A Relative Difference System Involving Newton's Equations of Celestial Mechanics

Harry Gingold and Jocelyn Quaintance
West Virginia University, Morgantown, WV, USA; hgingold@gmail.com
University of Pennsylvania, Philadelphia, PA, USA; jocelynq@seas.upenn.edu

Abstract

We point out an inconsistency in Newton's equations of celestial mechanics. A set of differential equations implied by Newton's equations are shown to be free of this inconsistency. We then investigate the integrals of motion associated with this relative difference system.

Keywords: Newton's celestial mechanics equations; N-body problem; Kepler problem; relative difference; origin invariance; integrals of motion; conservation of energy; conservation of angular momentum; inertia; Lagrange-Jacobi formula; total collapse.

1 Introduction

Over the course of the millennia of years, numerous astronomical observations and measurements of celestial bodies were made by earthbound humans. The twentieth century ushered in an era of human made objects orbiting earth or traveling in space. Observations and measurements from earth or from other moving bodies viewed theoretically as point masses require the utilization of coordinate systems in which the origin O coincides with the center of mass of one of these point masses. Note that this choice of origin does not necessarily coincide with the center of mass of the entire N-body system. It goes without saying that these point masses accelerate and decelerate during their travel. By fixing the origin O on the point mass, Newton's equations become an inconsistent system.

Newton's equations of the N-body problem are important theoretical and computational tools used throughout the study of celestial mechanics. As a case in point, Newton's equations

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

are the leading term in certain general relativity celestial mechanics equations proposed by Einstein, Infeld, and Hoffman [2]. In special relativity, Newton's equations may be viewed as the leading term in celestial mechanics equations that undergo expansion in terms of the parameter c^{-1}, where c is the speed of light. Compare e.g. with Feynman [[4], Chapter 3].

Surprisingly, an analysis of the Kepler problem for 2-body problem reveals a paradox in the simplest setting of the N-body problem.

2 A Paradox For the Two-body Problem

Suppose that we have a fixed origin O in space from which we take measurements of the position and the velocity of two point masses m_1 and m_2. If the position vector of m_1 is given by the 3×1 vector $r_1(t)$, (which for ease of exposition we denote as r_1), and the position vector of m_2 is given by the 3×1 vector $r_2(t)$ or r_2. Newton's equations for acceleration imply that

$$r_1''(t) = \frac{Gm_2(r_2 - r_1)}{\|r_2 - r_1\|^3}, \qquad r_2''(t) = \frac{Gm_1(r_1 - r_2)}{\|r_1 - r_2\|^3}, \tag{2.1}$$

where G is the gravitational constant and $\|r(t)\| = \sqrt{r^T r}$. Note that r^T is a 1×3 vector. Now, as in the case of a central force problem, we set in the equations of (2.1)

$$r_1(t) \equiv \overrightarrow{0} \Longrightarrow r_1'(t) = r_1''(t) \equiv \overrightarrow{0}, \tag{2.2}$$

where $\overrightarrow{0} = [0, 0, 0]^T$. The assumption of (2.2) is equivalent to placing m_1 on O, or in other words taking all measurements from m_1. See Figure 1. The second equation of (2.1) then

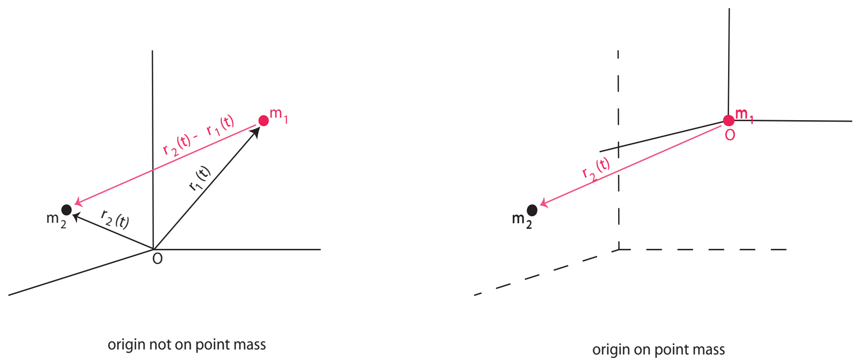

Figure 1: Change of Origin

becomes

$$r_2''(t) = -\frac{Gm_1 r_2}{\|r_2\|^3}. \tag{2.3}$$

Equation (2.3) is the Kepler problem and is known to have conic section $r_2(\theta) = \frac{ed}{1+e\cos\theta}$ as a solution. But what happens to the first equation of (2.1)? It becomes

$$\overrightarrow{0} = \frac{Gm_2 r_2}{\|r_2\|^3}. \tag{2.4}$$

The only possible solution to (2.4) is $\frac{r_2}{\|r_2\|^3} = \overrightarrow{0}$, which in turn implies $\|r_2\| = \infty$. To obtain a meaning for the solution $\frac{r_2}{\|r_2\|^3} = \overrightarrow{0}$, we proceed with the following heuristic description which is justified rigorously via compactification; see Y. I. Gingold and H. Gingold and/or H. Gingold, and D. Solomon [5, 6]. Supplement $\mathbb{R}^3$ by an ideal set

$$ID := \{\infty U\}, \qquad \text{where } U \in \mathbb{R}^3 \text{ and } \|U\| = 1, \tag{2.5}$$

and create

$$UE\mathbb{R}^3 := \{\mathbb{R}^3\} \cup ID. \tag{2.6}$$

Then ∞U may be considered a constant solution "at infinity" (critical point or equilibrium point) of (2.4), where $\|\infty U\| = \infty$. This way we can add elements like ∞U as legitimate constant solutions to the equations of celestial mechanics. This is consistent with setting $r_2 = \infty U$ in (2.4) since

$$\frac{Gm_2\|\infty U\|}{\|\infty U\|^3} = \frac{Gm_2}{\|\infty U\|^2} = \frac{Gm_2}{\infty} = 0.$$

Thus, if we supplement $\mathbb{R}^3$ with a collection of ideal " points at infinity", which we denote as ∞U (U any arbitrary unit vector in $\mathbb{R}^3$), and if we assume (2.3) and (2.4) form an initial value problem with finite initial conditions $r_1(t_0) \neq r_2(t_0)$, $r_t'(t_0)$, $r_2'(t_0)$, the simultaneous solution of (2.3) and (2.4) leads to the conclusion that $r_2(t)$ is both a conic section and a constant vector ∞U. Hence, for the aforementioned initial value problem, the system of (2.3) and (2.4) is an overdetermined system with no consistent solution.

Geometrically, there is a one-to-one mapping between $UE\mathbb{R}^3$ and a closed subset that is a "bowl" on the unit sphere S^4 [5, 7]. Alternatively, there is a one-to-one mapping between $UE\mathbb{R}^3$ and a "parabolic bowl" in $\mathbb{R}^4$ [3, 6]. For either one of these two compactified geometric realizations of $UE\mathbb{R}^3$, given $P_1, P_2 \in UE\mathbb{R}^3$, we determine $d(P_1, P_2)$ via the chordal distance. By construction, since S^4 and the "parabolic bowl" are compact in $\mathbb{R}^4$, $d(P_1, P_2)$ is always finite. And indeed the chordal metric in both instances makes $UE\mathbb{R}^3$ into a complete metric space [3, 5, 6, 7].

So how can one avoid this quandary? One possible way is to define the origin independent relative system

$$(r_1 - r_2)'' = -\frac{G(m_1 + m_2)(r_1 - r_2)}{\|r_1 - r_2\|^3}. \tag{2.7}$$

Then when $r_1 \equiv \overrightarrow{0}$, i.e. when m_1 corresponds with origin O, Equation (2.7) becomes

$$-r_2''(t) = \frac{G(m_1 + m_2)r_2}{\|r_2\|^3}, \tag{2.8}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

a modified Kepler problem. Observe that if $m_1 m_2^{-1}$ is very small, as is the case with the Earth as compared to the sun, Equation (2.8) is a very good approximation to (2.3).

3 A Paradox For the Three-body Problem

Now assume from a fixed origin O we record the position of three point masses, m_1, m_2, and m_3 as the 3×1 vectors $r_1(t)$, $r_2(t)$, and $r_3(t)$ respectively. Then Newton's law of planetary motion leads to the following system of three nonlinear second order differential equations.

$$r_1''(t) = \frac{Gm_2(r_2 - r_1)}{\|r_2 - r_1\|^3} + \frac{Gm_3(r_3 - r_1)}{\|r_3 - r_1\|^3} \tag{3.1}$$

$$r_2''(t) = \frac{Gm_1(r_1 - r_2)}{\|r_1 - r_2\|^3} + \frac{Gm_3(r_3 - r_2)}{\|r_3 - r_2\|^3} \tag{3.2}$$

$$r_3''(t) = \frac{Gm_1(r_1 - r_3)}{\|r_1 - r_3\|^3} + \frac{Gm_2(r_2 - r_3)}{\|r_2 - r_3\|^3} \tag{3.3}$$

Once again, let us see what happens to the above system when m_1 corresponds to O, i.e. when $r_1 \equiv \overrightarrow{0}$. Since $r_1 \equiv \overrightarrow{0}$ implies that $r_1' = r_1'' \equiv \overrightarrow{0}$, Equations (3.1) through (3.3) respectively become

$$\overrightarrow{0} = \frac{Gm_2 r_2}{\|r_2\|^3} + \frac{Gm_3 r_3}{\|r_3\|^3} \tag{3.4}$$

$$r_2''(t) = -\frac{Gm_1 r_2}{\|r_2\|^3} + \frac{Gm_3(r_3 - r_2)}{\|r_3 - r_2\|^3} \tag{3.5}$$

$$r_3''(t) = -\frac{Gm_1 r_3}{\|r_3\|^3} + \frac{Gm_2(r_2 - r_3)}{\|r_2 - r_3\|^3} \tag{3.6}$$

Equation (3.4) implies that

$$\frac{m_2 r_2}{\|r_2\|^3} = -\frac{m_3 r_3}{\|r_3\|^3} \iff r_2 = -\frac{m_3\|r_2\|^3}{m_2\|r_3\|^3} r_3. \tag{3.7}$$

If we take the norm of (3.7) we find that

$$\|r_2\| = \frac{m_3\|r_2\|^3}{m_2\|r_3\|^3}\|r_3\| \iff \|r_2\|^{-2} = \frac{m_3}{m_2}\|r_3\|^{-2} \iff \|r_2\| = \sqrt{\frac{m_2}{m_3}}\|r_3\|. \tag{3.8}$$

By substituting (3.8) into (3.7) we obtain the relation

$$r_2 = -\sqrt{\frac{m_2}{m_3}} r_3. \tag{3.9}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

We now place (3.7) and (3.9) into (3.5) to obtain

$$
\begin{aligned}
-\sqrt{\frac{m_2}{m_3}}r_3'' &= \frac{Gm_1m_3}{m_2}\frac{r_3}{\|r_3\|^3} + \frac{Gm_3\left(r_3+\sqrt{\frac{m_2}{m_3}}r_3\right)}{\left\|\left(r_3+\sqrt{\frac{m_2}{m_3}}r_3\right)\right\|^3} \\
&= Gm_3\left[\frac{m_1}{m_2} + \frac{1}{\left(1+\sqrt{\frac{m_2}{m_3}}\right)^2}\right]\frac{r_3}{\|r_3\|^3} \\
&= \frac{Gm_3}{m_2}\left[\frac{m_1(\sqrt{m_3}+\sqrt{m_2})^2+m_2m_3}{(\sqrt{m_3}+\sqrt{m_2})^2}\right]\frac{r_3}{\|r_3\|^3}.
\end{aligned}
$$

The above calculations show that

$$
r_3'' = -G\left(\frac{m_3}{m_2}\right)^{\frac{3}{2}}\left[\frac{m_1(\sqrt{m_3}+\sqrt{m_2})^2+m_2m_3}{(\sqrt{m_3}+\sqrt{m_2})^2}\right]\frac{r_3}{\|r_3\|^3}. \tag{3.10}
$$

However, if we substitute (3.9) into (3.6) we obtain

$$
\begin{aligned}
r_3'' &= -\frac{Gm_1r_3}{\|r_3\|^3} - \frac{Gm_2\left(1+\sqrt{\frac{m_2}{m_3}}\right)}{\left(1+\sqrt{\frac{m_2}{m_3}}\right)^3}\frac{r_3}{\|r_3\|^3} \\
&= -G\left[m_1 + \frac{m_2}{\left(1+\sqrt{\frac{m_2}{m_3}}\right)^2}\right]\frac{r_3}{\|r_3\|^3} \\
&= -G\left[\frac{m_1(\sqrt{m_3}+\sqrt{m_2})^2+m_2m_3}{(\sqrt{m_3}+\sqrt{m_2})^2}\right]\frac{r_3}{\|r_3\|^3}.
\end{aligned} \tag{3.11}
$$

Since (3.10) must equal (3.11) we have the relation

$$
\left(\frac{m_3}{m_2}\right)^{\frac{3}{2}}\left[\frac{m_1(\sqrt{m_3}+\sqrt{m_2})^2+m_2m_3}{(\sqrt{m_3}+\sqrt{m_2})^2}\right]\frac{r_3}{\|r_3\|^3} = \left[\frac{m_1(\sqrt{m_3}+\sqrt{m_2})^2+m_2m_3}{(\sqrt{m_3}+\sqrt{m_2})^2}\right]\frac{r_3}{\|r_3\|^3}. \tag{3.12}
$$

In order for (3.12) to be valid, either one of two possibilities occurs. First $\frac{r_3}{\|r_3\|^3} \equiv \overrightarrow{0}$, which implies that $r_3 = \infty U$, a contradiction to the assumption of finite initial conditions, or

$$
\left(\frac{m_3}{m_2}\right)^{\frac{3}{2}} = 1 \Longleftrightarrow m_3 = m_2. \tag{3.13}
$$

ec Thus if the system in question does not satisfy (3.13), Newton's equations (3.1) through (3.3) lead to a paradox if m_1 is taken to be the origin O. However if the system in question does satisfy (3.13) we can substitute (3.13) into (3.9) and (3.8) to obtain

$$
r_2(t) = -r_3(t) \qquad \text{and} \qquad \|r_2(t)\| = \|r_3(t)\|. \tag{3.14}
$$

Then the system of three equations given by Equations (3.4) through (3.6) reduces to

$$r_1 \equiv \overrightarrow{0} \qquad \text{and} \qquad r_2'' = -r_3'' = G\left[m_1 + \frac{m_3}{4}\right]\frac{r_3}{\|r_3\|^3}, \tag{3.15}$$

and the second equation of (3.15) is a modified Kepler equation which has a conic section curve as a solution. Furthermore, if $r_1 \equiv \overrightarrow{0}$ and r_2 and r_3 satisfy (3.14) we see that the center of mass of the three bodies

$$\frac{m_1r_1 + m_2r_2 + m_3r_3}{m_1 + m_2 + m_3} = \overrightarrow{0} = r_1.$$

In summary, the above discussion shows that we can not freely choose the origin O to be centered on a point mass m_1, since unless the other point masses obey "antipodal" symmetry conditions, the choice of $r_1 \equiv \overrightarrow{0}$ in the system of Newton's equations given by (3.1) through (3.3), leads to the paradoxical conclusion of m_2 and m_3 escaping to "infinity". In order to do so in a manner that avoids contradiction, we propose to use an origin invariant model of relative differences, namely

$$(r_1 - r_2)'' = -\frac{G(m_1 + m_2)(r_1 - r_2)}{\|r_1 - r_2\|^3} - \frac{Gm_3(r_1 - r_3)}{\|r_1 - r_3\|^3} + \frac{Gm_3(r_2 - r_3)}{\|r_2 - r_3\|^3} \tag{3.16}$$

$$(r_1 - r_3)'' = -\frac{Gm_2(r_1 - r_2)}{\|r_1 - r_2\|^3} - \frac{G(m_1 + m_3)(r_1 - r_3)}{\|r_1 - r_3\|^3} - \frac{Gm_2(r_2 - r_3)}{\|r_2 - r_3\|^3} \tag{3.17}$$

If we put $r_1 \equiv \overrightarrow{0}$ into (3.16) and (3.17),we obtain the consistent system

$$r_2'' = -\frac{G(m_1 + m_2)r_2}{\|r_2\|^3} - \frac{Gm_3r_3}{\|r_3\|^3} - \frac{Gm_3(r_2 - r_3)}{\|r_2 - r_3\|^3} \tag{3.18}$$

$$-r_3'' = \frac{Gm_2r_2}{\|r_2\|^3} + \frac{G(m_1 + m_3)r_3}{\|r_1 - r_3\|^3} - \frac{Gm_2(r_2 - r_3)}{\|r_2 - r_3\|^3} \tag{3.19}$$

If m_2 and m_3 are small compared to m_1, Equations (3.18) and (3.19) are perturbations of (3.5) and (3.6). Also since the system of equations given by (3.18) and (3.19) does not have an analog of Equation (3.4), it is not overdetermined.

4 Well Posed Origin Anywhere Consistent System

As the examples of the previous two section demonstrate, it is desirable to have a system of differential equations for the N-body problem that has the following properties:

a. The system of differential equations is consistent with any coordinates system whose origin is any point in space.

b. Any singularities free initial value problem has a unique solution with a continuous second derivative on some interval containing the initial point.

We propose to call such systems of differential equations *well posed origin anywhere consistent*. For the theory of ordinary differential equations see e.g. [1, 9, 10, 12].

Any Newtonian N-body problem has at least two inconsistencies built in it. Recall that the original Newton system of N nonlinear second order differential equations is given by

$$r_j''(t) = G\sum_{\substack{i=1\\ i\neq j}}^{N} \frac{m_i(r_i - r_j)}{\|r_i - r_j\|^3}, \qquad 1 \leq j \leq N. \tag{4.1}$$

Fix the origin of coordinate system at the point mass m_j, where $j \neq N$. Substitute $r_j''(t) \equiv r_j'(t) \equiv r_j(t) = \overrightarrow{0}$ into each of the N equations associated with (4.1). Then for any legitimate initial conditions, the jth differential equation becomes

$$\overrightarrow{0} = G\sum_{\substack{i=1\\ i\neq j}}^{N} \frac{m_i r_i(t_0)}{\|r_i(t_0)\|^3}, \qquad r_i(t_0) \neq r_k(t_0) \neq \overrightarrow{0} \text{ whenever } i \neq k. \tag{4.2}$$

On the other hand the initial values chosen with $r_N(t_0) \neq \overrightarrow{0}$ can be made contradictory to (4.2) by the additional choice

$$G\sum_{\substack{i=1\\ i\neq j}}^{N-1} \frac{m_i r_i(t_0)}{\|r_i(t_0)\|^3} = \frac{-2m_N r_N(t_0)}{\|r_N(t_0)\|^3}, \tag{4.3}$$

since (4.3) when substituted in (4.2) implies that

$$\overrightarrow{0} = G\sum_{\substack{i=1\\ i\neq j}}^{N-1} \frac{m_i r_i(t_0)}{\|r_i(t_0)\|^3} + \frac{m_N r_N(t_0)}{\|r_N(t_0)\|^3} = -\frac{m_N r_N(t_0)}{\|r_N(t_0)\|^3} \neq \overrightarrow{0}.$$

Thus we get a contradiction to the desired well posedness.

Another contradiction in Newton's N-body problem is obtained as follows. Recall that the origin is centered on the point mass m_j, where $j \neq N$. Choose initial values in (4.1) as $r_i(t_0) = \theta_i U$, where $\theta_i > 0$ and U is a constant unit vector. Then (4.2) becomes

$$\overrightarrow{0} = G\sum_{\substack{i=1\\ i\neq j}}^{N} \frac{m_i r_i(t_0)}{\|r_i(t_0)\|^3} = G\left[\sum_{\substack{i=1\\ i\neq j}}^{N} \frac{m_i\theta_i}{\|r_i(t_0)\|^3}\right] U \Longleftrightarrow \sum_{\substack{i=1\\ i\neq j}}^{N} \frac{m_i\theta_i}{\|r_i(t_0)\|^3} = 0.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

Since by construction $\sum_{\substack{i=1 \\ i\neq j}}^{N} \frac{m_i\theta_i}{\|r_i(t_0)\|^3} > 0$, we obtain a contradiction.

We now generalize the origin invariant model of the relative differences to N point masses m_i, where $1 \le i \le N$. When doing so we obtain a system of $N-1$ nonlinear second order differential equations which is well posed origin consistent anywhere. Let O be any origin, and $r_i(t)$ be the position of m_i. The origin invariant model of relative differences consists of $N-1$ nonlinear second order differential equations of the form

$$\begin{aligned}(r_1 - r_k)'' &= G\sum_{i=2}^{N} \frac{m_i(r_i - r_1)}{\|r_i - r_1\|^3} - G\sum_{\substack{i=1 \\ i\neq k}}^{N} \frac{m_i(r_i - r_k)}{\|r_i - r_k\|^3} \\ &= G\sum_{i=2}^{N} \frac{m_i(r_i - r_1)}{\|r_i - r_1\|^3} - G\sum_{\substack{i=1 \\ i\neq k}}^{N} \frac{m_i[r_1 - r_k - (r_1 - r_i)]}{\|r_1 - r_k - (r_1 - r_i)\|^3}, \qquad 2 \le k \le N, \qquad (4.4)\end{aligned}$$

where the dependent variables of (4.4) are $(\Delta_{1k} := r_1(t) - r_k(t))_{k=2}^{N}$. Contrast (4.4) with the original Newton system of N nonlinear second order differential equations (4.1). Unlike Newton's system (4.1) which is only invariant under inertial translations $c(t)$ (recall that this means that $c''(t) = 0$), the relative difference system (4.4) is invariant under any arbitrary translation $c(t)$. This is because

$$\begin{aligned}r_1'' - r_k'' &= (r_1 + c(t))'' - (r_k + c(t))'' \\ &= \sum_{i=2}^{N} \frac{Gm_i(r_i + c(t) - (r_1 + c(t)))}{\|r_i + c(t) - (r_1 + c(t))\|^3} - \sum_{\substack{i=1 \\ i\neq k}}^{N} \frac{Gm_i(r_i + c(t) - (r_k + c(t)))}{\|r_i(t) + c(t) - (r_k + c(t))\|^3} \\ &= G\sum_{i=2}^{N} \frac{m_i(r_i - r_1)}{\|r_i - r_1\|^3} - G\sum_{\substack{i=1 \\ i\neq k}}^{N} \frac{m_i(r_i - r_k)}{\|r_i - r_k\|^3}, \qquad 2 \le k \le N. \qquad (4.5)\end{aligned}$$

Furthermore, any solution to an initial value problem of (4.4) is also invariant under arbitrary translations $c(t)$.

The last sentence leads us to consider how a solution to an initial value problem of Newton's system (4.1) is related to an initial value problem of the relative difference system (4.4). Clearly any solution to an initial value problem of (4.1), where $(r_j(t_0))_{1\le j\le N}$ with $r_j(t_0) \neq r_k(t_0)$ for $j \neq k$ and $(r_j'(t_0))_{1\le j\le N}$ denote the $2N$ initial values, is a solution to the initial value problem of (4.4) with $2(N-1)$ initial conditions $(r_{1k}(t_0) := r_1(t_0) - r_k(t_0))_{2\le k\le N}$ and $(r_{1k}'(t_0) := r_1'(t_0) - r_k'(t_0))_{2\le k\le N}$.

On the other hand, if we start with the relative system and initial conditions $(r_k(t_0))_{1\le k\le N}$ and $(r_k'(t_0))_{1\le<k\le N}$ associated with Newton's system (4.1), we can

form the initial conditions $(r_{1k}(t_0))_{2\le k\le N}$ and $(r'_{1k}(t_0))_{2\le k\le N}$ and obtain an initial value problem associated with the relative difference system (4.4). Because (4.4) is independent of origin, we may solve the aforementioned initial value problem for $r_1 \equiv \overrightarrow{0}$ and find solutions for $(r_j)_{j=2}^N$ relative to the origin at m_1 since the $N-1$ equations of (4.4) become

$$-r_k'' = G\sum_{i=2}^{N}\frac{m_i r_i}{\|r_i\|^3} + \frac{Gm_1 r_k}{\|r_k\|^3} - G\sum_{\substack{i=2\\ i\neq k}}^{N}\frac{m_i(r_i - r_k)}{\|r_i - r_k\|^3}, \qquad 2\le k\le N \tag{4.6}$$

Then to find the position of m_1 relative to m_N, take the above initial value problem of (4.4) and using the independence of origin, set $r_N \equiv \overrightarrow{0}$ in the $N-1$ equations of (4.4) to obtain

$$(r_1 - r_k)'' = G\sum_{i=2}^{N-1}\frac{m_i(r_i - r_1)}{\|r_i - r_1\|^3} - \frac{Gm_N r_1}{\|r_1\|^3} - G\sum_{\substack{i=1\\ i\neq k}}^{N-1}\frac{m_i(r_i - r_k)}{\|r_i - r_k\|^3} + \frac{Gm_N r_k}{\|r_k\|^3}, \tag{4.7}$$

and

$$r_1'' = G\sum_{i=2}^{N-1}\frac{m_i(r_i - r_1)}{\|r_i - r_1\|^3} - \frac{Gm_N r_1}{\|r_1\|^3} - G\sum_{i=1}^{N-1}\frac{m_i r_i}{\|r_i\|^3}. \tag{4.8}$$

We then solve the system of differential equations given by (4.7) and (4.8) to determine the positions of m_1 through m_{N-1} relative to m_N. Thus (4.6) and (4.8), when used in succession, show how a solution of (4.4) is also a solution of (4.1).

The discussion in the preceding paragraphs demonstrate a *principle of indetermination* which states that unless we identify the origin O with a point mass m_j, we cannot determine the position, velocity, and acceleration of m_k, where $k \neq j$.

Another system, closely related to (4.4) and invariant under arbitrary translations $c(t)$, is

$$(r_j - r_k)'' = G\sum_{\substack{i=1\\ i\neq j}}^{N}\frac{m_i(r_i - r_j)}{\|r_i - r_j\|^3} - G\sum_{\substack{i=1\\ i\neq k}}^{N}\frac{m_i(r_i - r_k)}{\|r_i - r_k\|^3}, \qquad 1\le j<k\le N, \tag{4.9}$$

where the dependent variables are $(\Delta_{jk} := r_j(t) - r_k(t))_{1\le j<k\le N}$. System (4.9) consists of $\binom{N}{2}$ equations and is generated by the the $N-1$ equations $(r_1 - r_p)'', 2\le p\le N$ since

$$(r_j - r_k)'' = (r_1 - r_k)'' - (r_1 - r_j)'' \qquad \text{whenever } j\neq 1. \tag{4.10}$$

Because the right side of the equations of (4.1) involves differences of the form $r_i - r_j$, where $i\neq j$, System (4.9) could be technically preferable to (4.4) since the solutions to (4.9) will directly calculate these differences without involving the subtractive step of (4.10). Furthermore, given initial conditions $(r_k(t_0))_{1\le k\le N}$ and $(r_k'(t_0))_{1\le<k\le N}$ of (4.1), we form the initial conditions $(r_{jk}(t_0) := r_j(t_0)-r_k(t_0))_{1\le j<k\le N}$ and $(r_{jk}'(t_0) := r_j'(t_0)-r_k'(t_0))_{1\le j<k\le N}$ and obtain an initial value problem associated with (4.9). Since (4.9) is independent of origin, we can use the principle of indetermination, exactly as we did for System (4.4), and determine the position, velocity, and acceleration of each individual point mass m_j, where $1 \le j \le N$.

Because of this correspondence between solutions of (4.1), (4.4), and (4.9), and the fact that both (4.4) and (4.9) have the advantage of being origin independent, we propose utilizing both (4.4) and (4.9) as independent systems to model the N-body problem.

In the next four sections we analyze the integrals of motion associated with (4.9) and obtain conservation of energy and conservation of angular momentum results analogous to those of the System (4.1).

5 Conservation of Energy

Newton's laws of motion for the N-body problem result in a system of $\binom{N}{2}$ nonlinear second order differential equations of the form

$$
\begin{aligned}
(r_j - r_k)'' &= G\sum_{\substack{i=1\\ i\neq j}}^{N} \frac{m_i(r_i - r_j)}{\|r_i - r_j\|^3} - G\sum_{\substack{i=1\\ i\neq k}}^{N} \frac{m_i(r_i - r_k)}{\|r_i - r_k\|^3} \\
&= \frac{-G(m_j + m_k)(r_j - r_k)}{\|r_j - r_k\|^3} + G\sum_{\substack{i=1\\ i\neq j,k}}^{N} m_i\left[\frac{r_i - r_j}{\|r_i - r_j\|^3} - \frac{r_i - r_k}{\|r_i - r_k\|^3}\right],
\end{aligned}
\tag{5.1}
$$

where we assume $1 \le j < k \le N$.

Take each equation in (5.1) and multiply both sides by $m_j m_k[(r_j - r_k)']^T$. The resulting left side is

$$
\begin{aligned}
m_j m_k[(r_j - r_k)']^T (r_j - r_k)'' &= \frac{m_j m_k}{2}\frac{d[\,[(r_j - r_k)']^T (r_j - r_k)']}{dt} \\
&= \frac{m_j m_k}{2}\frac{d\|(r_j - r_k)'\|^2}{dt},
\end{aligned}
\tag{5.2}
$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

while the resulting right side is $T_1(j,k)+T_2(j,k)$, where

$$T_1(j,k) := \frac{-Gm_jm_k(m_j+m_k)[(r_j-r_k)']^T(r_j-r_k)}{\|r_j-r_k\|^3} \tag{5.3}$$

$$T_2(j,k) := Gm_jm_k[(r_j-r_k)']^T \sum_{\substack{i=1\\ i\neq j,k}}^{N} m_i \left[\frac{r_i-r_j}{\|r_i-r_j\|^3} - \frac{r_i-r_k}{\|r_i-r_k\|^3}\right] \tag{5.4}$$

Observe that

$$\begin{aligned} T_1(j,k) &= -Gm_jm_k(m_j+m_k)[(r_j-r_k)']^T \frac{r_j-r_k}{\|r_j-r_k\|^3} \\ &= Gm_jm_k(m_j+m_k)[(r_j-r_k)']^T \left[\nabla_{(r_j-r_k)} \frac{1}{\|r_j-r_k\|}\right] \\ &= G(m_j+m_k)\frac{d}{dt}\frac{m_jm_k}{\|r_j-r_k\|}. \end{aligned} \tag{5.5}$$

Then use (5.2) and (5.5) to sum together

$$\begin{aligned} \sum_{1\le j<k\le N} m_jm_k[(r_j-r_k)']^T(r_j-r_k)'' &= \sum_{1\le j<k\le N} \frac{m_jm_k}{2}\frac{d\|(r_j-r_k)'\|^2}{dt} \\ &= \sum_{1\le j<k\le N} T_1(j,k) + \sum_{1\le j<k\le N} T_2(j,k) \\ &= G\sum_{1\le j<k\le N}(m_j+m_k)\frac{d}{dt}\frac{m_jm_k}{\|r_j-r_k\|} + \sum_{1\le j<k\le N} T_2(j,k), \end{aligned} \tag{5.6}$$

where

$$\sum_{1\le j<k\le N} T_2(j,k) = G\sum_{1\le j<k\le N} m_jm_k[(r_j-r_k)']^T \sum_{\substack{i=1\\ i\neq j,k}}^{N} m_i \left[\frac{r_i-r_j}{\|r_i-r_j\|^3} - \frac{r_i-r_k}{\|r_i-r_k\|^3}\right]. \tag{5.7}$$

The goal is to show that

$$\sum_{1\le j<k\le N} T_2(j,k) = \sum_{1\le j<k\le N}(M-m_j-m_k)\frac{d}{dt}\frac{m_jm_k}{\|r_j-r_k\|}, \qquad M := \sum_{k=1}^{N} m_k. \tag{5.8}$$

In order to prove (5.8), temporarily fix an index pair (j,k) and recall that we are summing the $\binom{N}{2}$ equations $m_\ell m_p[(r_\ell-r_p)']^T(r_\ell-r_p)''$, where $1\le \ell<p\le N$. Look

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

at the sum of the $\binom{N}{2} - 1$ equations arising from $(r_\ell - r_p)''$, where $1 \leq \ell < p \leq N$ and $(j,k) \neq (\ell, p)$, and add together the terms which have a factor of $\frac{r_j - r_k}{\|r_j - r_k\|^3}$. This process is equivalent to interchanging the order of summation on the right side of (5.7). Observe that in order to obtain $\frac{r_j - r_k}{\|r_j - r_k\|^3}$ either $\ell = j,k$ or $p = j,k$. The number of such ordered pairs $(\ell, p) \neq (j,k)$ for which either $\ell = j,k$ or $p = j,k$ is

$$\binom{N}{2} - \binom{N-2}{2} - 1 = \frac{N(N-1)}{2} - \frac{(N-2)(N-3)}{2} - 1 = \frac{4N-8}{2} = 2N-4,$$

where $\binom{N-2}{2}$ is the number of ordered pairs containing neither j or k. These $2N-4$ ordered pairs will be paired up with opposite signs as will shortly discover. To explain how the pairing occurs we analyze four mutually exclusive cases.

Case 1: $\ell = j$ and $p \neq k, j$. Note this implies that $p > j$. Then

$$(r_\ell - r_p)'' = (r_j - r_p)'' = \frac{-G(m_j + m_p)(r_j - r_p)}{\|r_j - r_p\|^3} + G\sum_{\substack{i=1 \\ i \neq j,p}}^{N} \frac{m_i(r_i - r_j)}{\|r_i - r_j\|^3} - G\sum_{\substack{i=1 \\ i \neq j,p}}^{N} \frac{m_i(r_i - r_p)}{\|r_i - r_p\|^3}. \tag{5.9}$$

Since $p \neq j,k$, only $i = k$ in second summand on the right side of (5.9) gives rise to $r_j - r_k$, in which case we obtain

$$\frac{Gm_k(r_k - r_j)}{\|r_k - r_j\|^3} = \frac{-Gm_k(r_j - r_k)}{\|r_j - r_k\|^3}.$$

We then left multiply the above vector by $m_j m_p [r_j' - r_p']^T$ to obtain a summand of the form

$$\frac{-Gm_j m_k m_p [r_j' - r_p']^T (r_j - r_k)}{\|r_j - r_k\|^3}. \tag{5.10}$$

Case 2: $\ell = k$ and $p \neq j,k$. This implies that $p > k$. Then

$$(r_\ell - r_p)'' = (r_k - r_p)'' = \frac{-G(m_k + m_p)(r_k - r_p)}{\|r_k - r_p\|^3} + G\sum_{\substack{i=1 \\ i \neq k,p}}^{N} \frac{m_i(r_i - r_k)}{\|r_i - r_k\|^3} - G\sum_{\substack{i=1 \\ i \neq k,p}}^{N} \frac{m_i(r_i - r_p)}{\|r_i - r_p\|^3}. \tag{5.11}$$

Since $p \neq j,k$, only $i = j$ in second summand on the right side of (5.11) gives rise to $r_j - r_k$, in which case we obtain

$$\frac{Gm_j(r_j - r_k)}{\|r_j - r_k\|^3}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

We then left multiply the above vector by $m_k m_p [r'_k - r'_p]^T$ to obtain a summand of the form

$$\frac{G m_j m_k m_p [r'_k - r'_p]^T (r_j - r_k)}{\|r_j - r_k\|^3}. \tag{5.12}$$

Case 3: $\ell \neq k, j$ and $p = j$. This implies $\ell < j$. Then

$$(r_\ell - r_p)'' = (r_\ell - r_j)'' = \frac{-G(m_\ell + m_j)(r_\ell - r_j)}{\|r_\ell - r_j\|^3} + G \sum_{\substack{i=1 \\ i \neq \ell, j}}^{N} \frac{m_i (r_i - r_\ell)}{\|r_i - r_\ell\|^3} - G \sum_{\substack{i=1 \\ i \neq \ell, j}}^{N} \frac{m_i (r_i - r_j)}{\|r_i - r_j\|^3}. \tag{5.13}$$

Since $\ell \neq j, k$, only $i = k$ in the third summand on the right side of (5.13) gives rise to $r_j - r_k$, in which case we obtain

$$\frac{G m_k (r_j - r_k)}{\|r_j - r_k\|^3}.$$

We then left multiply the above vector by $m_\ell m_j [r'_\ell - r'_j]^T$ to obtain

$$\frac{G m_j m_\ell m_k [r'_\ell - r'_j]^T (r_j - r_k)}{\|r_j - r_k\|^3}. \tag{5.14}$$

Case 4: $\ell \neq k, j$ and $p = k$. This implies $\ell < k$. Then

$$(r_\ell - r_p)'' = (r_\ell - r_k)'' = \frac{-G(m_\ell + m_k)(r_\ell - r_k)}{\|r_\ell - r_j\|^3} + G \sum_{\substack{i=1 \\ i \neq \ell, k}}^{N} \frac{m_i (r_i - r_\ell)}{\|r_i - r_\ell\|^3} - G \sum_{\substack{i=1 \\ i \neq \ell, k}}^{N} \frac{m_i (r_i - r_k)}{\|r_i - r_k\|^3}. \tag{5.15}$$

Since $\ell \neq j, k$, only $i = j$ in the third summand on the right side of (5.13) gives rise to $r_j - r_k$, in which case we obtain

$$\frac{-G m_j (r_j - r_k)}{\|r_j - r_k\|^3}.$$

We then left multiply the above vector by $m_\ell m_k [r'_\ell - r'_k]^T$ to obtain

$$\frac{-G m_j m_\ell m_k [r'_\ell - r'_k]^T (r_j - r_k)}{\|r_j - r_k\|^3}. \tag{5.16}$$

In all four cases, as evidenced by (5.10), (5.12), (5.14), and (5.16), there is a factor of the form $m_j m_k m_\alpha$. We want to pairwise combine via the value of α. The above four cases imply that $\alpha \neq j, k$. However, α is free to be any other value from the set

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

$\{1, ..., N\}$. This leads to following considerations: $\alpha < j$, $j < \alpha < k$, and $\alpha > k$. Suppose $\alpha > k$. This occurs for all of the p in Case 2 and those p in Case 1 for which $p > k$. Since the trace is linear, we can pairwise add (5.10) and (5.12) to obtain

$$\begin{aligned}\frac{-Gm_j m_k m_p[r_j' - r_p']^T(r_j - r_k)}{\|r_j - r_k\|^3} + \frac{Gm_j m_k m_p[r_k' - r_p']^T(r_j - r_k)}{\|r_j - r_k\|^3}&\\ = \frac{-Gm_j m_k m_p[r_j' - r_k']^T(r_j - r_k)}{\|r_j - r_k\|^3}&\\ = Gm_p \frac{d}{dt}\frac{m_j m_k}{\|r_j - r_k\|}, \qquad p > k.& \qquad (5.17)\end{aligned}$$

Next consider $\alpha < j$. This occurs for all of the ℓ in Case 3 and those ℓ in Case 4 for which $\ell < j$. We pairwise add (5.14) and (5.16) to obtain

$$\begin{aligned}\frac{Gm_j m_\ell m_k[r_\ell' - r_j']^T(r_j - r_k)}{\|r_j - r_k\|^3} + \frac{-Gm_j m_\ell m_k[r_\ell' - r_k']^T(r_j - r_k)}{\|r_j - r_k\|^3}&\\ = \frac{-Gm_j m_k m_\ell[r_j' - r_k']^T(r_j - r_k)}{\|r_j - r_k\|^3}&\\ = \frac{-Gm_j m_k m_p[r_j' - r_k']^T(r_j - r_k)}{\|r_j - r_k\|^3}, \text{ renamed } \ell \text{ as } p&\\ = Gm_p \frac{d}{dt}\frac{m_j m_k}{\|r_j - r_k\|}, \qquad p < j.& \qquad (5.18)\end{aligned}$$

Finally we have to consider when $j < \alpha < k$. This occurs in the remaining p and ℓ of Cases 1 and 4 not covered by (5.17) and (5.18) respectively. We can pairwise add (5.10) to (5.16) to obtain

$$\begin{aligned}\frac{-Gm_j m_k m_p[r_j' - r_p']^T(r_j - r_k)}{\|r_j - r_k\|^3} + \frac{-Gm_j m_\ell m_k[r_\ell' - r_k']^T(r_j - r_k)}{\|r_j - r_k\|^3}&\\ \frac{-Gm_j m_k m_p[r_j' - r_p']^T(r_j - r_k)}{\|r_j - r_k\|^3} + \frac{-Gm_j m_p m_k[r_p' - r_k']^T(r_j - r_k)}{\|r_j - r_k\|^3}, \text{ renamed } \ell \text{ to } p&\\ = \frac{-Gm_j m_k m_p[r_j' - r_k']^T(r_j - r_k)}{\|r_j - r_k\|^3}&\\ = Gm_p \frac{d}{dt}\frac{m_j m_k}{\|r_j - r_k\|}, \qquad j < p < k.& \qquad (5.19)\end{aligned}$$

If we add (5.17) through (5.19) together, we get

$$G \sum_{\substack{i=1,\\ i \neq j,k}}^{N} m_i \frac{d}{dt}\frac{m_j m_k}{\|r_j - r_k\|}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

The above term is true for an arbitrary yet fixed (j, k), where $1 \leq j < k \leq N$. By summing over $1 \leq j < k \leq N$, we obtain (5.8) as desired.

By combining (5.6) with (5.8) we obtain

$$\frac{1}{2} \sum_{1 \leq j < k \leq N} m_j m_k \frac{d}{dt} \|(r_j - r_k)'\|^2 = GM \sum_{1 \leq j < k \leq N} \frac{d}{dt} \frac{m_j m_k}{\|r_j - r_k\|}.$$

We then integrate the above results to obtain the conservation of energy formula

$$T = GM[U + h], \qquad h \text{ constant}, \qquad M = \sum_{i=1}^{N} m_i \tag{5.20}$$

$$U := \sum_{1 \leq j < k \leq N} \frac{m_j m_k}{\|r_j - r_k\|} \qquad \text{and} \qquad T := \frac{1}{2} \sum_{1 \leq j < k \leq N} m_j m_k \|(r_j - r_k)'\|^2.$$

6 Inertia and the Lagrange-Jacobi Formula

We will use (5.20) to simplify the second derivative of inertia I, where

$$I := \frac{1}{2} \sum_{1 \leq j < k \leq N} m_j m_k \left\|r_j - r_k\right\|^2. \tag{6.1}$$

We claim that

$$I'' = 2GM[U + h] - GMU, \tag{6.2}$$

a result known as the Lagrange-Jacobi formula. To prove (6.2) first observe that

$$I' = \sum_{1 \leq j < k \leq N} m_j m_k [r_j - r_k]^T (r_j - r_k)'. \tag{6.3}$$

Then

$$\begin{aligned} I'' &= \sum_{1 \leq j < k \leq N} m_j m_k [(r_j - r_k)']^T (r_j - r_k)' + \sum_{1 \leq j < k \leq N} m_j m_k [r_j - r_k]^T (r_j - r_k)'' \\ &= 2GM[U + h] + \sum_{1 \leq j < k \leq N} m_j m_k [r_j - r_k]^T (r_j - r_k)'', \qquad \text{by (5.20)} \\ &= 2GM[U + h] \\ &+ G \sum_{1 \leq j < k \leq N} m_j m_k [r_j - r_k]^T \left[\frac{-(m_j + m_k)(r_j - r_k)}{\left\|r_j - r_k\right\|^3} + \sum_{\substack{i=1 \\ i \neq j,k}}^{N} \frac{m_i (r_i - r_j)}{\left\|r_i - r_j\right\|^3} - \sum_{\substack{i=1 \\ i \neq j,k}}^{N} \frac{m_i (r_i - r_k)}{\left\|r_i - r_k\right\|^3} \right], \end{aligned}$$

where the final sum made use of (5.1). It remains to simplify the expression in the large square bracket. To do so, we will temporarily fix (j,k), vary $(\ell,p) \neq (j,k)$ (where $1 \leq \ell < p \leq N$), collect and add together all the terms that have a factor of $\frac{r_j - r_k}{\|r_j - r_k\|^3}$. In other words, we must look at a typical summand in $\sum_{1\leq j<k\leq N}[r_j - r_k]^T(r_j - r_k)''$ with index (ℓ,p), namely

$$m_\ell m_p[r_\ell - r_p]^T(r_\ell - r_p)'' = -\frac{Gm_\ell m_p(m_\ell + m_p)}{\|r_\ell - r_p\|} + Gm_\ell m_p[r_\ell - r_p]^T[S_1 + S_2], \quad (6.4)$$

where

$$S_1 := \sum_{\substack{i=1 \\ i\neq \ell,p}} \frac{m_i(r_i - r_\ell)}{\|r_i - r_\ell\|^3}, \qquad S_2 := \sum_{\substack{i=1 \\ i\neq \ell,p}} \frac{m_i(r_i - r_p)}{\|r_i - r_p\|^3}.$$

This calculation utilizes the four case argument.

Case 1: $\ell = j$ and $p \neq j,k$. Note that $p > j$. Set $i = k$ in S_1 to obtain

$$-\frac{Gm_j m_k m_p[r_j - r_p]^T(r_j - r_k)}{\|r_j - r_k\|^3}. \quad (6.5)$$

Case 2: $\ell = k$ and $p \neq j,k$. Note that $p > k$. Set $i = j$ in S_1 to obtain

$$\frac{Gm_j m_k m_p[r_k - r_p]^T(r_j - r_k)}{\|r_j - r_k\|^3}. \quad (6.6)$$

We can add each term of Case 2 to a corresponding term of Case whenever $p > k$ to obtain

$$-Gm_j m_k m_p \left[r_j^T - r_p^T - (r_k^T - r_p^T)\right] \frac{r_j - r_k}{\|r_j - r_k\|^3} = -\frac{Gm_j m_k m_p[r_j - r_k]^T(r_j - r_k)}{\|r_j - r_k\|^3} = -\frac{Gm_j m_k m_p}{\|r_j - r_k\|}, \qquad p > k. \quad (6.7)$$

The remaining terms of Case 1 satisfy $j < p < k$.

Case 3: $\ell \neq j,k$, and $p = j$. Note that $\ell < j$. Set $i = k$ in S_2 to obtain

$$\frac{Gm_\ell m_j m_k[r_\ell - r_j](r_j - r_k)}{\|r_j - r_k\|^3}. \quad (6.8)$$

Case 4: $\ell \neq j,k$ and $p = k$. Note that $\ell < k$. Set $i = j$ in S_2 to obtain

$$\frac{-Gm_\ell m_j m_k[r_\ell - r_k](r_j - r_k)}{\|r_j - r_k\|^3}. \quad (6.9)$$

We add each term of Case 3 to a corresponding term of Case 4 with $\ell < j$ to obtain

$$\begin{aligned} -Gm_jm_km_\ell\left[-(r_\ell^T - r_j^T) + (r_\ell^T - r_k^T)\right]\frac{r_j - r_k}{\|r_j - r_k\|^3} &= -\frac{Gm_jm_km_\ell[r_j - r_k]^T(r_j - r_k)}{\|r_j - r_k\|^3} \\ &= -\frac{Gm_jm_km_\ell}{\|r_j - r_k\|}, \qquad \ell < j \\ &= -\frac{Gm_jm_km_p}{\|r_j - r_k\|}, \qquad p < j. \end{aligned} \tag{6.10}$$

where in the last equality we renamed ℓ as p. The remaining terms of Case 4 (where ℓ is renamed as p) satisfy $j < p < k$ and can be added to the corresponding remaining terms of Case 1 to obtain

$$\begin{aligned} -Gm_jm_km_p\left[r_j^T - r_p^T + (r_p^T - r_k^T)\right]\frac{r_j - r_k}{\|r_j - r_k\|^3} &= -\frac{Gm_jm_km_p[r_j - r_k]^T(r_j - r_k)}{\|r_j - r_k\|^3} \\ &= -\frac{Gm_jm_km_p}{\|r_j - r_k\|}, \qquad j < p < k. \end{aligned} \tag{6.11}$$

The results of (6.7), (6.10), and (6.11) imply that

$$\begin{aligned} I'' &= \sum_{1\le j<k\le N} m_jm_k[(r_j - r_k)']^T(r_j - r_k)' + \sum_{1\le j<k\le N} m_jm_k[r_j - r_k]^T(r_j - r_k)'' \\ &= 2GM[U + h] - G\sum_{1\le j<k\le N}\frac{m_jm_k(m_j + m_k)[r_j - r_k]^T(r_j - r_k)}{\|r_j - r_k\|^3} \\ &+ G\sum_{1\le j<k\le N}\sum_{\substack{i=1\\ i\ne j,k}}^{N}\frac{m_jm_km_i[r_j - r_k]^T(r_i - r_j)}{\|r_i - r_j\|^3} - G\sum_{1\le j<k\le N}\sum_{\substack{i=1\\ i\ne j,k}}^{N}\frac{m_jm_km_i[r_j - r_k]^T(r_i - r_k)}{\|r_i - r_k\|^3} \\ &= 2GM[U + h] - G\sum_{1\le j<k\le N}\frac{m_jm_k(m_j + m_k)}{\|r_j - r_k\|} - G\sum_{1\le j<k\le N}\sum_{\substack{i=1\\ i=\ne j,k}}\frac{m_jm_km_i}{\|r_j - r_k\|} \\ &= 2GM[U + h] - GM\sum_{1\le j<k\le N}\frac{m_jm_k}{\|r_j - r_k\|} = 2GM[U + h] - GMU, \end{aligned}$$

which is precisely (6.2).

By using the definitions of T and U provided by (5.20), we may rewrite (6.2) as

$$I'' = 2GM[U + h] - GMU = GMU + 2GMh = 2T - GMU = T + GMh. \tag{6.12}$$

7 Conservation of Angular Momentum

Next we prove the following identity which shows that the sum of the angular momentum is constant, namely that

$$\sum_{1\le j<k\le N} m_j m_k[\,(r_j - r_k)\times(r_j - r_k)'\,] = c, \qquad c \text{ constant vector.} \tag{7.1}$$

In particular, by using the same four case pairing argument as previously discussed, we will prove

$$\sum_{1\le j<k\le N} m_j m_k[\,(r_j - r_k)\times(r_j - r_k)''\,] = 0. \tag{7.2}$$

Since

$$\frac{d}{dt} m_j m_k[\,(r_j - r_k)\times(r_j - r_k)'\,] = m_j m_k[\,(r_j - r_k)\times(r_j - r_k)''\,],$$

by integrating both sides of (7.2) we obtain (7.1) as desired.

To prove (7.2), we will regroup the terms in the sum via common denominators of the form $\|r_j - r_k\|^3$. For an arbitrary yet fixed pair (j,k) with $1 \le j < k \le N$, we look for terms in (7.2) of the form $\frac{S_{i,j}}{\|r_j - r_k\|^3}$ and calculate their sum. In order to efficiently find these terms, we vary (ℓ,p) with $1\le \ell < p \le N$, look at $m_\ell m_p(r_\ell - r_p)\times(r_\ell - r_p)''$ on the right side of (7.2), expand $(r_\ell - r_p)''$ via Newton's equations and show that the sum of all terms of the form

$$\frac{\gamma(r_\ell - r_p)\times(r_j - r_k)}{\|r_j - r_k\|^3}, \qquad \gamma \text{ constant.} \tag{7.3}$$

is indeed zero. By then varying (j,k) over the range of $1\le j<k\le N$, we will have accounted for all the terms in $\sum_{1\le j<k\le N} m_j m_k[\,(r_j - r_k)\times(r_j - r_k)''\,]$ and will have proven (7.2).

First let $(\ell,p) = (j,k)$, go back to (5.1) and observe that

$$\begin{aligned}(r_j - r_k)\times(r_j - r_k)'' &= \frac{-G(m_j + m_k)(r_j - r_k)\times(r_j - r_k)}{\|r_j - r_k\|^3} \\ &\quad + G\sum_{\substack{i=1\\ i\ne j,k}}^{N} m_i\left[\frac{(r_j - r_k)\times(r_i - r_j)}{\|r_i - r_j\|^3} - \frac{(r_j - r_k)\times(r_i - r_k)}{\|r_i - r_k\|^3}\right] \\ &= G\sum_{\substack{i=1\\ i\ne j,k}}^{N} m_i\left[\frac{(r_j - r_k)\times(r_i - r_j)}{\|r_i - r_j\|^3} - \frac{(r_j - r_k)\times(r_i - r_k)}{\|r_i - r_k\|^3}\right].\end{aligned} \tag{7.4}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

None of the above terms of $(r_j - r_k) \times (r_j - r_k)''$ have denominator $\|r_j - r_k\|^3$. To figure out how the other $\binom{N}{2} - 1$ terms $m_\ell m_p(r_\ell - r_p) \times (r_\ell - r_p)$ contribute terms of the form (7.3), we use a familiar case/parity argument.

Case 1: $\ell = j$ and $p \neq j, k$. Note that $p > j$. Observe that

$$m_j m_p (r_j - r_p) \times (r_j - r_p)'' = G m_j m_p \left[\sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{m_i (r_j - r_p) \times (r_i - r_j)}{\|r_i - r_j\|^3} - \sum_{\substack{i=1 \\ i \neq p}}^{N} \frac{m_i (r_j - r_p) \times (r_i - r_p)}{\|r_i - r_p\|^3} \right].$$

The only way to obtain a term of type (7.3) is to set $i = k$ in the first summand, in which case we get the following contribution

$$-\frac{G m_j m_k m_p (r_j - r_p) \times (r_j - r_k)}{\|r_j - r_k\|^3}, \qquad p > j \tag{7.5}$$

Case 2: $\ell = k$ and $p \neq j, k$. Note that $p > k$. Then

$$m_k m_p (r_k - r_p) \times (r_k - r_p)'' = G m_k m_p \left[\sum_{\substack{i=1 \\ i \neq k}}^{N} \frac{m_i (r_k - r_p) \times (r_i - r_k)}{\|r_i - r_k\|^3} - \sum_{\substack{i=1 \\ i \neq p}}^{N} \frac{m_i (r_k - r_p) \times (r_i - r_p)}{\|r_i - r_p\|^3} \right].$$

The only way to obtain a term of type (7.3) is to set $i = j$ in the first summand in which case we get

$$\frac{G m_j m_k m_p (r_k - r_p) \times (r_j - r_k)}{\|r_j - r_k\|^3}, \qquad p > k. \tag{7.6}$$

Note that if $p > k$, we can pair up each term of Case 2 with a corresponding term of Case 1, use the bilinearity of the cross product, and get a contribution of zero as seen by the following calculation.

$$\begin{aligned} -\frac{G m_j m_k m_p}{\|r_j - r_k\|^3}(r_j - r_p) \times (r_j - r_k) &+ \frac{G m_j m_k m_p}{\|r_j - r_k\|^3}(r_k - r_p) \times (r_j - r_k) \\ &= \frac{-G m_j m_k m_p}{\|r_j - r_k\|^3}\left[(r_j - r_p - (r_k - r_p)) \times (r_j - r_k)\right] \\ &= \frac{-G m_j m_k m_p}{\|r_j - r_k\|^3}\left[(r_j - r_k) \times (r_j - r_k)\right] = 0. \end{aligned}$$

The remaining contributions from Case 1 are the $k - j - 1$ terms for which $j + 1 \leq p \leq k - 1$, namely

$$-\frac{G m_j m_k m_p (r_j - r_p) \times (r_j - r_k)}{\|r_j - r_k\|^3}, \qquad j + 1 \leq p \leq k - 1. \tag{7.7}$$

Case 3: $\ell \neq j, k$ and $p = j$. Note that $\ell < j$. Note that

$$m_\ell m_j (r_\ell - r_j) \times (r_\ell - r_j)'' = G m_\ell m_j \left[\sum_{\substack{i=1 \\ i \neq \ell}}^{N} \frac{m_i (r_\ell - r_j) \times (r_i - r_\ell)}{\|r_i - r_\ell\|^3} - \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{m_i (r_\ell - r_j) \times (r_i - r_j)}{\|r_i - r_j\|^3} \right].$$

The only way to obtain a term of type (7.3) is to set $i = k$ in the second sum, in which case (after renaming ℓ as p) we get the following contribution.

$$\frac{G m_j m_k m_p (r_p - r_j) \times (r_j - r_k)}{\|r_j - r_k\|^3}, \qquad p < j. \tag{7.8}$$

Case 4: $\ell \neq j, k$ and $p = k$. Note that $\ell < k$. Note that

$$m_\ell m_k (r_\ell - r_k) \times (r_\ell - r_k)'' = G m_\ell m_k \left[\sum_{\substack{i=1 \\ i \neq \ell}}^{N} \frac{m_i (r_\ell - r_k) \times (r_i - r_\ell)}{\|r_i - r_\ell\|^3} - \sum_{\substack{i=1 \\ i \neq k}}^{N} \frac{m_i (r_\ell - r_k) \times (r_i - r_k)}{\|r_i - r_k\|^3} \right].$$

The only way to obtain a term of type (7.3) is to set $i = j$ in the second sum, in which case (after renaming ℓ as p) we get the following contribution.

$$-\frac{G m_j m_k m_p (r_p - r_k) \times (r_j - r_k)}{\|r_j - r_k\|^3}, \qquad p < k. \tag{7.9}$$

Notice that each term of Case 3 can be added to a corresponding term of Case 4 in which $p < j$. Then by using the bilinearity of the cross product, we see from the calculation below the result of this addition is zero.

$$\begin{aligned} \frac{G m_p m_j m_k}{\|r_j - r_k\|^3} (r_p - r_j) \times (r_j - r_k) - \frac{G m_p m_j m_k}{\|r_j - r_k\|^3} (r_p - r_k) \times (r_j - r_k) \\ = \frac{-G m_p m_j m_k}{\|r_j - r_k\|^3} \left[(-(r_p - r_j) + r_p - r_k) \times (r_j - r_k) \right] \\ = \frac{-G m_p m_j m_k}{\|r_j - r_k\|^3} \left[(r_j - r_k) \times (r_j - r_k) \right] = 0. \end{aligned}$$

The remaining contributions from Case 4 are the are the $k - j - 1$ terms for which $j + 1 \leq p \leq k - 1$, namely

$$-\frac{G m_j m_k m_p (r_p - r_k) \times (r_j - r_k)}{\|r_j - r_k\|^3}, \qquad j + 1 \leq p \leq k - 1. \tag{7.10}$$

Then it is a matter of adding together the corresponding terms of (7.7) and (7.10) and show they the resulting sum is zero. In particular, for $j+1 \leq p \leq k-1$, we find that

$$-\frac{Gm_j m_k m_p}{lr_j - r_l\|^3}(r_j - r_p) \times (r_j - r_k) - -\frac{Gm_j m_k m_p}{\|r_j - r_l\|^3}(r_p - r_k) \times (r_j - r_k)$$
$$= \frac{-Gm_j m_k m_p}{\|r_j - r_l\|^3}\left[(r_j - r_p + r_p - r_k) \times (r_j - r_k)\right]$$
$$= \frac{-Gm_j m_k m_p}{\|r_j - r_l\|^3}\left[(r_j - r_k) \times (r_j - r_k)\right] = 0.$$

8 Sundman's Inequality and Total Collapse

Let us obtain Sundman's two theorems regarding total collapse for our system of differences. Following the exposition of Pollard [[11], Page 64], we must first obtain Sundman's inequality by estimating the norm of the angular momentum. Start with (7.1) and apply the triangle inequality to obtain

$$\|c\| := \mathrm{c} \leq \sum_{1\leq j<k\leq N} m_j m_k \|(r_j - r_k) \times (r_j - r_k)'\|$$
$$= \sum_{1\leq j<k\leq N} m_j m_k \|(r_j - r_k)\|, \|(r_j - r_k)'\|,$$

since $\|(r_j - r_k) \times (r_j - r_k)'\| = \|(r_j - r_k)\| \, \|(r_j - r_k)'\| \sin\theta$ where θ is the angle between $\|(r_j - r_k)\|$ and $\|(r_j - r_k)'\|$. Let

$$r_{jk} := \|r_j - r_k\|, \qquad v_{jk} := \|(r_j - r_k)'\|, \qquad 1 \leq j < k \leq N$$

The above becomes

$$\mathrm{c} \leq \sum_{1\leq j<k\leq N} m_j m_k r_{jk} v_{jk} = \sum_{1\leq j<k\leq N} (\sqrt{m_j m_k} r_{jk})(\sqrt{m_j m_k} v_{jk}).$$

If we take the preceding inequality, square it, and apply the Cauchy-Schwarz inequality, we find that

$$\mathrm{c}^2 \leq \left[\sum_{1\leq j<k\leq N} (\sqrt{m_j m_k} r_{jk})(\sqrt{m_j m_k} v_{jk})\right]^2$$
$$\leq \left[\sum_{1\leq j<k\leq N} m_j m_k r_{jk}^2\right]\left[\sum_{1\leq j<k\leq N} m_j m_k v_{jk}^2\right], \qquad \text{Cauchy-Schwarz inequality}$$
$$= 2I \cdot 2T = 4IT = 4I[I'' - GMh], \tag{8.1}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 October 2022 doi:10.20944/preprints202210.0277.v1

where the last equality follows from (6.12). Inequality (8.1) is known as Sundman's inequality.

Next we prove the first of Sundman's theorems regarding total collapse, namely that total collapse occurs in finite time. Or as Pollard says, " $I \to 0$ as $t \to \infty$ is impossible" [[11], Page 66]. The proof proceeds by contradiction. Suppose that $I \to 0$ as $t \to \infty$. By the definition of I, this implies that $\|r_j - r_k\| \to 0$ as $t \to \infty$ whenever $1 \leq j < k \leq N$. Then the definition of U implies that $U \to \infty$ as $t \to \infty$. Since $I'' = GMU + 2GMh$, we deduce that $I'' \to \infty$ as $t \to \infty$. Hence there exists $t_1 \in \mathbb{R}$ such that $I'' \geq 1$ whenever $t \geq t_1$. If we twice integrate both sides of this inequality we obtain

$$I \geq \frac{1}{2}t^2 + at + b, \qquad a, b \in \mathbb{R}, \qquad t \geq t_1. \tag{8.2}$$

But Inequality (8.2) implies that $I \to \infty$ as $t \to \infty$, a contradiction of the assumption that $\lim_{t\to\infty} I = 0$.

We next prove the second of Sundman's theorem, which says that if total collapse occurs, then the angular momentum is zero. The proof of this theorem make use of the following lemma.

Lemma 8.1 *Let* $f \in C^2[a, b]$ *and assume that* $f(x) > 0$ *whenever* $x \in [a, b)$*, that* $f''(x) > 0$ *for* $x \in [a, b]$*, and that* $f(b) = 0$*. Then* $f'(x) \leq 0$ *whenever* $x \in [a, b]$*.*

Proof: First observe that $f'(b) \leq 0$ since by assumption $a \leq x \leq b^-$, $f(x) \geq 0$, $f(b) = 0$, and

$$f'(b) = \lim_{x\to b^-} \frac{f(x) - f(b)}{x - b} = \lim_{x\to b^-} \frac{f(x)}{x - b}.$$

Since $f'' > 0$, we know $g(x) := \int_x^b f''(s)\,ds \geq 0$, or equivalently that $-g(x) = -\int_x^b f''(s)\,ds \leq 0$. But since $f'(b) - f'(x) = g(x)$, we deduce that

$$f'(x) = f'(b) - g(x) = f'(b) - \int_x^b f''(s)\,ds \leq 0,$$

whenever $x \in [a, b]$. □

Once again we follow Pollard [[11], pp. 65-67] and suppose that total collapse occurs at time t_1. Without loss of generality we may assume that $t_1 > 0$. This implies that $\lim_{t\to t_1} I = 0$. Also, as discussed above, we deduce that $\lim_{t\to t_1} U = \infty$ and that $\lim_{t\to t_1} I'' = \infty$. Therefore there exists a finite positive closed interval $[t_2, t_1]$ such that $I''(t) > 0$ whenever $t \in [t_2, t_1]$. By definition, $I \geq 0$ and in particular $I(t) > 0$ for $t \in [t_2, t_1)$ with $I(t_1) = 0$. Thus we may apply Lemma 8.1 to deduce that $-I'(t) \geq 0$

whenever $t \in [t_1, t_2]$. Now we take Inequality (8.1) and multiply it by the positive quantity $-I'(t)I^{-1}(t)$ to obtain

$$-\frac{\mathrm{c}^2}{4} I'\, I^{-1} \leq -I'\, I'' + I' GMh, \qquad 0 < t_2 \leq t \leq t_1. \tag{8.3}$$

Now integrate (8.3) to obtain

$$-\frac{\mathrm{c}^2}{4} \int_{t_2}^{t} I'(s)\, I^{-1}(s)\, ds \leq \int_{t_2}^{t} \left[-I'(s)\, I''(s) + I'(s) GMh\right] ds. \tag{8.4}$$

Since

$$-\frac{\mathrm{c}^2}{4} \int_{t_2}^{t} I'(s)\, I^{-1}(s)\, ds = \frac{\mathrm{c}^2}{4} \ln I^{-1}(t) + K_1, \qquad K_1 = \frac{\mathrm{c}^2}{4} \ln I(t_2),$$

and since

$$\int_{t_2}^{t} \left[-I'(s)\, I''(s) + I'(s) GMh\right] ds = GMhI(t) - \frac{1}{2}(I'(t))^2 + K_2, \quad K_2 = -GMh(I(t_2)) + \frac{1}{2}(I'(t_2))^2,$$

we may rewrite (8.4) as

$$\frac{\mathrm{c}^2}{4} \ln I^{-1}(t) \leq GMhI(t) - \frac{1}{2}(I'(t))^2 + K \leq GMhI(t) + K, \qquad K = K_2 - K_1. \tag{8.5}$$

Inequality (8.5) is equivalent to

$$\frac{\mathrm{c}^2}{4} \leq \frac{GMhI(t) + K}{\ln I^{-1}(t)}, \qquad 0 < t_2 \leq t \leq t_1. \tag{8.6}$$

Since the numerator of the right side of (8.6) is bounded (recall $I(t) \to 0$ as $t \to t_1$), we deduce that

$$0 \geq \lim_{t \to t_1} \frac{\mathrm{c}^2}{4} \leq \lim_{t \to t_1} \frac{GMhI(t) + K}{\ln I^{-1}(t)} = 0.$$

Hence $\mathrm{c} = \|c\| = 0$, which in turn implies that the angular momentum $c = 0$.

References

[1] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. L. Maier, *Qualitative Theory of Second–order Dynamic Systems*, Wiley, New York, (1973).

[2] A. Einstein, L. Infeld, and B. Hoffmann, The gravitational equations and the problem of motion. *Ann. of Math,*, (2) 39 (1938), no. 1, 65–100.

[3] U. Elias and H. Gingold, Critical points at infinity and blow up of solutions of autonomous polynomial differential systems via compactification, *J. Math. Anal.Appl.*, 318, no. 1 (2006), 305-322.

[4] R. P. Feynman, *Six Not so Easy Pieces*, Perseus Books, Reading Massachusetts, 1997.

[5] Y. I. Gingold and H. Gingold, Geometric Properties of a Family of Compactifications, *Balkan Journal of Geom. Appl.*, 12, no. 1 (2007), 44-55.

[6] H. Gingold and D. Solomon, *Advances in Pure Mathematics*, Vol.3 No.1A, 2013. https://www.scirp.org/(S(i43dyn45teexjx455qlt3d2q))/reference/referencespapers.aspx?referenceid=706616

[7] Y. I. Gingold and H. Gingold, Simulation of perspective by nonlinear transformations, *Mathematical Modeling and Analysis*, 18 3 (2013), 346-357; https://doi.org/10.3846/13926292.2013.804887

[8] H. Gingold and D. Solomon, Celestial mechanics solutions that escape, *Journal of Differential Equations*, Volume 263, No. 3 , 2017, 1695-1946.

[9] P. F. Hsieh and Y. Sibuya, *Basic Theory of Ordinary Differential Equations*, Universitext, Springer-Verlag, New York, 1999.

[10] L. Perko, *Differential equations and dynamical systems*, Springer Verlag, New York, USA, (1996).

[11] H. Pollard, *Celestial Mechanics*, Carus Mathematical Monographs, No. 18, Mathematical Association of America, 1976.

[12] G. Sansone and R. Conti, *Non-linear differential equations*, Pergamon Press, Oxford, United Kingdom, (1964).