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Article

Kaleidoscope states as a statistical mixture in the Jaynes-Cummings model: Entropy and purification

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Abstract: We investigate some of the fundamental features of the interaction of mixed kaleidoscope states, namely, particular statistical mixtures of coherent states, with two-level atoms in the Jaynes-Cummings model framework. We begin our analysis by calculating the von Neumann entropy of the field which is determined with help of the virtual atom method. Oscillations appear in this entropy that indicate a state of purity greater than the initial state, it i.e., a purification of the initial state is achieved. In this oscillatory region, we obtain the field Wigner function that resembles Schrodinger's cats.

Keywords: Kaleidoscope-States; Mixed-States; Virtual-Atom.

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1. Introduction

Schrödinger cat [1–3] states, or superpositions of coherent states, have attracted the attention of researchers due to their fundamental features. Of particular interest is the case of the superposition of two (or more) coherent states [4,5]. Where, because of quantum interference, their properties are very different from the properties of constituent coherent states, as well as from incoherent superposition or statistical mixtures of such states. For example, the superposition exhibits statistics for subpoissonian photon compression, higher-order compression, and oscillations in the photon number distribution [5], and these properties clearly differentiate the state of superposition and statistical mixing from two coherent states [4]. Because superpositions of macroscopically distinguishable states (or Schrödinger cat-like states) may be produced by using coherent states, the problem is important for the quantum theory of measurement. Several schemes have already been proposed to produce a superposition of coherent states, for instance the non-linear interaction of the field in a coherent state with a Kerr-like medium can produce their superposition [1]. Another possible way would be through the interaction between quantized fields, initially prepared in coherent states, with two-level atoms [4,5] or ion laser interactions.

One of the main tasks in the present manuscript is to calculate the entropy of the field for kaleidoscope states or a statistical mixture of coherent states, in the Jaynes-Cummings framework, which we will do with the aid of the Araki-Lieb inequality [6]. Because we will consider mixtures as initial states [7], in principle, it will not be possible to use the Araki-Lieb inequality to calculate the entropies, especially the field entropy. However via purification of the mixed density matrix of the quantized field [8], we will be able to use such inequality in order to calculate the field von Neumann entropy even in the case of initial statistical mixtures, either for the atom or the field.

In the next section, we define the Kaleidoscope states or initial mixed states for the field, used as initial states in their interaction with a two-level atom. Section 3 deals precisely with this interaction and there we calculate the field entropies and their Wigner functions [9], and finally, in section 4 we summarize our conclusions.

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Citation: Lastname, F.; Lastname, F.; Lastname, F. Kaleidoscope states. *Preprints* **2022**, *1*, 0. https://doi.org/

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2. Kaleidoscope states

Kaleidoscope states are a particular superposition of n coherent states, and are defined in reference [10], as

$$\begin{pmatrix} \sqrt{\lambda_{1\alpha}}|\psi_{1\alpha}\rangle \\ \sqrt{\lambda_{2\alpha}}|\psi_{2\alpha}\rangle \\ \sqrt{\lambda_{3\alpha}}|\psi_{3\alpha}\rangle \\ \vdots \\ \sqrt{\lambda_{n\alpha}}|\psi_{n\alpha}\rangle \end{pmatrix} = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{*} & \omega^{*\,2} & \cdots & \omega^{*\,(n-1)} \\ 1 & \omega^{*\,2} & \omega^{*\,4} & \cdots & \omega^{*\,2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{*\,(n-1)} & \omega^{*\,2(n-1)} & \cdots & \omega^{*\,(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} |\alpha\rangle \\ |\omega\alpha\rangle \\ |\omega^2\alpha\rangle \\ \vdots \\ |\omega^{(n-1)}\alpha\rangle \end{pmatrix}, \tag{1}$$

where $\sqrt{\lambda_{k\alpha}}$ are normalization constants, $\omega = \exp\left(i\frac{2\pi}{n}\right)$ and ω^k is the nth root of unity, for $1 \leq k \leq n$, with n, k integers.

We can observe that the Vandermonde matrix that transforms the vectors in equation (1) is proportional to the so-called discrete Fourier transform, also known as the quantum Fourier transform [11]. On the other hand, an initial statistical mixture of n coherent states density of the electromagnetic field may be written as

$$\hat{\rho}_F(0) = \frac{1}{n}\sum_{k=1}^{n} |\omega^{(k-1)}\alpha\rangle\langle\omega^{(k-1)}\alpha| \,. \tag{2}$$

This density matrix can be diagonalized by the virtual atom method [8], where the important issue is to establish a connection between the virtual (V) pure state,

$$|\psi_V\rangle = \frac{|\alpha\rangle}{\sqrt{n}}|a_1\rangle + \frac{|\omega\alpha\rangle}{\sqrt{n}}|a_2\rangle + \frac{|\omega^2\alpha\rangle}{\sqrt{n}}|a_3\rangle + \cdots + \frac{|\omega^{(n-1)}\alpha\rangle}{\sqrt{n}}|a_n\rangle \,, \tag{3}$$

with the state $|\omega^{(k-1)}\alpha\rangle$ used to obtain the density (2). Here

$$|a_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, |a_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, |a_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, |a_n\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{4}$$

In order to do that, from equation (1) we obtain that

$$\begin{pmatrix} |\alpha\rangle/\sqrt{n} \\ |\omega\alpha\rangle/\sqrt{n} \\ |\omega^2\alpha\rangle/\sqrt{n} \\ \vdots \\ |\omega^{(n-1)}\alpha\rangle/\sqrt{n} \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{(n-1)} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_{1\alpha}}|\psi_{1\alpha}\rangle \\ \sqrt{\lambda_{2\alpha}}|\psi_{2\alpha}\rangle \\ \sqrt{\lambda_{3\alpha}}|\psi_{3\alpha}\rangle \\ \vdots \\ \sqrt{\lambda_{n\alpha}}|\psi_{n\alpha}\rangle \end{pmatrix}, \tag{5}$$

and substituting equation (5) into equation (3) and so we have

$$|\psi_V\rangle = \sqrt{\lambda_{1\alpha}}|\psi_{1\alpha}\rangle|A_1\rangle + \sqrt{\lambda_{2\alpha}}|\psi_{2\alpha}|A_2\rangle + \sqrt{\lambda_{3\alpha}}|\psi_{3\alpha}\rangle|A_3\rangle + \cdots + \sqrt{\lambda_{n\alpha}}|\psi_{n\alpha}\rangle|A_n\rangle, \tag{6}$$

where

$$\begin{pmatrix} |A_1\rangle \\ |A_2\rangle \\ |A_3\rangle \\ \vdots \\ |A_n\rangle \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{(n-1)} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} |a_1\rangle \\ |a_2\rangle \\ |a_3\rangle \\ \vdots \\ |a_n\rangle \end{pmatrix}, \tag{7}$$

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which are the virtual atom basis $\{|A_k\rangle\}$ for $k = 1, .., n$. After tracing the density operator $|\psi_V\rangle\langle\psi_V|$, over the virtual atom states $\{|A_k\rangle\}$, we obtain the diagonal density

$$\hat{\rho}_F(0) = \sum_{k=1}^{n} \lambda_{k\alpha} |\psi_{k\alpha}\rangle\langle\psi_{k\alpha}| \,, \tag{8}$$

where

$$\lambda_{k\alpha} = \frac{e^{-|\alpha|^2}}{n} \sum_{m=1}^{n} \omega^{*\,m(k-1)} \exp\left(\omega^m |\alpha|^2\right). \tag{9}$$

Similarly, by tracing the density $|\psi_V\rangle\langle\psi_V|$, over the field states we obtain the virtual atom (VA) density

$$\hat{\rho}_{VA} = \frac{1}{n} \begin{pmatrix} 1 & \langle\alpha|\omega\alpha\rangle^* & \langle\alpha|\omega^2\alpha\rangle^* & \cdots & \langle\alpha|\omega^{(n-1)}\alpha\rangle^* \\ \langle\alpha|\omega\alpha\rangle & 1 & \langle\alpha|\omega\alpha\rangle^* & \cdots & \langle\alpha|\omega^{(n-2)}\alpha\rangle^* \\ \langle\alpha|\omega^2\alpha\rangle & \langle\alpha|\omega\alpha\rangle & 1 & \cdots & \langle\alpha|\omega^{(n-3)}\alpha\rangle^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle\alpha|\omega^{(n-1)}\alpha\rangle & \langle\alpha|\omega^{(n-2)}\alpha\rangle & \langle\alpha|\omega^{(n-3)}\alpha\rangle & \cdots & 1 \end{pmatrix}, \tag{10}$$

and taking into account that $\langle\alpha|\omega^k\alpha\rangle = \langle\alpha|\omega^{(n-k)}\alpha\rangle^*$ and $\langle\alpha|\omega^k\alpha\rangle = \exp(-|\alpha|^2)\exp(\omega^k|\alpha|^2)$, equation (10) may be rewritten as

$$\hat{\rho}_{VA} = \frac{\exp(-|\alpha|^2)}{n} \begin{pmatrix} \exp(|\alpha|^2) & \exp(\omega^*|\alpha|^2) & \exp(\omega^{*\,2}|\alpha|^2) & \cdots & \exp(\omega^{*\,(n-1)}|\alpha|^2) \\ \exp(\omega^{*\,(n-1)}|\alpha|^2) & \exp(|\alpha|^2) & \exp(\omega^*|\alpha|^2) & \cdots & \exp(\omega^{*\,(n-2)}|\alpha|^2) \\ \exp(\omega^{*\,(n-2)}|\alpha|^2) & \exp(\omega^{*\,(n-1)}|\alpha|^2) & \exp(|\alpha|^2) & \cdots & \exp(\omega^{*\,(n-3)}|\alpha|^2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \exp(\omega^*|\alpha|^2) & \exp(\omega^{*\,2}|\alpha|^2) & \exp(\omega^{*\,3}|\alpha|^2) & \cdots & \exp(|\alpha|^2) \end{pmatrix}. \tag{11}$$

Finally from (7) we write

$$|A_k\rangle = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^{(k-1)} \\ \omega^{2(k-1)} \\ \omega^{3(k-1)} \\ \vdots \\ \omega^{(n-1)(k-1)} \end{pmatrix}, \tag{12}$$

by noting that $|A_k\rangle$ is an eigenvector of $\hat{\rho}_{VA}$, whose eigenvalue is represented by (9).

3. Interaction of kaleidoscope states with a two-level atom

The interaction between a quantized field and a two-level atom (under rotating wave approximation) is given by the Jaynes-Cumming interaction Hamiltonian [12] (for simplicity we have set $\hbar = 1$)

$$\hat{H}_I = \lambda\left(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+\right), \tag{13}$$

where we have considered on-resonance conditions (equal field and atomic transition frequencies). In the above equation, λ is the coupling constant, and $\hat{a}$ and $\hat{a}^\dagger$ and the annihilation and creation operators, respectively, and σ_+ and σ_- are the raising and lowering Pauli matrices, respectively. The evolution operator, $\hat{U} = \exp(-i\hat{H}_I t)$, in the 2×2 basis is given by

$$\hat{U} = \begin{pmatrix} \cos\left(\lambda t\sqrt{\hat{a}\hat{a}^\dagger}\right) & -i\,\hat{V}\sin\left(\lambda t\sqrt{\hat{a}^\dagger\hat{a}}\right) \\ -i\,\hat{V}^\dagger\sin\left(\lambda t\sqrt{\hat{a}\hat{a}^\dagger}\right) & \cos\left(\lambda t\sqrt{\hat{a}^\dagger\hat{a}}\right) \end{pmatrix}, \tag{14}$$

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with $\hat{V}$ and $\hat{V}^\dagger$ the London phase operator [13].

If we consider the atom initially prepared in the state $|e\rangle$ and the field is a statistical mixture of n coherent states, i.e, the system is initially prepared in, $\hat{\rho}(0) = \hat{\rho}_F(0)|e\rangle\langle e|$, with $\hat{\rho}_F(0)$ defined by (2). Then the time evolution density matrix is given by

$$\hat{\rho} = \begin{pmatrix} \sum_{k=1}^{n} |C_k\rangle\langle C_k| & |C_n\rangle\langle S_1| + \sum_{k=2}^{n} |C_{k-1}\rangle\langle S_k| \\ |S_1\rangle\langle C_n| + \sum_{k=2}^{n} |S_k\rangle\langle C_{k-1}| & \sum_{k=1}^{n} |S_k\rangle\langle S_k| \end{pmatrix} \tag{15}$$

where

$$|C_k\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{+\infty} \frac{\alpha^{nm+k}}{\sqrt{(nm+k)!}} \cos\left(\lambda t \sqrt{nm+k+1}\right) |nm+k\rangle\,, \tag{16}$$

$$|C_n\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{+\infty} \frac{\alpha^{nm}}{\sqrt{(nm)!}} \cos\left(\lambda t \sqrt{nm+1}\right) |nm\rangle\,, \tag{17}$$

$$|S_k\rangle = -i\, e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{+\infty} \frac{\alpha^{nm+k-1}}{\sqrt{(nm+k-1)!}} \sin\left(\lambda t \sqrt{nm+k}\right) |nm+k\rangle\,, \tag{18}$$

whereas the equation (16) is valid whenever $k = 1, 2, \cdots n-1$; and for equation (18), k satisfying the condition $k = 1, 2, \cdots n$.

Therefore, tracing over the field states and over the atomic states, we obtain the reduced density matrix for the atom and the field respectively as,

$$\hat{\rho}_A = \begin{pmatrix} \sum_{k=1}^{n} \langle C_k|C_k\rangle & 0 \\ 0 & \sum_{k=1}^{n} \langle S_k|S_k\rangle \end{pmatrix}, \tag{19}$$

$$\hat{\rho}_F = \sum_{k=1}^{n} |C_k\rangle\langle C_k| + \sum_{k=1}^{n} |S_k\rangle\langle S_k|\,. \tag{20}$$

As two coherent states are sufficiently apart when $\alpha \approx 2$, they may be considered orthogonal. On the other hand, the Kaleidoscope-States are orthogonal for any $\alpha > 0$, and $\hat{\rho}_A$ will be diagonalize as was shown in equation (19).

Within the present formalism, it is straightforward to calculate the atomic von Neumann entropy as,

$$S_A = -\left(\sum_{k=1}^{n} \langle C_k|C_k\rangle\right) \ln\left(\sum_{k=1}^{n} \langle C_k|C_k\rangle\right) - \left(\sum_{k=1}^{n} \langle S_k|S_k\rangle\right) \ln\left(\sum_{k=1}^{n} \langle S_k|S_k\rangle\right), \tag{21}$$

and by using the method of the virtual atom as has been proposed in Ref. [8] the field entropy can be written as,

$$S_F = -\sum_{k=1}^{n} \lambda_k^+ \ln \lambda_k^+ - \sum_{k=1}^{n} \lambda_k^- \ln \lambda_k^- \tag{22}$$

where the eigenvalues are

$$\lambda_k^\pm = \frac{1}{2}(\langle C_k|C_k\rangle + \langle S_k|S_k\rangle) \pm \frac{1}{2}\sqrt{(\langle C_k|C_k\rangle - \langle S_k|S_k\rangle)^2 + 4|\langle C_k|S_k\rangle|^2}\,, \tag{23}$$

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where k satisfying the condition $k = 1, 2, \cdots n$. And also

$$
\begin{aligned}
\langle C_n | C_n \rangle &= e^{-|\alpha|^2} \sum_{m=0}^{+\infty} \frac{|\alpha|^{2nm}}{(nm)!} \cos^2\left(\lambda t \sqrt{nm+1}\right), \\
\langle C_n | S_n \rangle &= -i\alpha^* e^{-|\alpha|^2} \sum_{m=0}^{+\infty} \frac{|\alpha|^{2n(m+1)-2}}{(n(m+1)-1)!\sqrt{n(m+1)}} \cos\left(\lambda t \sqrt{n(m+1)+1}\right) \sin\left(\lambda t \sqrt{n(m+1)}\right), \\
\langle S_n | S_n \rangle &= e^{-|\alpha|^2} \sum_{m=0}^{+\infty} \frac{|\alpha|^{2n(m+1)-2}}{(n(m+1)-1)!} \sin^2\left(\lambda t \sqrt{n(m+1)}\right), \\
\langle C_k | C_k \rangle &= e^{-|\alpha|^2} \sum_{m=0}^{+\infty} \frac{|\alpha|^{2nm+2k}}{(nm+k)!} \cos^2\left(\lambda t \sqrt{nm+k+1}\right), \\
\langle C_k | S_k \rangle &= -i\alpha^* e^{-|\alpha|^2} \sum_{m=0}^{+\infty} \frac{|\alpha|^{2nm+2k-2}}{(nm+k-1)!\sqrt{nm+k}} \cos\left(\lambda t \sqrt{nm+k+1}\right) \sin\left(\lambda t \sqrt{nm+k}\right), \\
\langle S_k | S_k \rangle &= e^{-|\alpha|^2} \sum_{m=0}^{+\infty} \frac{|\alpha|^{2nm+2k-2}}{(nm+k-1)!} \sin^2\left(\lambda t \sqrt{nm+k}\right),
\end{aligned}
\tag{24}
$$

which are valid whenever $k = 1, 2, \cdots n-1$.

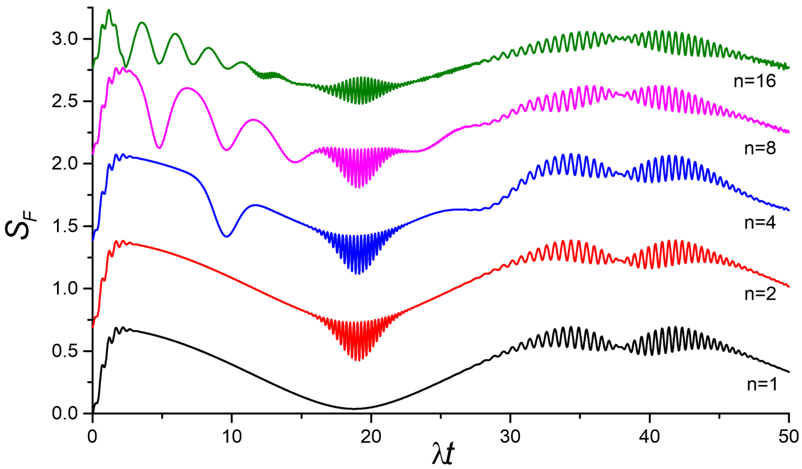

Figure 1. The evolution of the field entropy of Kaleidoscope-States as a function of the scaled time λt and different values of statistical mixture of coherent states $n = 1, 2, 4, 8$ and 16, with $\alpha = 6.0$.

The field entropy is plotted as a function of the scaled time λt in figure 1, for Kaleidoscope-States with different values of statistical mixture of coherent states $n = 1, 2, 4, 8$ and 16, and $\alpha = 6.0$. It may be seen that the entropy has similar behaviour, namely, each one possesses a global minimum of about $\lambda t \approx 19$, for all values of n before commenting. In this region and in each case, the field becomes purer than its initial state and oscillations appear for $n \geq 2$. Note that for all values of n the entropy reaches its maximum value quickly due to the interaction with the atom, and the coherence between the n coherent states is lost, and a maximal mixed of a statistical mixture state will be the new state of the field.

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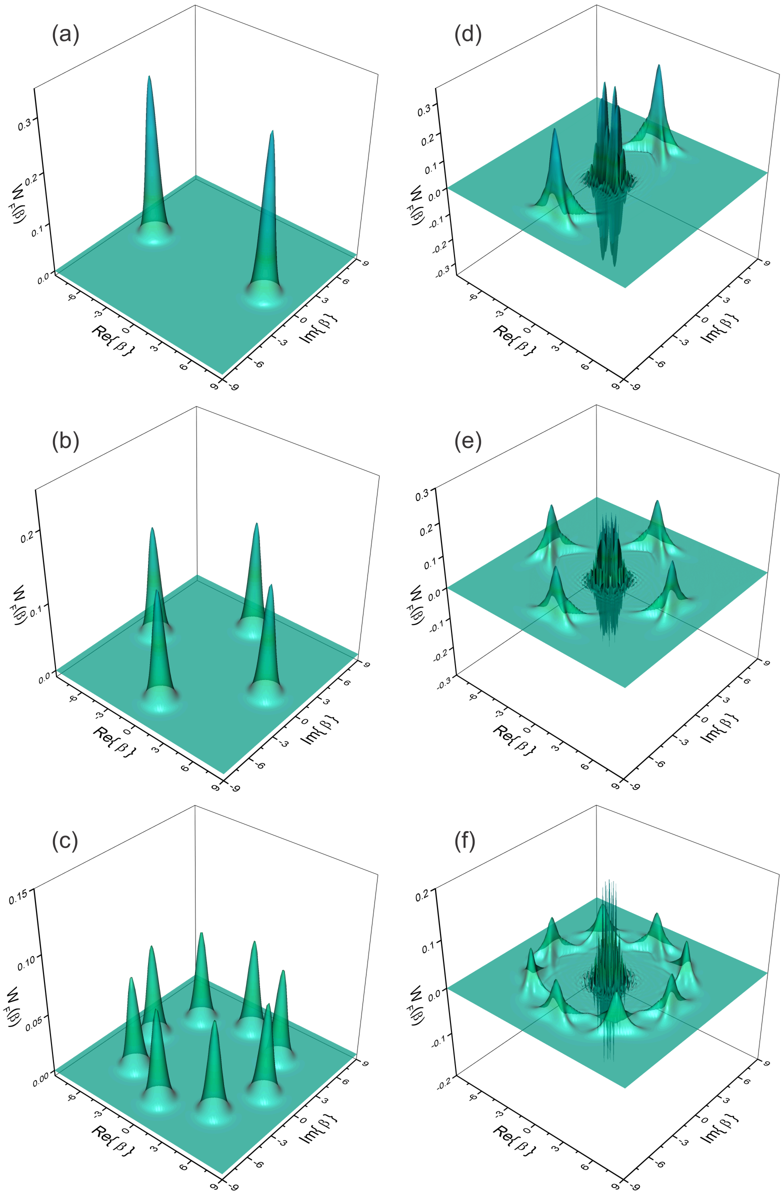

Figure 2. Wigner function for Kaleidoscope-State with $\alpha = 6.0$ for several values of the time λt and n. For $\lambda t = 0$: (a) $n = 2$, (b) $n = 4$ and (c) $n = 8$; for $\lambda t = 19.15$: (d) $n = 2$, (e) $n = 4$ and (f) $n = 8$.

Here a question arises: What will be the form of the above-mentioned purer states for different values of a statistical mixture of coherent states n about $\lambda t \approx 19$? In order to ask the above question in Fig. 2 (d), (e), and (f) we show the field Wigner function corresponding to the Kaleidoscope-State with $n = 2, 4$ and 8 respectively when the field would become a purer state at time $\lambda t = 19.15$. These Wigner functions resemble an Schrödinger cat state of $2, 4$ and 8 components, where we note the characteristic interference structure. We clearly see the formation of the quantum interference structure halfway between the n humps. The frequency of the interference structure increases with the separation distance α increases [14]. For example, setting $\alpha = 4$ and $n = 2$, we see that the entropy has a similar behaviour as in figure 1, but now its minimum is around $\lambda t \approx 12.5$, and its corresponding field Wigner function has interference structure with a lower frequency as it is shown in figure 3. Finally, when the time goes to $\lambda t \approx 19$, the initial Kaleidoscope States, (as we showed in figure 2 (a), (b), and (c)), gain purity as was suggested by the entropy behaviour, and the negativity of the field Wigner function are an indicator of the non-classical properties of the state at

$\lambda t \approx 19.15$.

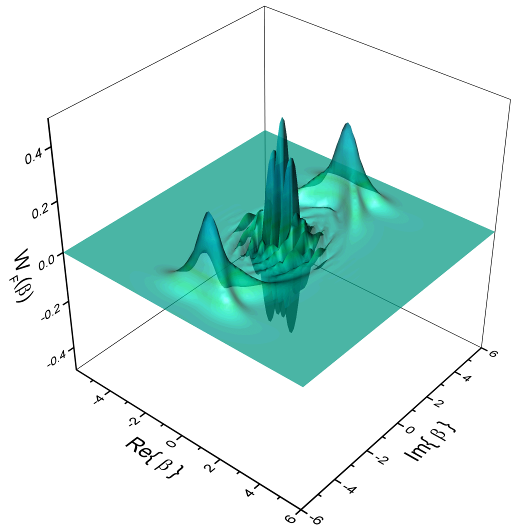

Figure 3. Wigner function for Kaleidoscope-State with $\alpha = 4.0$ for the time $\lambda t = 12.59$ and $n = 2$.

4. Conclusions

We have shown that the Jaynes-Cummings interaction with an initial Kaleidoscope mixture may be modeled by the virtual Hamiltonian method by extending the atomic Hilbert space such that a virtual pure state mya be associated as initial wavefunction. In particular, we have seen that the purification procedure takes us from a mixed field density matrix to a pure wave function that involves a virtual $2n$-level atom, as we can see in the $2n$ term in equation (22). Finally, we should mention that the effects presented in the field entropy for the initial field state given by a statistical mixture of constituent states, are reflected in the appearence of which produces Wigner functions resembling Schrödinger's cats.

Conflicts of Interest: The authors declare no conflict of interest.

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