Proof of The Collatz Conjecture

Bülent SUKUŞU
Department of Mathematics, Faculty of Science, Maltepe University, Istanbul, Turkey
bsukusu.blsu@gmail.com

Abstract

Collatz conjecture (or 3n+1 problem) has been explored for about 85 years. In this article, we prove the Collatz conjecture. We will show that this conjecture is valid for all positive integers by performing the Collatz inverse operation on the numbers that comply with the rules of the Collatz conjecture. Finally, it will be proved that there are no positive integers that do not comply with this conjecture.

Keywords: Collatz operation, Collatz inverse operation and Collatz numbers.

1 Introduction

The Collatz conjecture is one of the unsolved problems in mathematics. Introduced by German mathematician Lothar Collatz in 1937 [1], it is also known as the 3n + 1 problem, 3x + 1 mapping, Ulam conjecture (Stanisław Ulam), Kakutani's problem (Shizuo Kakutani), Thwaites conjecture (Sir Bryan Thwaites), Hasse's algorithm (Helmut Hasse), or Syracuse problem [2,3].

The Collatz Conjecture or 3n+1 problem can be summarized as follows:

Take any positive integer n. If n is even, divide n by 2. If n is odd, multiply n by 3 and add 1. Repeat this process continuously. The conjecture states that no matter which number you start with, you will always reach 1 eventually.

For example, if we start with 17, multiply by 3 and add 1, we get 52. If we divide 52 by 2, 26, and so on, the rest of the sequence is: 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.Or if we start 76, the sequence is: 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.

 doi:10.20944/preprints202210.0432.v1

This sequence of numbers involved is sometimes referred to as the hailstone sequence, hailstone numbers or hailstone numerals (because the values are usually subject to multiple descents and ascents like hailstones in a cloud) [4], or as wondrous numbers [5].

2 The Conjecture and Releated Conversions

Definition 2.1. N^+ the set of all positive integers, $n \in N^+$ Collatz defined the following map:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if n is even} \\ 3n+1, & \text{if n is odd} \end{cases}$$

The Collatz conjecture states that the value of every positive integer in the f function eventually reaches 1. In the following sections, In the following sections we will refer to the function f as Collatz operations.(CO).

Remark 2.2 According to definition of the Collatz conjecture, if the number we choose at the beginning is even, by continuing to divide all even numbers by 2, one of the odd numbers is achieved. For this reason, It is only sufficient to check whether all odd numbers reach 1 by the Collatz operation.

Therefore, If we prove that it reaches 1, when we apply the Collatz operation to all the elements of the set N^+_{odd} $=\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25...\}$, we have proved it for all positive integers. (The notaion of N^+_{odd} is positive odd numbers.)

Remark 2.3. If the Collatz operation is applied to the number 2^n ($n \in N^+$), then eventually 1 is reached. When we apply the Collatz operation to all the elements of the N^+_{odd} set, if we can convert them to 2^n numbers, we reach the result.

2.1 Collatz Inverse Operation (CIO)

Let $n \in N^+, a \in N^+_{odd}$; if a is converted to 2^n by the Collatz operation (CO), it must satisfy the following equation,

$$3.a + 1 = 2^n$$

then,

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 October 2022 doi:10.20944/preprints202210.0432.v1

$$a = \frac{2^n - 1}{3} \tag{1}$$

When we apply the collatz operation to the numbers a in (1), we always get 1.But,

Lemma 2.4. In $a = \frac{2^n-1}{3}$ (1), a cannot be an integer if n is a positive odd integer.

Proof. Let $n = 2m + 1$ and $m \in N$, (n is a positive odd integer), then substituting n to $2m + 1$ in (1); We get,

$$a = \frac{2^{2m+1} - 1}{3} \tag{2}$$

The factorization of $2^{2m+1} + 1$;

$2^{2m+1} + 1 = (2 + 1)(2^{2m} - 2^{2m-1} + 2^{2m-2} - \ldots + 1) = \mathbf{3}.k$, that is multiples of 3 are obtained. (k is positive odd numbers).

Since $2^{2m+1} - 1 = (2^{2m+1} + 1) - 2 = 3.k - 2$,

$3k - 2$ ($k \in N^+_{odd}$) is not a multiple of 3.Therefore a is not integer, for all m.

If we substitute $2n$ for n in (2), we get equation

$$a = \frac{2^{2n} - 1}{3} \tag{3}$$

Lemma 2.5. If $n \in N^+$ in (3), $a = \frac{2^{2n}-1}{3}$, we can find positive odd numbers a for all numbers of n.

Proof. Factorization of 2^{2n-1} for all n, ($n \in N^+$), if

$n = 1, \quad (2^2 - 1) = (2 - 1)(2 + 1) = \mathbf{3}.1$
$n = 2, \quad (2^4 - 1) = (2 - 1)(2 + 1)(2^2 + 1) = \mathbf{3}.5$
$n = 3, \quad (2^6 - 1) = (2^3 - 1)(2^3 + 1) = \mathbf{3.3}.7$
$n = 4, \quad (2^8 - 1) = (2 - 1)(2 + 1)(2^2 + 1)(2^4 + 1) = \mathbf{3}.(\ldots)$
$n = 5, \quad (2^{10} - 1) = (2^5 - 1)(2^5 + 1) = (2 - 1)(2^4 + \ldots)(2 + 1)(2^4 + \ldots.) = \mathbf{3}.(\ldots)$
$n = 6, \quad (2^{12} - 1) = (2^3 - 1)(2^3 + 1)(2^6 + 1) = \mathbf{3.3}.(\ldots)$
$n = 7, \quad (2^{14} - 1) = (2^7 - 1)(2 + 1)(2^6 - 2^5 + \ldots..) = \mathbf{3}.(\ldots)$
$\ldots$
$\ldots$

Subsituting n to 2^m, $\quad$ (m $\in \mathbb{N}^+$);

$(2^{2^m} - 1) = (2 - 1)(2 + 1)(2^2 + 1)(2^4 + 1)(2^8 + 1)(2^16 + 1)\ldots\ldots(2^{2^m-1} - 1) = \mathbf{3}.(\ldots)$

...

Since each of these infinite numbers has a factor of 3, we can find infinite positive odd numbers a, and when the Collatz operation is applied to these numbers, 1 is always obtained. In (3),

$$a = \frac{2^{2n} - 1}{3};$$

If $n = 1,\quad a_1 = 1$
$n = 2,\quad a_2 = 5$
$n = 3,\quad a_3 = 21 = 3.7$
$n = 4,\quad a_4 = 85$
$n = 5,\quad a_5 = 341$
$\dots\quad\dots$

$$\begin{array}{ccccccccccc} & 2^2 & 2^4 & 2^6 & 2^8 & 2^{10} & 2^{12} & 2^{14} & 2^{16} & 2^{18} & \dots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{A}=\{ & 1, & 5, & 21, & 85, & 341, & 1365, & 5461, & 21845, & 87381 & \dots \} \end{array}$$

Corollary 2.6. We get a set of A with infinite elements,these numbers reach 1 when we apply the collatz operation. (In the following sections, we will call the elements of the set A and the another numbers that fit the Collatz conjecture as Collatz numbers.)

Example 2.7. 5 $\rightarrow$ odd number, 16 $\rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$.
21$\rightarrow$ odd number, 64$\rightarrow$32$\rightarrow$16$\rightarrow$8$\rightarrow$4$\rightarrow$2$\rightarrow$1.

If we can generalize the elements of the set $A = \{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381 \dots\}$ to all positive odd numbers, the Collatz conjecture is proven.

2.2 Transformations in the Set A with Infinite Elements

Let the elements of set $A = \{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381 \dots\}$ be $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots\}$ respectively.

Lemma 2.8. In the set of A$\setminus\{a_0\}$, if $a_n \equiv 1 \pmod 3$

$$b_n = \frac{2^{2m}.a_n - 1}{3} \tag{4}$$

 doi:10.20944/preprints202210.0432.v1

$m \in N^+$, from each a_n we get infinite different B_n sets, that have infinite elements b_n (Collatz numbers), these numbers fit the conjecture. And than, from each b_n to C_n (with infinite elements), from each c_n to D_n ... similarly goes on forever.

Proof. If $a_n \equiv 1 \pmod 3$, we can take a_n as $3.p+1$. ($p \in N^+_{odd}$)

$a_n = 3.p+1$, substituting in (4),

$$b_n = \frac{2^{2m}.(3.p+1)-1}{3} = \frac{2^{2m}3p+2^{2m}-1}{3} = 2^{2m}p + \frac{2^{2m}-1}{3}$$

is divisible by 3 (Lemma 2.5.). Therefore, we get set of b_n with infinite different elements, which can be converted to a_n, so 1 by the Collatz operation. The elements of b_n are odd numbers.

Example 2.9. Let $a_1 = 85$, $a_1 \equiv 1 \pmod 3$,

in (4), $a_1 = 85 \to \; b_1 = \frac{2^2.85-1}{3} = 113, b_2 = \frac{2^4.85-1}{3} = 453, b_3 = \frac{2^6.85-1}{3} = 1813$

$b_4 = \frac{2^8.85-1}{3} = 7253, b_5 = \frac{2^{10}-1}{3} = 29013, b_6 = \frac{2^{12}.85-1}{3} = 116053$

$B = \{113, 453, 1813, 7253, 29013, 116053...\}$

Lemma 2.10. In the set of A$\backslash\{a_0\}$, if $a_n \equiv 2 \pmod 3$,

$$b_n = \frac{2^{2m-1}.a_n - 1}{3} \tag{5}$$

$m \in N^+$, from each a_n we get infinite sets with infinite elements b_n, that fit the collatz conjecture.

Proof. If $a_n \equiv 2 \pmod 3$, we can take a_n as $3.p+2$. ($p \in N^+_{odd}$)
$a_n = 3.p+2$, substituting in (5),

$$b_n = \frac{2^{2m-1}.(3p+2)-1}{3} = \frac{2^{2m-1}.3p+2^{2m}-1}{3} = 2^{2m-1}p + \frac{2^{2m}-1}{3}$$

$\frac{2^{2m}-1}{3}$ is divisible by 3 (Lemma 2.5). Therefore, we get set of b_n with infinite different elements, which can be converted to a_n, so 1 by the Collatz operation. The elements of b_n are odd numbers.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 October 2022 doi:10.20944/preprints202210.0432.v1

Example 2.11. Let $a_1 = 5$ and $a_1 \equiv 2 \pmod 3$;

$a_1 = 5 \rightarrow\ b_1 = \frac{2^1.5-1}{3} = 3,\ b_2 = \frac{2^3.5-1}{3} = 13,\ b_3 = \frac{2^5.5-1}{3} = 53$

$b_4 = \frac{2^7.5-1}{3} = 213,\ b_5 = \frac{2^9.5-1}{3} = 853,\ b_6 = \frac{2^{11}.5-1}{3} = 3413\ldots$

$B = \{3, 13, 53, 213, 853, 3413, 13653, 54613\ldots\}$

Lemma 2.12. In the set of A$\backslash\{a_0\}$, if $a_n \equiv 0 \pmod 3$,

$$b_n = \frac{2^m.a_n - 1}{3} \tag{6}$$

$m \in N^+$, there is no such integer b_n.

Proof . If $a_n \equiv 0 \pmod 3$, we can take a_n as $3.p$ ($p \in N^+_{odd}$)
$a_n = 3.p$, substituting in (6),

$$b_n = \frac{2^m(3.p) - 1}{3} = \frac{2^m 3.p - 1}{3} = 2^m.p - \frac{1}{3},$$

is not integer. We can apply Collatz inverse operation again to each element of the B_n set, we obtained above.

2.3 Converting the Collatz Numbers to all Positive Odd Integers

In the previous sections, when we applied the Collatz operation, we had named the numbers that reached 1 as the Collatz numbers. Now let's see how these collatz numbers convert to all integers.

$$\begin{array}{lccccccccccl} & 2^2 & 2^4 & 2^6 & 2^8 & 2^{10} & 2^{12} & 2^{14} & 2^{16} & 2^{18} & \ldots & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \text{A}=\{ & 1, & 5, & 21, & 85, & 341, & 1365, & 5461, & 21845, & 87381 & \ldots & \}\ \text{(Collatz Numbers)} \end{array}$$

 doi:10.20944/preprints202210.0432.v1

Example 2.13. A small fraction of the Collatz numbers that convert to 2^4

	...		...		...		...		...	...
	46421		2389		4949		1077		34581	69173
	11605		597		1237		269		8645	17293
	2901		149		309		67		2161	4323
	725		37		77		↑		↑	↑
	181,		9,		19 → 25, 101, 405, 1621,					6485, ...
5	45,		↑		↑					
↓	11	→	7,		29, 117, 469, 1877, 7509, 30037, ...					
3	↑									
13→	17,		69,	277,		1109,	4437,	17749,		70997, ...
53				↓		↓		↓		↓
213				151		739		23665		47331
853				605		2957		94661		189325
3413				2421		11829		378645		757301
13653				9685		47317		1514581		3029205
54613				38741		189269		6058325		12116821
218453				154965		757073		24233331		48467285
...				...		...		...		...

Example 2.14. Collatz numbers converting to 2^6

$2^6 \rightarrow 21$ (There are no other Collatz numbers. Lemma 2.12)

Lemma 2.15. There is only one different Collatz number which converts into each of 2^{6n}; (2^6, 2^{12}, 2^{18}, 2^{24} ...) numbers.

Proof. Factorization of 2^{6n} - 1,

$2^{6n}-1=(2^{3n}-1)(2^{3n}+1)$, In the expression 2^{6n}- 1, there is always a factor of (2^3+1),
Because when factoring $(2^{3n}-1)$ and $(2^{3n}+1)$,
if n is even, $(2^{3n}-1) = \ldots \ (2^{3f}+1)$ $3f$ is odd integer.
if n is odd, in $(2^{3n}+1)$, $3n$ is odd integer.
And if $3f$, $3n$ are odd,
$2^{3n}+1=(2^3+1)(2^{3n-3}-2^{3n-6}+2^{3n-9}-2^{3n-12}+2^{3n-15}\ldots+1)$
Therefore $2^{6n}-1=(2^3+1).(\ldots)=9.$(odd integer)

And, when we divide $(2^{6n}-1)$ by 3, only one collatz number is obtained. We can't obtain another collatz number because it is a multiple of 3.(Lemma 2.12)

Example 2.16. There is only one Collatz number of each of 2^{6n}, because the numbers are the multiple of 3.

$2^6 \rightarrow 21$

$2^{12} \rightarrow 1365$

$2^{18} \rightarrow 87381$

$2^{24} \rightarrow 5592405$

Example 2.17. A small fraction of the Collatz numbers that convert to 2^8

85								
$\downarrow$	... ...	...	...	...	...	...	...	
113	$\rightarrow$ 75,	301,	1205,	4821,	19285,	77141,	308565	...
453	$\downarrow$							
1813	401	$\rightarrow$ 267,	1069,	4277,	17109	....		
7253	1605		$\downarrow$	$\downarrow$				
29013	6421		1425	2851	$\rightarrow$ 3801 ...			
116053	25685		5701	11405	$\rightarrow$ 7603 ...			
464213	102741		22805	45621	$\downarrow$			
...	...		...	...	... ...			
...	...		...	...	... ...			

Lemma 2.18. We obtain new Collatz numbers by applying the Collatz inverse operation $(\frac{2^m.a_n-1}{3})$ $(m \in N^+)$, to the Collatz numbers. All of these numbers are different from each other.

Proof. Let a_1 and a_2 be any Collatz numbers, when we apply the collatz inverse operation to each of them, the resulting numbers are b_1 and b_2. If $b_1 = b_2$ then,

$\frac{2^m.a_1-1}{3} = \frac{2^t.a_2-1}{3}$ and $2^m.a_1 = 2^t.a_2$ for odd positive integers, must be $a_1 = a_2$ and $m = t$.

Lemma 2.19. Let $a_n \equiv 0 \pmod 3$ and a_n Collatz numbers. We can derive a_n, from other collatz numbers.

Proof. If $a_n \equiv 0 \pmod 3$, we can write $a_n = 3.k$, $k \in N^+_{odd}$, then let $b_n \not\equiv 0 \pmod 3$, With the Collatz operation to a_n is

$$\frac{3a^n+1}{2^n} = b_n = \frac{3.3.k+1}{2^n} = \frac{9k+1}{2^n}$$

Then, we get $k = 2.m+1$, for divide b_n by 2;

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 27 October 2022 doi:10.20944/preprints202210.0432.v1

$b_n = \frac{9.(2.m+1)+1}{2^n} = 9m+5$ if $b_n = 9m+5$ is odd, $b_n \not\equiv 0 \pmod 3$

if $b_n = 9m+5$ is even, m is odd, take that $m = 2y+1$ and then,
substituting m to $2y+1$, $b_n = 18y+14$ when we divide b_n by 2,
$b_n = 9y+7$ if b_n is odd, $b_n \not\equiv 0 \pmod 3$
if $b_n = 9y+7$ is even, y is odd , take that $y = 2x+1$ and then,
substitutingn$y$ to $2x+1$, $b_n = 18x+16$ when we divide b_n by 2, $b_n = 9x+8$ if b_n is odd, $b_n \not\equiv 0 \pmod 3$
if $b_n = 9x+8$ is even,x is even, take that $x = 2.z$ and then,
substituting x to $2z$, $b_n = 18z+8$ when we divide b_n by 2, $b_n = 9z+4$ if b_n is odd, $b_n \not\equiv 0 \pmod 3$
when we go on like this $b_n = 9s+2$, $b_n \not\equiv 0 \pmod 3$
and $b_n = 9r+1$,
consequantly $b_n \not\equiv 0 \pmod 3$
so we aplly Collatz inverse operaton to b_n, we get a_n, $\frac{2^n.b_n-1}{3} = a_n$

Corollary 2.20. The Collatz conjecture is valid for all positive odd integers.

Corollary 2.21. From each element of the infinitely element set A (the set of collatz numbers), we create new Collatz sets with infinite elements. [Note: If the elements of set A are a_n, must be $a_n \not\equiv 0 \pmod 3$] From the new infinite Collatz numbers that have been formed, infinite new numbers are formed, and it goes on like this forever and ever without stopping. So we get the whole set of positive odd numbers and we prove the Collatz conjecture for all N^+ (Remark 2.2).

3 The Absence of Any Positive Integer Other Than Collatz Numbers

In this section, we prove that there are no positive integers that do not comply with this conjecture.

Lemma 3.1. Cannot be any positive integer other than Collatz numbers.

Proof. Let's a number t_0 that is not Collatz number, $t_0 \in N^+$, then
When we apply the collatz inverse operation to t_0,

$t_0 \rightarrow \frac{2^n \quad .t_0-1}{3}$ we get T $=$\{ t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10} ...\}, and the elements of set T are not Collatz numbers.

Also, apply the collatz operation to t_0, until finding odd numbers;

$$t_0 \rightarrow \frac{3.\ t_0\ +1}{2^n}\ , \quad s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow s_5 \rightarrow s_6 \rightarrow s_7 \rightarrow s_8 \rightarrow s_9 \rightarrow s_{10} \quad \ldots$$

We get $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10} \ldots\}$ and the elements of set S are not Collatz numbers.

When we apply the Collatz inverse operation to each number that is not divisible by 3 in the sets T and S, we get new numbers that is not Collatz numbers. If t_0 is multiple of 3, we take s_1 instead of set T.

$$\begin{array}{llllllllll}
t_0 & & & & & & & & & \\
\downarrow & & & & & & & & & \\
t_1 \rightarrow t_{1(1)}, & t_{1(2)}, & t_{1(3)}\ , & t_{1(4)}, & t_{1(5)}, & t_{1(6)}, & t_{1(7)}, & t_{1(8)}, & t_{1(9)}, & t_{1(10)} \ \ldots \\
t_2 & \downarrow & & & & & & & & \\
t_3 & t_{11(1)} \rightarrow t_{111(1)}, & t_{111(2)}, & t_{111(3)}, & t_{111(4)}, & t_{111(5)}, & t_{111(6)}, & t_{111(7)}, \ \ldots & & \\
t_4 & t_{11(2)} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
t_5 & t_{11(3)} & t_{1111(1)} & t_{1112(1)} & t_{1113(1)} & t_{1114(1)} & t_{1115(1)} & t_{1116(1)} & t_{1117(1)} & \ldots \\
t_6 & t_{11(4)} & t_{1111(3)} & t_{1112(3)}. & t_{1113(3)} & t_{1114(3)} & t_{1115(3)} & t_{1116(3)} & t_{1112(3)} & \\
t_7 & t_{11(5)} & t_{1111(2)} & t_{1112(2)}. & t_{1113(2)} & t_{1114(2)} & t_{1115(2)} & t_{1116(2)} & t_{1112(2)} & \ldots \\
t_8 & t_{11(6)} & t_{1111(4)} & t_{1112(4)} & t_{1113(4)} & t_{1114(4)} & t_{1115(4)} & t_{1116(4)} & t_{1112(4)} & \ldots \\
t_9 & t_{11(7)} & t_{1111(5)} & t_{1112(5)}. & t_{1113(5)} & t_{1114(5)} & t_{1115(5)} & t_{1116(5)} & t_{1112(5)} & \ldots \\
t_{10} & t_{11(8)} & t_{1111(6)} & t_{1112(6)} & t_{1113(6)} & t_{1114(6)} & t_{1115(6)} & t_{1116(6)} & t_{1112(6)} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

$$\begin{array}{llllllllll}
s_1 \rightarrow t_0 & & & & & & & & & \\
\downarrow & & & & & & & & & \\
s_{1(1)} \rightarrow s_{11(1)}, & s_{11(2)}, & s_{11(3)}\ , & s_{11(4)}, & s_{11(5)}, & s_{11(6)}, & s_{11(7)}, & s_{11(8)}, & s_{11(9)}, & s_{11(10)} \ \ldots \\
s_{1(2)} & \downarrow & & & & & & & & \\
s_{1(3)} & s_{111(1)} \rightarrow s_{1111(1)}, & s_{1111(2)}, & s_{1111(3)}\ , & s_{1111(4)}, & s_{1111(5)}, & s_{1111(6)}, & s_{1111(7)} \ \ldots & & \\
s_{1(4)} & s_{111(2)} \rightarrow s_{1112(1)}, & s_{1112(2)}, & s_{1112(3)}\ , & s_{1111(4)}, & s_{1111(5)}, & s_{1111(6)}, & s_{1111(7)} \ \ldots & & \\
s_{1(5)} & s_{111(3)} \rightarrow s_{1113(1)}, & s_{1113(2)}, & s_{1113(3)}\ , & s_{1113(4)}, & s_{1113(5)}, & s_{1113(6)}, & s_{1113(7)} \ \ldots & & \\
s_{1(6)} & s_{111(4)} \rightarrow s_{1114(1)}, & s_{1114(2)}, & s_{1114(3)}\ , & s_{1114(4)}, & s_{1114(5)}, & s_{1114(6)}, & s_{1114(7)} \ \ldots & & \\
s_{1(7)} & s_{111(5)} \rightarrow s_{1115(1)}, & s_{1115(2)}, & s_{1115(3)}\ , & s_{1115(4)}, & s_{1115(5)}, & s_{1115(6)}, & s_{1115(7)} \ \ldots & & \\
s_{1(8)} & s_{111(6)} \rightarrow s_{1116(1)}, & s_{1116(2)}, & s_{1116(3)}\ , & s_{1116(4)}, & s_{1116(5)}, & s_{1116(6)}, & s_{1116(7)} \ \ldots & & \\
s_{1(9)} & s_{111(7)} \rightarrow s_{1117(1)}, & s_{1117(2)}, & s_{1117(3)}\ , & s_{1117(4)}, & s_{1117(5)}, & s_{1117(6)}, & s_{1117(7)} \ \ldots & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

 doi:10.20944/preprints202210.0432.v1

Lemma 3.2. The elements of set S do not loop with any element of set S or T.

Proof. We assume that such a loop exist.

$t_0 \to s_1 \to s_2 \to s_3 \to s_4 \to s_5 \to s_6 \to s_7 \to s_8 \to s_9$

$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

CIO$\downarrow$ $\ldots$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $s_{10} \downarrow$ CO

$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$s_n \leftarrow \ldots\ldots\ldots\ldots \leftarrow \ldots\ldots \leftarrow \ldots\ldots\ldots\ldots \leftarrow s_{12} \leftarrow s_{11}$

For such a loop to exist, must have the same value, any number in the loop is applied the Collatz inverse operation and the Collatz operation. This is not possible.

Example 3.3. Lets take t_0 in the loop and $t_0 \not\equiv 0 \pmod 3$, then for such a loop to occur, must be $t_0 \to^{CIO} = t_0 \to^{CO} = s_1$.

If $t_0 \equiv 2 \pmod 3$ $\qquad \frac{2t_0-1}{3} = \frac{3t_0+1}{2^n} \qquad 2^{n+1}.t_0 - 2^n = 9t_0 + 3$

$t_0 = \frac{2^n+3}{2^{n+1}-9}$

or if $t_0 \equiv 1 \pmod 3$ $\qquad \frac{4t_0-1}{3} = \frac{3t_0+1}{2^n} \qquad 2^{n+2}.t_0 - 2^n = 9t_0 + 3$

$t_0 = \frac{2^n+3}{2^{n+2}-9}$

There is no such a positive integer t_0. Therefore, a loop cannot exist.

As a result, since we assume that there is a number t_0 that is not a Collatz number, we obtained two sets (T and S) with infinitely different elements from this number. The elements of T and S sets are not Collatz number. From the new infinite numbers that have been formed, infinite new numbers are formed, and it goes on like this forever and ever without stopping. And they don't form a loop. If there was such a number t_0, there would be no Collatz number. This result contradicts the result we found above (Corollary2.21). Therefore, there cannot be any number that is not the Collatz number.

4 Conclusion

We proved the Collatz conjecture with the Collatz inverse operation method. It is shown that all positive integers reach 1 as indicated in the Collatz conjecture. In this study, with the methods described for $3n+1$, numbers such as $5n+1$, $7n+1$, $9n+1 \ldots$ etc. It can be found whether it reaches 1.

References

[1] O' Connor, J.J.; Robertson, E.F. (2006). "Lothar Collatz". St Andrews University School of Mathematics and Statistics, Scotland.

[2] Maddux, Cleborne D.; Johnson, D. Lamont (1997). Logo: A Retrospective. New York: Haworth Press. p. 160. ISBN 0-7890-0374-0. The problem is also known by several other names, including: Ulam's conjecture, the Hailstone problem, the Syracuse problem, Kakutani's problem, Hasse's algorithm, and the Collatz problem.

[3] Lagarias, Jeffrey C. (1985)."The 3x + 1 problem and its generalizations". The American Mathematical Monthly. 92(1):3–23.doi:10.1080/00029890.1985.11971528.JSTOR 2322189.

[4] Pickover, Clifford A. (2001). Wonders of Numbers. Oxford: Oxford University Press. pp. 116–118. ISBN 0-19-513342-0.

[5] "Hailstone Number". MathWorld. Wolfram Research.

Author : Bülent SUKUŞU

Address: Yeniçağ Mahallesi Ragıp Tüzün Cad. No: 144 Yenimahalle/ANKARA-TURKEY

E-mail : bsukusu.blsu@gmail.com