assume that*

(i) $x^* \in I$ is a simple zero of f,

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(ii) f is two times differentiable on I,
(iii) $\lambda_n f(x) - f'(x) \neq 0$, for all $x \in N(x^*)$, where $N(x^*)$ is neighborhood of x^* and $\mid \lambda_n \mid \leq 1$.
Then, the Newton's like normal S-iteration method defined by the (7) is quadratically convergent locally to the zero of f.

Proof: Let $x^* \in I$ be a simple zero of a function f, $e_n = x_n - x^*$ and $A_k = \left(\frac{1}{k!}\right) f^{(k)}(x^*)/f'(x^*)$. Using Taylor expansion about x^* and using $f(x^*) = 0$, we get

$$f(x_n) = f'(x^*)\left[e_n + A_2 e_n^2 + A_3 e_n^3 + O(e_n^4)\right], \tag{9}$$

$$f'(x_n) = f'(x^*)\left[1 + 2A_2 e_n + 3A_3 e_n^2 + 4A_4 e_n^3 + O(e_n^4)\right]. \tag{10}$$

Now, from above two equations we get

$$\begin{aligned} f'(x_n) - \lambda_n f(x_n) &= f'(x^*)[1 + (2A_2 - \lambda_n)e_n + (3A_3 - \lambda_n A_2)e_n^2 + (4A_4 - \lambda_n A_3)e_n^3 \\ &+ O(e_n^4)] \end{aligned} \tag{11}$$

and from (9) and (11), we get

$$\frac{f(x_n)}{\lambda_n f(x_n) - f'(x_n)} = -e_n + (A_2 - \lambda_n)e_n^2 + \left(2A_2\lambda_n - \lambda_n^2 - 2A_2^2 + 2A_3\right)e_n^3 + O(e_n^4).$$

Using above in (8), we obtain,

$$G(x_n) = x^* + (A_2 - \lambda_n)e_n^2 + \left(2A_2\lambda_n - \lambda_n^2 - 2A_2^2 + 2A_3\right)e_n^3 + O(e_n^4). \tag{12}$$

Now, on using (12) in the first substep of (7), we get

$$y_n = x^* + (1 - \beta_n)e_n + \beta_n(A_2 - \lambda_n)e_n^2 + \beta_n\left(2A_2\lambda_n - \lambda_n^2 - 2A_2^2 + 2A_3\right)e_n^3 + O(e_n^4). \tag{13}$$

On expanding $f(y_n)$ and $f'(y_n)$ about x_n, we obtain

$$\begin{aligned} f(y_n) &= f'(x^*)\Big[(1 - \beta_n)e_n + \Big\{A_2(1 - \beta_n)^2 + \beta_n(A_2 - \lambda_n)\Big\}e_n^2 \\ &+\beta_n\Big\{2A_2\lambda_n - \lambda_n^2 - 2A_2^2 + 2A_3 + 2A_2(1 - \beta_n)(A_2 - \lambda_n)\Big\}e_n^3 \\ &+O(e_n^4)\Big], \end{aligned} \tag{14}$$

$$\begin{aligned} f'(y_n) &= f'(x^*)\Big[1 + 2A_2(1 - \beta_n)e_n + \Big\{3A_3(1 - \beta_n)^2 + 2A_2\beta_n(A_2 - \lambda_n)\Big\}e_n^2 \\ &+\beta_n\Big\{2A_2\Big(2A_2\lambda_n - \lambda_n^2 - 2A_2^2 + 2A_3\Big) + 6A_3(1 - \beta_n)(A_2 - \lambda_n)\Big\}e_n^3 \\ &+O(e_n^4)\Big]. \end{aligned} \tag{15}$$

Now, from (14) and (15), we have

$$\begin{aligned} \lambda_n f(y_n) - f'(y_n) &= f'(x^*)[-1 + (1 - \beta_n)(\lambda_n - 2A_2)e_n \\ &+\lambda_n\Big\{(1 - \beta_n)^2(A_2 - 3A_3) + \beta_n(A_2 - \lambda_n)(1 + 2A_2)\Big\}e_n^2 \\ &+O(e_n^3)\Big]. \end{aligned} \tag{16}$$

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Furthermore, from (14) and (16), we have

$$\frac{f(y_n)}{\lambda_n f(y_n) - f'(y_n)} = -(1-\beta_n)e_n + \{\lambda_n - 3A_2 - \beta_n(\lambda_n - 5A_2) + \beta_n^2(\lambda_n - 3A_2)\Big\}e_n^2 + O(e_n^3). \tag{17}$$

With the help of (17), the second equation of (7) becomes

$$x_{n+1} = x^* + \Big\{\lambda_n - 3A_2 - \beta_n(2\lambda_n - 6A_2) + \beta_n^2(\lambda_n - 3A_2)\Big\}e_n^2 + O(e_n^3)$$

$$\Rightarrow e_{n+1} = Ce_n^2 + O(e_n^3) \tag{18}$$

where $C = \lambda_n - 3A_2 - \beta_n(2\lambda_n - 6A_2) + \beta_n^2(\lambda_n - 3A_2)$.

Hence, the Newton's like normal S-iteration method proposed in (7) has second order convergence.

4. Numerical Results

In this section, we present some numerical tests to show the applicability of the proposed method by considering two categories of functions namely (i) functions which are differentiable three times and (ii) functions which are differentiable only two times. Numerical computations have been carried out in MATLAB 2007 and stoping criteria has been taken as (i) $|f'(x_k)| \le \varepsilon$, (ii) $|x_k - x_{k-1}| \le \varepsilon$, where $\varepsilon = 10^{-15}$. We have applied Newton's like normal S-iteration method for the following three values of λ_n

(i) $|\lambda_n| = 0.5$
(ii) $|\lambda_n| = 1.0$ and
(iii) $\lambda_n = -sign(f(x_n)f'(x_n))min\{1, |f(x_n)|\}$ (λ_n is taken as in Wang and Liu [10]).

(i) ***Functions with third order differentials***

Here, we have considered those example which were taken by Wang and Liu [10] as follows:

$$F_1(x) = x\sin x + \cos x - 0.6, \quad x^* = -2.54623173142842,$$

$$F_2(x) = x^3 - 2x^2 + x - 1, \quad x^* = 1.75487766624669,$$

$$F_3(x) = \ln x, \quad x^* = 1.0000,$$

$$F_4(x) = \arctan x, \quad x^* = 0.0000,$$

$$F_5(x) = x + 1 - \exp(\sin x), \quad x^* = 1.69681238680975,$$

$$F_6(x) = x\exp(-x^2) - (\sin x)^2 + 3\cos x + 5, \quad x^* = -1.20764782713092,$$

$$F_7(x) = 10x\exp(-x^2) - 1. \quad x^* = 1.67963061042845.$$

For the two values of $\lambda_n = 0.5$ and λ_n as wang, we have considered $\beta_n = 0.5$ and 0.9 in Table 1. On starting with the same initial point as in Wang and Liu [10] in all test problems, we observe that for the both values of λ_n our normal S-iteration method takes less number of iterations than the Wang and Liu method [10] for the value of $\beta_n = 0.9$. Thus in spite of being second order convergence it performs better than third order Wang and Liu method [10]. Also, It may be noted that in all test problems, the classical Newton's method is either fail or diverge in most of the cases. In Table 1 F, D and NC denote failure of the method, divergence of the method and not converging to the

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Table 1: Functions whose third order differentials exist

$f(x)$	x_0	Newton Method	Wang and Liu Method	Normal S-iteration method			
				$\lvert\lambda_n\rvert = 0.5$		λ_n as Wang and Liu	
				$\beta_n = 0.5$	$\beta_n = 0.9$	$\beta_n = 0.5$	$\beta_n = 0.9$
F_1	0	F	5	7	5	5	4
	-4	6	5	5	4	6	5
F_2	1	F	7	5	5	5	4
	3	7	6	6	6	6	5
F_3	5	D	5	5	4	7	6
	2	6	4	3	3	5	4
F_4	3	D	4	5	4	5	4
	-1	5	3	4	3	4	3
F_5	4	NC	6	6	5	7	6
	2	5	4	4	4	4	4
F_6	0.73	D	8	6	4	8	4
	-3	23	15	11	9	11	9
F_7	0.7	D	5	4	4	4	4
	2	6	4	4	3	4	3

desired root respectively.

(ii) ***Functions which are differentiable only two times***

We have considered the following real functions from $I \subset \Re \to \Re$ and the results are shown in Table 2.

$$f_1(x) = x^{\frac{5}{2}} - \exp x + 1, \quad x^* = 0.0,$$

$$f_2(x) = x^4 \sin\frac{1}{x}, x \neq 0, \quad x^* = 0.31830988618379 (x_0 = 1),$$

$$x^* = 0.106103295394597 (x_0 = 0.1),$$

$$f_3(x) = x^{\frac{7}{3}} \sin x, \quad x^* = 0.0,$$

$$f_4(x) = (x-2)^{\frac{7}{3}} - x^3 + 3x^2 - 2, \quad x^* = 2.475200396019297,$$

$$f_5(x) = x^{\frac{7}{3}} \exp x, \quad x^* = 0.0,$$

$$f_6(x) = (x+2)^{\frac{5}{2}} + \exp x - 1, \quad x^* = -1.142466838767107.$$

Table 2: Functions whose third order differential does not exist

$f(x)$	x_0	Newton Method	Fang et al. Method	Normal S-iteration method					
				λ_n as Wang and Liu		$\lvert\lambda_n\rvert = 0.5$		$\lvert\lambda_n\rvert = 1$	
				$\beta_n = 0.9$	$\beta_n = 0.5$	$\beta_n = 0.9$	$\beta_n = 0.5$	$\beta_n = 0.9$	$\beta_n = 0.5$
f_1	0.5	F	7	3	4	3	4	4	5
f_2	1.0	9	9	6	7	6	7	6	8
	0.1	5	5	3	4	3	4	3	4
f_3	0.3	85	58	47	60	47	60	47	60
	1.0	88	61	49	62	49	62	49	62
f_4	2.0	F	9	4	5	5	4	4	4
f_5	1.0	89	43	33	42	33	42	33	43
f_6	-2.0	10	F	3	4	5	4	3	4

As we know from the condition *(W2)* of Theorem 3 that the cubically convergent Wang and Liu method will converge to the root only if the third order differential to the function would exist in the neighbourhood of the root. Hence, Wang's method is no more applicable in this case. Therefore, we have compared the present method with quadratically convergent same order Newton's method and Fang et al. method [15] for the different values of λ_n and β_n ($\lambda_n = 0.5, \lambda_n = 1$, λ as in Wang and Liu [10] and

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$\beta_n = 0.5, \beta_n = 0.9$) in Table 2. In all test problems, for all the values of λ_n and β_n, we can see that the present new Newton's like normal S-iteration method is alwasys taking less number of iterations except for the example 3 (case $\beta_n = 0.5$) in comparison to the quadratically convergent methods. Hence, we conclude that the present method is more effective robust and stable.

4.1. *Behavior of normal S-iteration method for different value of λ_n and β_n*

We have considered the function F_6 to see the empirical behavior of proposed normal S-iteration method for different value of λ_n and β_n starting with the initial points 0.73 and -3.0. Numerical results in Table 3 shows that the proposed method is not affected much due to the variation in value of λ_n. But the value of β_n play crucial role as we take its different values in the interval (0, 1). We can see that ranging the value of β_n from 0.1 to 0.9, the optimum value of β_n comes out to be 0.9 for which the proposed method is taking the least number of iterations.

Table 3: Proposed method for different value of λ_n and β_n

$f(x)$	x_0	β_n	Normal S-iteration method		
			$\lvert\lambda_n\rvert = 0.5$	$\lvert\lambda_n\rvert = 1$	λ_n as Wang and Liu
		0.1	13	9	9
		0.3	7	8	8
	0.73	0.5	6	8	8
		0.7	5	5	5
		0.9	4	4	4
F_6					
		0.1	14	15	14
		0.3	12	13	13
	-3.0	0.5	11	11	11
		0.7	10	10	10
		0.9	8	9	9

5. Normal-S iteration method with variable value of β

We consider the two sequence of β_n as $\beta_n^1 = 0.1 + 1/2(n+2)$ and $\beta_n^2 = 1 - 1/2(n+2)$ to solve following two test functions:

(a) $F_1(x) = xsinx + cosx - 0.6$ $x^* = -2.54623173142842$ and
(b) $f_2(x) = x^4 sin(1/x)$ $x^* = 0.31830988618379$.

We observe from the Table 4, that the second sequence $\beta_n^2 = 1 - 1/2(n+2)$ is taking less number of iterations in comparison to the first sequence $\beta_n^1 = 0.1 + 1/2(n+2)$ in converging to the root for the both examples. Hence, we conclude that the sequence which converges near 1, ($\beta_n^2 = 1 - 1/2(n+2)$) gives the faster convergence.

6. Average number of iterations in Normal-S iteration method

Table 5 and Table 6 show the average number of iterations denoted by ANI of 50 tests done for different values of β_n [7]. For this purpose, we have considered the following two test functions, which are three times differentiable:
Example $F_2(x) = x^3 - 2x^2 + x - 1.0 = 0$.
It has root $x^* = 1.75487766624669$. We have taken the initial approximations in the grid as follows: $x_0 = 0.25 + ih, i = 1, \cdots, 50$ and $h = 0.03$ (see Table 5). Allowed error is 10^{-14}.

Example $F_6(x) = x\exp(-x^2) - (\sin x)^2 + 3\cos x + 5$.
It has root $x^* = -1.20764782713092$. We have taken the initial approximations x_0 in the

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Table 4: Normal-S iteration method with variable value of β

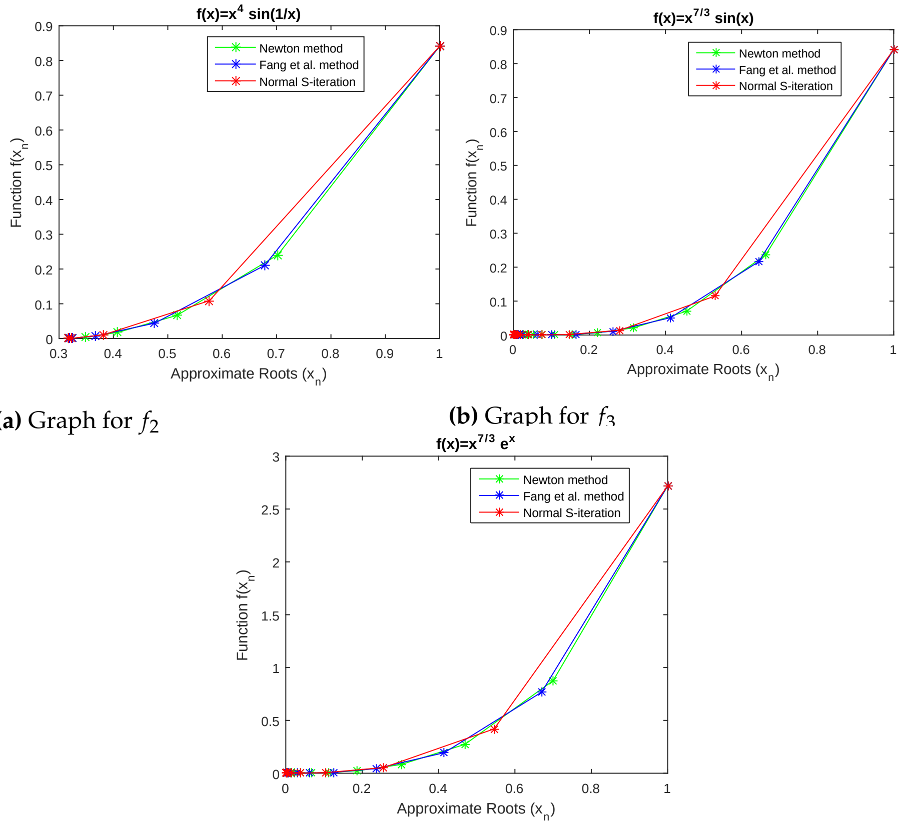
$f(x)$	Normal S-iteration for sequence β_n^1		Normal S-iteration for sequence β_n^2	
	$\lvert\lambda_n\rvert = 0.5$	λ_n as Wang and Liu	$\lvert\lambda_n\rvert = 0.5$	λ_n as Wang and Liu
$F_1(x)$	-4.00000000000000	-4.00000000000000	-4.00000000000000	-4.00000000000000
	-3.019890471239318	-3.269614812666443	-2.787748595141695	-3.031336002398129
	-2.647689829523139	-2.830596759888509	-2.550732240466982	-2.602227130430227
	-2.552574309373607	-2.597269129310047	-2.546231963106547	-2.546267106449917
	-2.546259317314531	-2.547305870288047	-2.546231731428419	-2.546231731433164
	-2.546231731968219	-2.546232155885697		-2.546231731428418
	-2.546231731428418	-2.546231731428486		
		-2.546231731428418		
$f_2(x)$	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
	0.690862097114279	0.713251419170333	0.588489366623379	0.613537499145787
	0.500158984920628	0.506202917231944	0.388129276855868	0.391429495638206
	0.391718592801076	0.390681052832476	0.323527870651833	0.323501217147855
	0.338547374719877	0.337061756299449	0.318314259700137	0.318313848040219
	0.320626856711258	0.320222652750527	0.318309886184780	0.318309886184561
	0.318346374736978	0.318333591884421	0.318309886183791	0.318309886183791
	0.318309895590689	0.318309889954683		
	0.318309886183791	0.318309886183791		

Table 5: Average number of iterations in Normal-S iteration method

β	Average number of iterations in Normal-S iteration method (ANI)		
	$\lvert\lambda_n\rvert = 0.5$	$\lvert\lambda_n\rvert = 1$	λ_n as Wang and Liu
0.1	5.340000	5.080000	5.100000
0.2	5.040000	4.920000	4.920000
0.3	4.800000	4.720000	4.600000
0.4	4.360000	4.480000	4.420000
0.5	4.240000	4.300000	4.280000
0.6	4.100000	4.080000	4.140000
0.7	3.800000	3.760000	3.640000
0.8	3.700000	3.600000	3.540000
0.9	3.620000	3.300000	3.340000

Table 6: Average number of iterations in Normal-S iteration method (ANI)

β	Average number of iterations in Normal-S iteration method		
	$\lvert\lambda_n\rvert = 0.5$	$\lvert\lambda_n\rvert = 1$	λ_n as Wang and Liu
0.1	5.725490	5.411765	5.333333
0.2	5.411765	5.196078	5.137255
0.3	5.176471	4.941176	4.882353
0.4	4.980392	4.823529	4.764706
0.5	4.705882	4.666667	4.607843
0.6	4.431373	4.549020	4.450980
0.7	4.254902	4.352941	4.294118
0.8	3.764706	4.137255	4.058824
0.9	3.803922	3.764706	3.666667



(a) Graph for f_2

(b) Graph for f_3

(c) Graph for f_5

Figure 1. Graph between value of functions and roots

grid as follows: $x_0 = -2.0 + ih, i = 1, \cdots, 50$ and $h = 0.03$ (see Table 6). Allowed error is 10^{-14}.

7. Convergence behaviour of Newton's, Fang et al. and present method

Convergence behaviour of Newton's method, Fang et al. method [15] and new Newton's like normal S-iteration method are shown in Fig. 1-3. To study the convergence behavior we have taken the test functions f_2, f_3 and f_5 and for each test functions, we have considered the three cases as:

Case 1: The graph between function and root for f_2, f_3, f_5.

Here, from the Fig. 1(a), it is clear that for $x_0 = 1.0$, we have $f_2(x_0) = 0.841470984807896$. Starting with this initial approximation x_0, the value of x_1 for Newton's method, Fang et al. method [15] and present method are 0.702195479022049, 0.677964714450141 and 0.576332178830878 respectively. Clearly the present method (red line) is better in its very first iteration among the all three methods. After successive iterations starting with $x_0 = 1.0$, present method converges to the root $x^* = 0.318309886183791$ in very fast manner as shown in figure. Similarly, we can see Fig. 1(b) for the function f_3 and Fig. 1(c) for function f_5, that the present method converges to the root $x^* = 0.318309886183791$ faster than others.

Case 2: The graph between number of iterations and root for f_2, f_3, f_5.

For the function f_3, we have $f_3(x_0) = 0.841470984807896$ for $x_0 = 1.0$. It is clear from Fig. 2(b) that starting with the initial approximation x_0, Newton's method, Fang et al. method [15] and present method converge to the root $x^* = 0.0$ in 88, 61 and 49 iterations respectively. Hence, new Newton's like normal S-iteration method takes less number of iterations in a very efficient manner. Similarly, we see the same pattern for f_2 and f_5 in Fig. 2(a) and Fig. 2(c) also.

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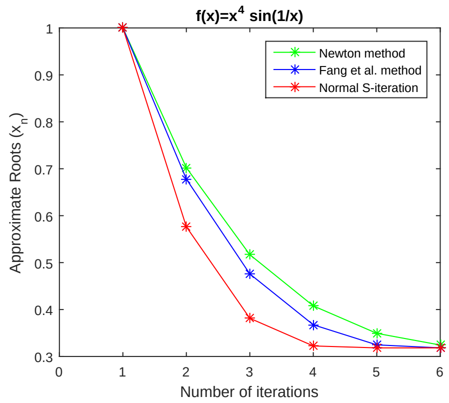
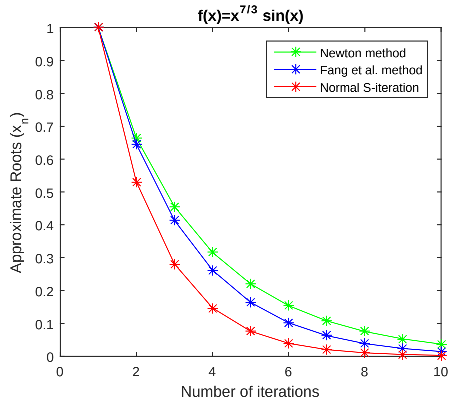
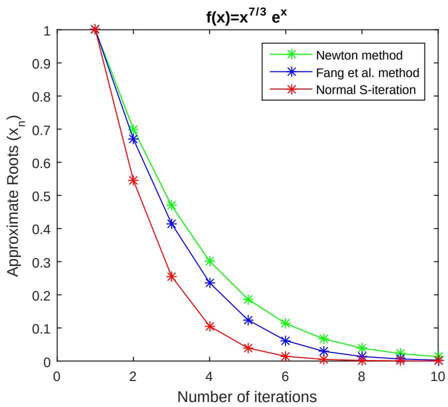


(a) Graph for f_2

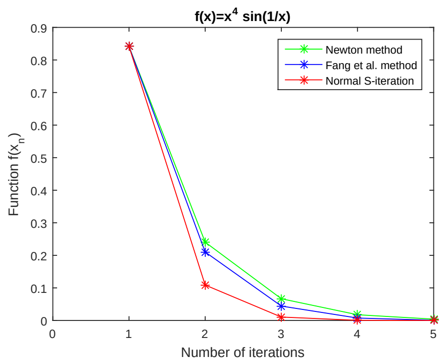


(b) Graph for f

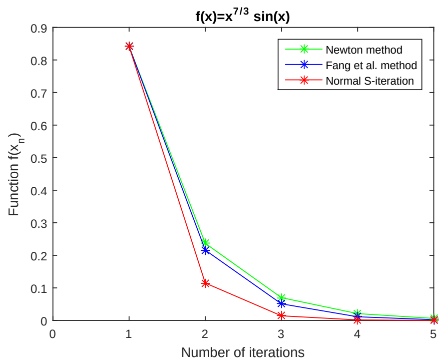


(c) Graph for f_5

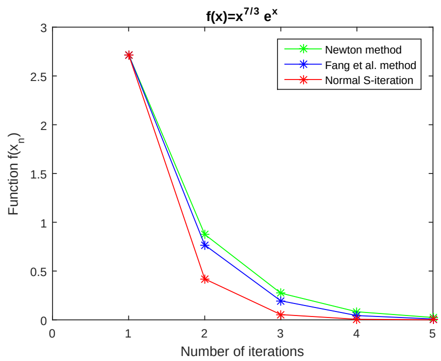
Figure 2. Graph between root and number of iterations



(a) Graph for f_2



(b) Graph for f



(c) Graph for f_5

Figure 3. Graph for value of functions and number of iterations

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Case 3: The graph between number of iterations and function for f_2, f_3, f_5.

In Fig. 3(c), we have $f_5(x_0) = 2.718281828459045$ for $x_0 = 1.0$. Starting with x_0, we can see from the graph that the value of the function f_5 in present method becomes 0 in 33 iterations while the Newton's method and Fang et al. method [15] takes 89 and 43 respectively, which shows that the present method converging to the root $x^* = 0.0$ faster than the Newton's method and Fang et al. method. Fig. 3(a) and Fig. 3(b) shows the same thing for the functions f_2 and f_3 respectively.

8. Dynamical Results of Methods for f_1, f_2, F_1, F_2

Now, we will define the following definitions but in the extended complex plane.
Definition 2 (see [20], [21]) Let us consider $g : I \to \mathbb{C}$ be a rational map on the Riemann sphere, where I is a subset of of the complex numbers $\mathbb{C}$. Then a point z_0 is said to be a fixed point of g if

$$g(z_0) = z_0.$$

Again for any point $z \in \mathbb{C}$, the Orbit of the point z can be difined as the set

$$Orb(z) = \{z, g(z), g^2(z), \cdots, g^n(z), \cdots\}.$$

Definition 3 (see [20], [21]) A periodic point z_0 is said to be of period k if $\exists$ a smallest positive integer k i.e. $g^k(z_0) = z_0$.
Remark If z_0 is periodic point of period k, then clearly it is a fixed point for g^k.

Definition 4 (see [20], [21]) Let z^* be a zero of the function F, then the basin of attraction of the zero z^* is defined as the set of all initial approximations z_0 such that any numerical iterative method starting with z_0 converges to z^*. It can be written as

$$B(z^*) = \{z_0 : z_{n+1} = g^n(z_0) \; converges \; \to \; z^*\}. \tag{19}$$

Here g^n is any fixed point iterative method.
Remark For example in case of Newton's method

$$z_{n+1} = g(z_n),$$

$$g(z_n) = z_n - \frac{F(z_n)}{F'(z_n)}, \quad n = 0, 1, 2, \cdots .$$

We can write the basin of attraction of the zero z^* for the Newton's method as follows:

$$B(z^*) = \{z_0 : z_{n+1} = g^n(z_0) \; converges \; \to \; z^*\}.$$

Definition 5 (see [20], [21]) The Julia set of a nonlinear map $g(z)$ is denoted as $J(g)$ and is defined as a set consisting of the closure of its repelling periodic points. The complement of Julia set $J(g)$ is called as the Fatou set $f(g)$.
Remark
(i) Some times Julia set of a nonlinear map may also be defined as the common boundary shared by basins of the roots and the Fatou set may also be defined as the set which contains the basin of attraction.
(ii) Fractals are very complicated phenomenon that may be defined as a self-similar surprising geometric object which repeated at every small scale ([4]).

We have studied the dynamical analysis of the rational functions using iterative methods. Then, we examined the theoretical and numerical results with the help of dynamical results. Dynamical study helps to understand the convergence and stabilty of the methods [20]. We apply our method on a square $\Re \times \Re = [-5, 5] \times [-5, 5]$ of 700×700 points with a tolerance- $|f(z_n)| < 5 \times 10^{-2}$ and a maximum of 30 iterations.

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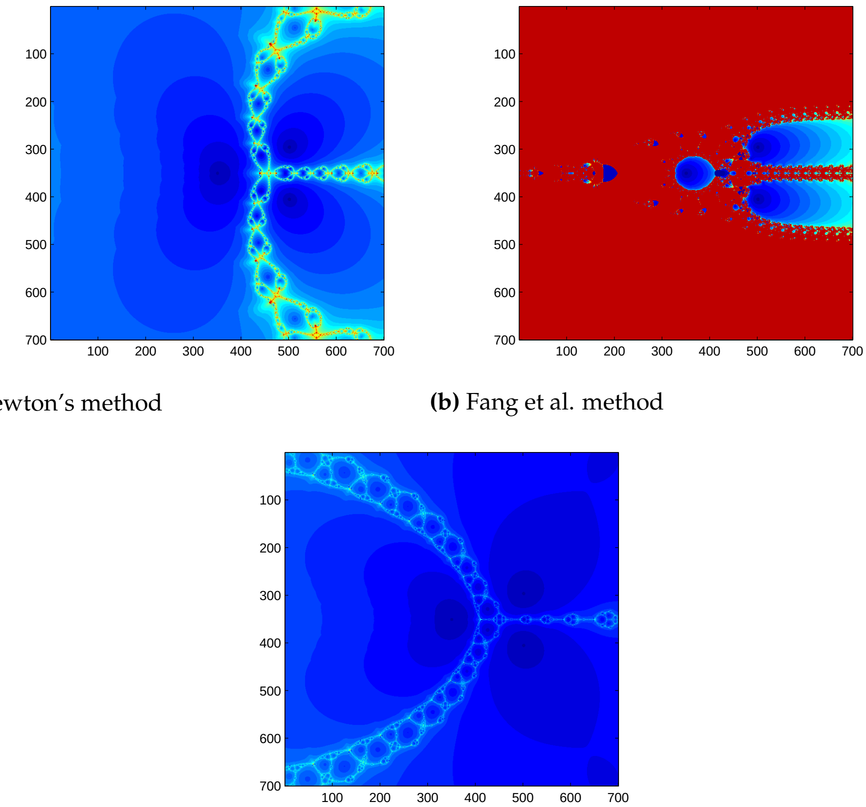

(a) Newton's method

(b) Fang et al. method

(c) Proposed method

Figure 4. Dynamics of different methods for $f_1(x) = x^{\frac{5}{2}} - \exp x + 1$

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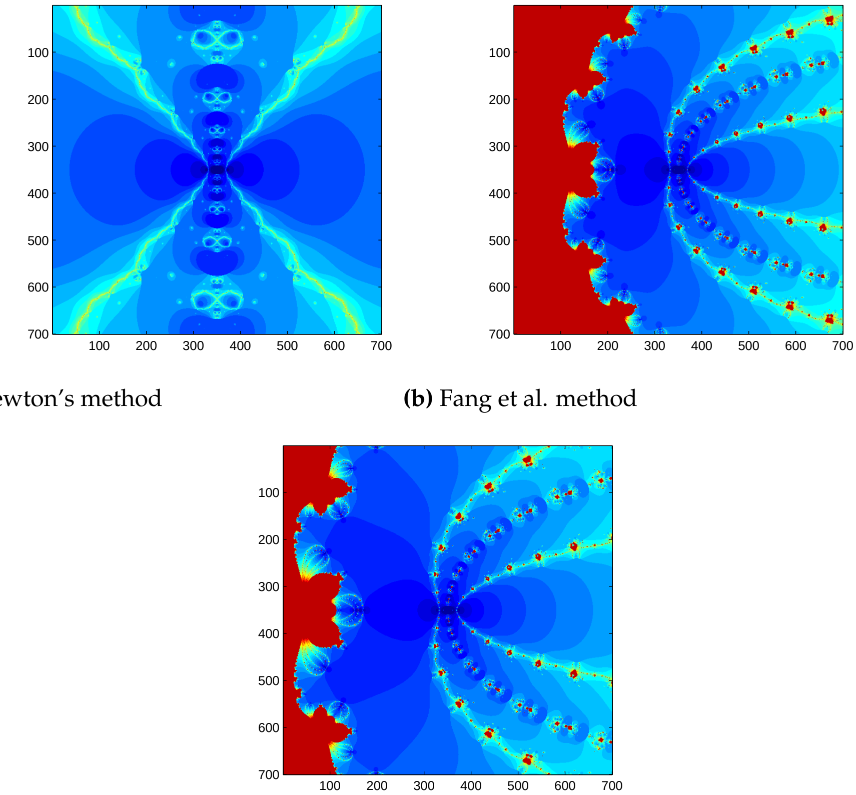


(a) Newton's method **(b)** Fang et al. method

(c) Proposed method

Figure 5. Dynamics of different methods for $f_2(x) = x^4 sin(1/x)$

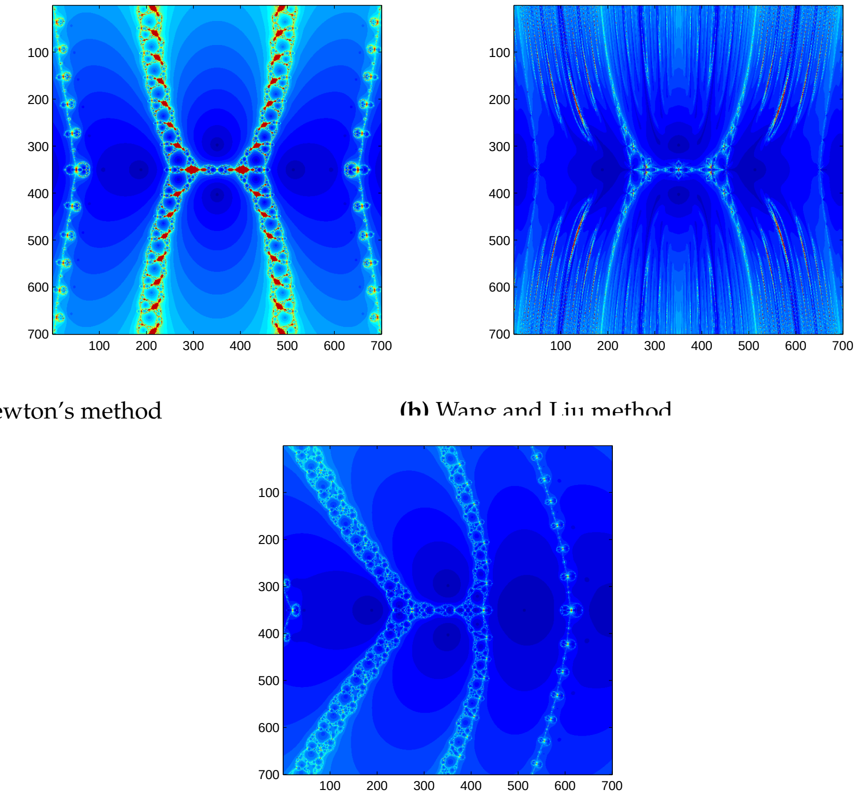
For any function, if the sequence generated by the iterative methods with any initial point z_0 converge to a zero z^* in the square, then we say that the point z_0 will lie in the basins of attraction of this zero and we assign a fixed color to this point z_0. In the following, we have described the speed of convergence and dynamics of the considered methods under two cases for finding complex roots of functions. In first case we have plotted the speed of convergence and dynamics of Newton's method, Fang et al. method [15] and the proposed method for functions f_1, f_2 (whose third order differential does not exist) . In the second case we have shown the speed of convergence and dynamics of Newton's method, Wang and Liu method [10] and the proposed method for functions F_1, F_2 (whose third order differentials exist).

8.1. Functions whose third order differential does not exist

$$f_1(x) = x^{\frac{5}{2}} - \exp x + 1, \quad x^* = 0.0,$$
$$f_2(x) = x^4 sin(1/x), \quad x^* = 0.31830988618379.$$

For the function $f_1 = x^{\frac{5}{2}} - \exp x + 1, \quad x^* = 0.0$, the dynamics and speed of convergence for various methods are shown in Fig. (4a), (4b), and (4c). It is clear from Fig. 4 that the proposed method with $|\lambda_n| = 0.5$ and $\beta_n = 0.9$ generate bigger orbits and darker color having less fractal boundaries and chaotic behavior. Newton's method show some type of chaotic behavior. Dynamics of Fang et al. method [15] generate smaller orbits but bigger Julia set showing the worst method.

The the dynamics and speed of convergence of Newton's method, Fang et al. method and the proposed method for $f_2 = x^4 sin(1/x)$, have been plotted in Fig. 5(a), 5(b), and 5(c) respectively. Clearly, fractal patterns of Newton's method contains large



(a) Newton's method

(b) Wang and Liu method

(c) Proposed method

Figure 6. Dynamics of different methods for $F_1(x) = x \sin x + \cos x - 0.6$

size of julia set having fractal bounadries and chaotic behaviour. Whereas proposed mathod and Fang et al. method [15] contains large size of Fotou set with basins but both the method have some non converging region in the left side.

8.2. *Functions whose third order differentials exist*

$$F_1(x) = x \sin x + \cos x - 0.6, \quad x^* = -2.54623173142842,$$
$$F_2(x) = x^3 - 2x^2 + x - 1, \quad x^* = 1.75487766624669.$$

For $F_1 = xsinx + cosx - 0.6$, the dynamics of Newton's method, Wang and Liu method [10] and the proposed method can be seen in Fig. 6(a), 6(b), 6(c) respectively. Here Fig. 6 show that the proposed method with $|\lambda_n| = 0.5$ and β_n = 0.9 is the best estimation because of bigger orbits and darker color having less fractal boundaries and chaotic behavior. Wang and Liu method [10] generate some type of chaotic behavior and Newton's method generate smaller orbits having fractal Julia set. This is the reason why Newton's method take several iterations and some times get failed.

The dynamics of Newton's method, Wang and Liu method and the proposed method for function $F_2 = x^3 - 2x^2 + x - 1$ have been shown in Fig. 7(a), 7(b), and 7(c). Failure of Newton's method with starting point $x_0 = 1.0$, as shown in the Table 3 is proved by fractal patterns Fig. 7(a). The speed of convergence of the Newton method and Wang and Liu method [10] is slow having fractal Julia set, chaotic behavior in comparison to the proposed method.

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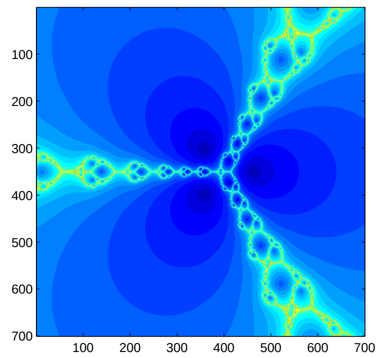
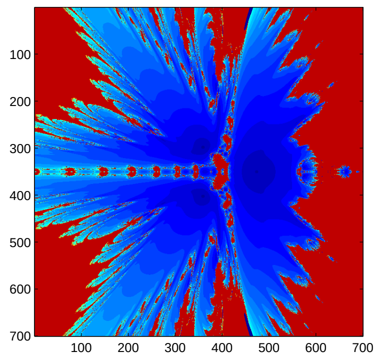
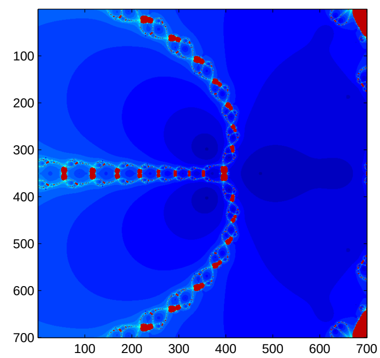


(a) Newton's method



(b) Wang and Liu method



(c) Proposed method

Figure 7. Dynamics of different methods for $F_2(x) = x^3 - 2x^2 + x - 1$

8.3. Dynamics of proposed method with variable value of β for example F_2

We have plotted the speed of convergence and dynamics of proposed method with variable value of β for $F_2(x) = x^3 - 2x^2 + x - 1, \quad x^* = 1.75487766624669$. Results are shown in Fig 8. It is clear from figure that speed of convergence is increasing with increase in value of β and for the value of $\beta = 0.9$, speed of convergence is optimal with bigger orbits and less chaotic behavior in comparison to the other value of $\beta = 0.1, 0.3, 0.5$ and 0.7.

9. Conclusion

We have obtained a new Newton's like normal S-iteration method for finding the root of the non-linear equation $f(x) = 0$. Theoretical results shows that it requires only second-order differentiability rather than third-order differentiability just like other methods. Numerical results and graphical illustration shows that the present method (7) is most effective and superior when Newton's method fails as well as it performs better than same order Fang et al. method [15] and third order Wang and Liu method [10] as it converges to the root much faster in very efficient manner for different values of λ_n with $\beta_n = 0.9$. Further, we have shown that the proposed method (7) converges to the root much faster for different values of λ_n with a sequence of variable value of β, which converges to one. Dynamical analysis also support the theoretical and Numerical results related to the convergence and stability behaviour of proposed method. Thus, from a practical point of view, the new Newton's like normal S-iteration method has the definite practical utility.

10. Conflict of interest

The authors declare that they have no conflict of interest.

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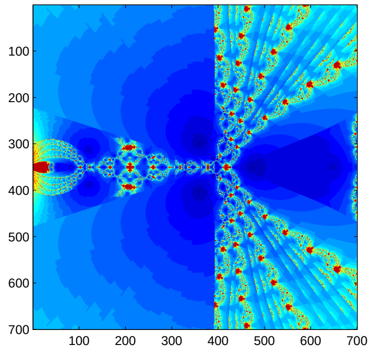
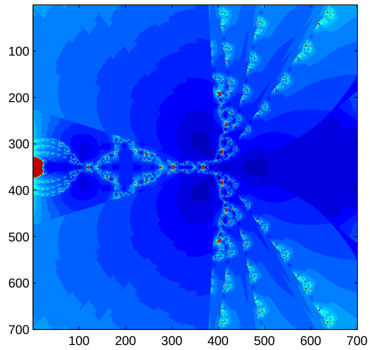
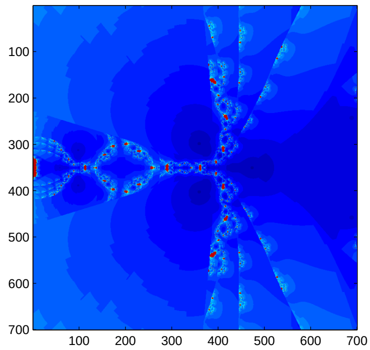


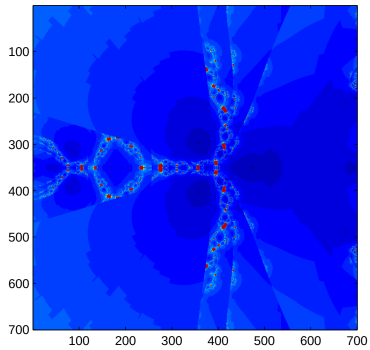
(a) Proposed method at $\beta = 0.1$



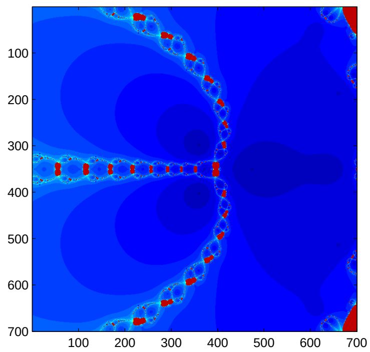
(b) Proposed method at $\beta = 0.3$



(c) Proposed method at $\beta = 0.5$



(d) Proposed method at $\beta = 0.7$



(e) Proposed method at $\beta = 0.9$

Figure 8. Dynamics of proposed method with variable value of β for $F_2(x) = x^3 - 2x^2 + x - 1$

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