Article

A Taxonomy of Elementary Behavior Modes for Novice Modelers

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Abstract: This article addresses a challenge modeling novices face in model conceptualization: the recognition of behavior modes. Despite behavior modes being at the heart of system dynamics modeling, there is no unifying taxonomy for them, making them harder to learn for beginners. The article proposes criteria a taxonomy should satisfy, critically reviews previous taxonomies in the literature, and then introduces a new consistent taxonomy based on slope and curvature with mode names that refer only to visual cues and free of references to prior mathematical or domain-specific knowledge. Evidence from an exploratory experiment suggests that novices actually have difficulties classifying curves when using previous taxonomies. Links from these elementary modes to analogous terms in other taxonomies in diverse disciplines allow this taxonomy to be related to other ones. The article concludes by mentioning relevant limitations and future steps.

Keywords: System dynamics; Education; Teaching; Learning; Systemsthinking

Introduction

This article focuses on one important aspect of learning system dynamics: the elementary behavior modes and their recognition in the behavior of variables. The term *variable* refers to a quantity that changes over time (Richardson and Pugh, 1989). When observing these quantities—each referring to a specific moment or interval of measurement—over a sequence of measurements, they appear as a shape representing the variable's behavior. This shape has been called *behavior pattern* (Barlas and Kanar, 1999; Boğ and Barlas, 2005; Ford, 1999), *reference mode* (Ford, 2019; Randers, 1980; Richardson and Pugh, 1989; Sterman, 2000), and *behavior mode* (Ford, 2019). Dictionaries usually define the term *pattern* as "a natural […] configuration," a "reliable sample of traits, acts, tendencies, or other observable characteristics of a person, group, or institution"; a *mode* is "a particular form or variety of something" or "a particular functioning arrangement or condition" (Merriam-Webster, 2022). We use *behavior* for the original data—the quantities reported per measurement. The *behavior modes* are generic classes of *behavior*, according to a specific set of defining feature types and feature values. *Behavior patterns* are the sets of feature values a person can detect in *behaviors* when applying (intuitively or deliberately) a taxonomy of *behavior modes*. Recognizing the *reference (behavior) modes* that a model shall replicate and explain is part of the model conceptualization phase (Randers, 1980; Richardson and Pugh, 1989; Sterman, 2000).

Since one cannot recognize a behavior mode without knowing the modes, recognizing a certain behavior mode can be anything from trivial to challenging, depending on (a) prior knowledge and (b) possible classification ambiguities or conflicts when applying a taxonomy to recognize patterns. An experienced modeler with personal knowledge of the modes of behavior used in system dynamics (SD) will intuitively recognize them when looking at a behavior over time graph; not so a novice coming from an area with a different taxonomy of behavior modes, or someone with no prior behavior classification schema. Allegedly, newcomers may know alternative behavior modes used in other areas, such as

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economics, management, ecology, or mathematics. Yet, just as novices, they cannot perceive system dynamic behavior modes in observed behaviors without resorting to guidance from the literature or from lecturers and experienced SD practitioners.

The SD literature disseminates different taxonomies in its discussion of behavior patterns and modes, developed for varying purposes. While Sterman (2000) addressed learners, others had more specific aspects of the modeling process in mind (Barlas and Kanar, 1999; Ford, 1999), or strived to give a terminological overview (Ford, 2019). This enables experienced modelers to apply the taxonomy which best fits their needs. However, it also increases the effort required from novices to choose a convenient taxonomy and get acquainted with the modes. As an initial step toward a terminology that best fits the circumstances and needs of modeling novices, this paper proposes a conceptual framework for discussing behavior mode taxonomies, critically analyzes three highly visible behavior mode taxonomies, and then proposes one set of elementary behavior modes which turn out to be conceptually consistent with the framework.

This contribution is mainly conceptual, but data from an initial experiment with our students suggest that a large share of undergraduate students fail in simple classification exercises of elementary behavior modes prior to courses in SD.

The remainder of this article is organized as follows. The second section introduces a conceptual background, discussing how patterns are intuitively recognized or have to be consciously categorized and proposing criteria for assessing behavior mode taxonomies. The third section applies these criteria to three previously published taxonomies; it also provides additional arguments for why some terms in these taxonomies are confounding for novices. The fourth section develops the taxonomy we propose, showing how the name labels for the elementary modes exclude any reference to aspects other than the visual cues of slope and curvature. The fifth section describes the two experimental tasks and their results. The discussion in section six provides some relevant theoretic connections into the literature of learning and teaching mathematical modeling, and then we conclude by summarizing the main points and some limitations and further steps.

Conceptual background

Recognition of behavior patterns implies prior knowledge

Consider the following exemplary case of behavior (Figure 1): the daily number of new COVID cases reported in Chile from the start of 2022 until Feb. 16, 2022.

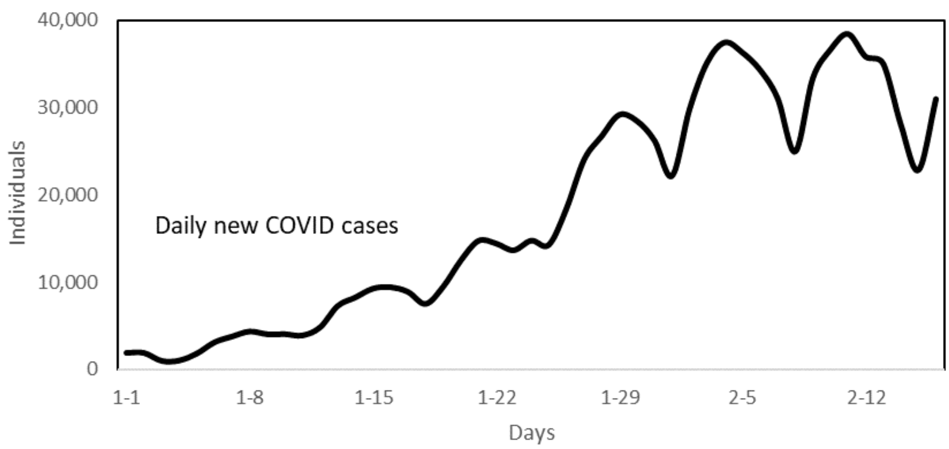

Figure 1: An exemplary behavior graph representing the daily number of new COVID cases reported in Chile during the initial weeks of 2022 (data from https://www.gob.cl/coronavirus/cifrasoficiales).

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Readers with SD background will effortlessly perceive patterns and immediately think of certain behavior modes, but readers without such prior knowledge will need to reflect, construct their own categories for classifying the shapes, and likely find themselves in need of guidance. Cognitive scientists have developed dual-processing theories of cognition to account for these two kinds of judging a situation. They distinguish between two distinct processing systems, where system 1 is active when the situation is familiar because it is quick and intuitive. The deliberative system 2 is only activated to process surprising or unknown situations for the additional effort it requires (Kahneman, 2011; Stanovich, 2012). Figure 2 depicts the two cognitive processes for assigning a shape to a behavior mode. Prior knowledge may exist as familiarity with concrete processes, mechanisms, or structures that are usually articulated as diagrams, as in most professional areas and scientific disciplines. However, it may also be abstract mathematical knowledge applicable to a class of concrete cases.

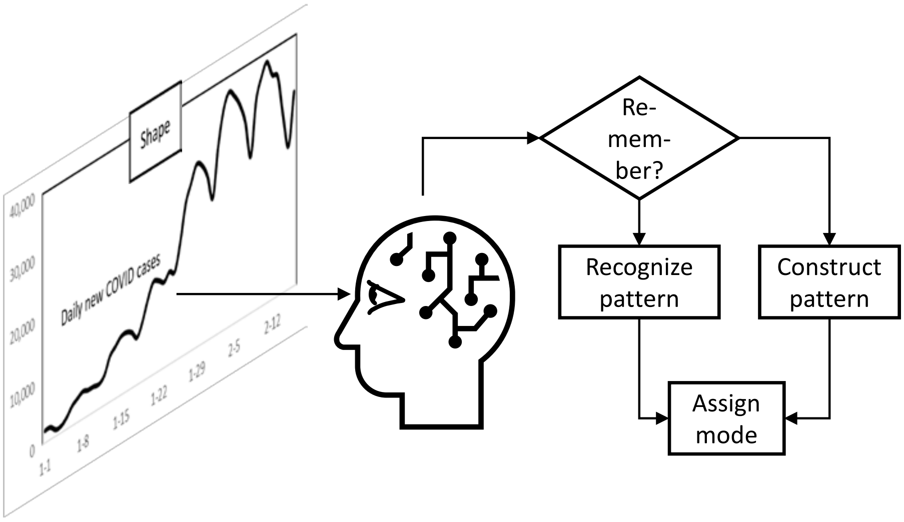

Figure 2: Prior knowledge leads to recognition of behavior modes; the alternative is to analyze the shape deliberately, define characteristic features, and then assign a behavior mode.

Prior knowledge builds up through repetition of experiences and is retrievable from long-term memory. Therefore, people are more likely to feel familiar with some terms than with others because they have seen them more frequently. Consider the differences in how frequently certain behavior mode labels appear on the Internet. Table 1 shows the number of general web pages, videos, and news entries replied by Google (on February 25, 2021).

Table 1: Frequency of name labels for behavior patterns on the Internet.

Search term	Hits at large	Videos	News entries
Exponential growth	65,400,000	769,000	1,120,000
Exponential decay	48,500,000	131,000	7,710
Exponential decline	32,000,000	192,000	155,000
Logarithmic growth	19,300,000	83,900	9,930
Exponential collapse	16,800,000	121,000	70,200
Asymptotic growth	13,500,000	40,100	1,820
Asymptotic decay	6,370,000	9,010	433
Asymptotic decline	4,610,000	8,860	1,480
Logarithmic decay	7,680,000	34,300	124,000
Logarithmic decline	7,220,000	16,800	3,430

The typical terms to describe the shape of a behavior combine one word referring to the direction of change ("growth" when the slope is positive and "decay," "decline," or "collapse" when it is negative) and one word for the change of slope ("exponential," "asymptotic," or "logarithmic"). These frequencies suggest different degrees of availability of these terms when someone looks at a graph showing a variable's behavior. On average, people will be more likely to recall "exponential growth" than any other term; this means that the intuitive system 1 will not recover and process the meaning of other terms, and so system 1 must construct it just-in-time. While many Internet users will recognize "exponential growth," many other terms will demand cognitive effort to decide what they mean.

"Growth" is generally used to describe curves with positive slope. When dealing with how variables behave over time, our mind handles time using spatial metaphors, as suggested by neuroscience (see chapter 10 of Buonomano, 2017); time becomes tangible through the analogy with movement in space. Frequently, these are orientation metaphors (refer to chapter 4 of Lakoff and Johnson, 1980), and one of them is "more is up." Therefore, quantities that become greater go up, and it seems straightforward to say "it grows". However, we do not use "growth" for everything that goes up; the interest rate, the unemployment rate, prices, the sea level, global temperatures, and the fraction of intensive care beds occupied by patients with COVID can increase, but we do not say "they grow." Further, when such quantities become less, we do not speak of decay or crisis or even collapse, but of decrease.

Desirable qualities of a taxonomy for classifying behaviors

Since words are important features tagged onto concepts, and concepts are what we have available to recognize things in the world and make sense of them, it is important to have a set of defining items that comply to rules minimizing the cognitive load imposed on newcomers. A *well-defined* and *clearly understandable* set of modes—here referred to as *taxonomy* because it establishes a set of categories—can provide truly useful guidance. We describe a behavior mode using three items:

- A name.
- A description (text, possibly including mathematical expressions) of the mode's characteristic features that are used to identify the mode's pattern in the behavior of a variable.
- A behavior graph with the characteristic shape(s) as an iconic representation of the mode's characteristic pattern(s).

Then, a taxonomy of behavior modes should satisfy some criteria related to usefulness (necessary) and usability (desirable).

1. *Consistency*. To be well-defined, two aspects of consistency are necessary.
   1.1. *Matching consistency*. Any behavior observed over a time interval must be matched by exactly one *mode*. Thus, the set of modes must be (a) exhaustive and (b) free of superpositions. Therefore, the *description* needs to contain actionable information for each of the feature types (e.g., "slope" or "change of slope"). The only exception could be that in a hierarchical taxonomy, one mode and one submode match a behavior.
   1.2. *Internal consistency*. Further, the defining items need to be internally consistent (free of contradictions) and free of ambiguities. The *names* of distinct modes shall only have similarities among one another if these similarities exist in the respective patterns. There also should be a 1:1 relationship between pattern feature and name.
2. *Usability*. To be clearly understandable for newcomers, the taxonomy should be free of unnecessary aspects (for the sake of cognitive economy).
   2.1. *Perceptive congruence*. The mode *descriptions* should describe salient features of behaviors such as visual cues, which are usually accounted for intuitively by newcomers, which spares newcomers unnecessary cognitive load.

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2.2. *Domain compatibility*. For new modelers with prior knowledge from other domains (e.g., management, economics, health, environment), the *name* labels need to be minimally different from existing taxonomies used in these domains.
2.3. *Detachment from causes*. For complete novices (e.g., school kids, users and clients, or citizens at large), the *names* and the *descriptions* should minimize dependency on prior technical knowledge to avoid unnecessary activation of system 2.

Ford (1999) contributed to criteria 2.1 and 2.3 when he argued that definitions of behavior patterns should be free of reference to the generating structure; this clearly favors consistency and avoids having to know the standard formulations of causal structure. Yet, the SD literature contains several taxonomies of behavior modes that are not mutually compatible, going counter to criterion 1, as shown below. Independently from SD, other areas of study use different sets of modes; even if the difference is only the name given to a behavior mode, the intention to establish a single name label will increase the cost of learning the SD behavior modes and collide with criterion 2.

Critical examination of published taxonomies for classifying behaviors

Three published taxonomies

The SD literature consistently distinguishes between simple behavior modes that cannot be decomposed in simpler components and composite modes consisting of concatenated, simple modes. Richardson and Pugh (1989) already mentioned that one variable can display a series of behavior modes in consecutive time intervals. Taxonomies of behavior modes are implicit in the entire SD literature; most textbooks progress from simple structures that drive simple behaviors toward more complex structures and behaviors. However, the names given to modes and the defining descriptions are not always the same. This becomes visible when comparing the taxonomies in several publications that this section analyzes.

Ford's atomic behavior patterns

By the end of the 20th century, (Ford, 1999) proposed a set of three "atomic" patterns based on the second derivative of the behavior, which "describes the movement of the net rate of change," as shown in Table 2.

Table 2: Ford's atomic behavior modes.

Label	Description
Linear	The absolute value of the net rate of change remains constant; therefore, the *speed of change* over time is *constant*. (Second derivative = 0.)
Exponential	The absolute value of the net rate of change increases; therefore, the *speed of change* over time is *increasing*. (Second derivative >0.)
Logarithmic	The absolute value of the net rate of change decreases; therefore, the *speed of change* over time is *decreasing*. (Second derivative <0.)

This taxonomy was meant for the search for dominant feedback loops in the causal structure of a problem because of the direct link from reinforcing and balancing loops to the *exponential* and *logarithmic* behavior patterns, respectively. Yet, the "atomic" taxonomy is also an attractive information source for newcomers; but for them, it has several problematic aspects.

Ford's exemplars accompanying the definitions considered only cases of increase (positive slope), but the definitions also apply for negative slopes. The increasing curves in Figure 3 (solid lines) correspond to Ford's exemplars; the decreasing curves (dashed lines) are added for the discussion:

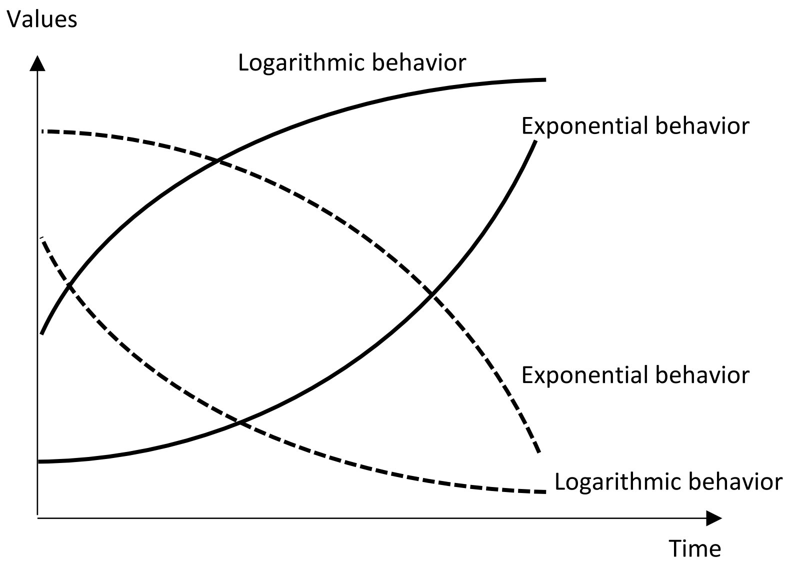

Figure 3: Ford's atomic behavior patterns with positive and negative slopes.

The descending version of the behavior patterns shows an accelerating decrease as *exponential* behavior pattern and a decelerating decrease as *logarithmic* behavior pattern. The adjective describing the acceleration of change over time is used consistently; acceleration is called *exponential*, and *logarithmic* describes deceleration. However, this collides with the two other taxonomies analyzed here (below).

Table 3: The atomic behavior patterns as seen by the quality criteria.

Criterion	Evaluation
Consistency	
Matching consistency	Covers all elementary possibilities of second derivative.
Internal consistency	No contradiction detected.
Usability	
Perceptive congruence	First derivative (slope or direction of change) is salient for people, but not accounted for.
Domain compatibility	The descriptions are abstract. Newcomers will not need to replace their usual vocabulary. However, *exponential behavior* with a negative slope contradicts the widespread definition of exponential decay.
Detachment from causes	No links to causal structure detected.

Table 3 shows how the atomic pattern taxonomy appears through the lens of the criteria introduced above. The taxonomy is internally consistent, but its usability is debatable because it makes no use of the slope (first derivative), which is arguably a salient feature of curves. Additionally, the use of *exponential* collides with the widely used *exponential decay* for negative slopes. Finally, the label *logarithmic* may cause some confusion for *goal-seeking* behavior; if one understands *logarithmic* in its strict mathematical sense, then *goal-seeking* behavior would not classify for this label.

Barlas and co-authors on basic behavior patterns

Behavior patterns also play an important role in model validation tests to check if a variable's simulated behavior replicates important features of a reference mode sufficiently well. Barlas and co-authors have therefore developed a method for automated pattern recognition based on a set of basic dynamic patterns which they ordered in six classes (Barlas and Kanar, 1999; Boğ and Barlas, 2005):

- Constant

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- Growth: linear, exponential, negative exponential and s-shaped.
- Decline: linear, exponential, negative exponential and s-shaped (this class was not spelled out in the original articles, so we filled the labels in).
- Growth-then-decline: growth-then-exponential decay to zero, growth-then-exponential decay to non-zero and growth-then-crash.
- Decline-then-growth_ s-shaped decline and s-shaped growth.
- Oscillatory: constant, with growth and with decline.

The pattern recognition method extracts the first and second derivative of the data describing a variable's behavior, building one separate segment for each pattern. First order and second order polynomials were used to approach the shape of each original segment without letting random variations misjudge the respective pattern. Such processing is certainly beyond the possibilities of modeling novices, but these authors' purpose was automating the process rather than making it easier to learn for novices. However, novices can learn and practice with stylized data and curves, and a simpler method can be applied to recognize slope and curvature in stylized curves.

Table 4: The atomic behavior patterns as seen by the quality criteria.

Criterion	Evaluation
Consistency	
Matching consistency	Covers all elementary possibilities of the first and second derivative.
Internal consistency	S-shaped patterns can be further decomposed, and so can the more complex patterns with diverse sequences of growth and decline.
Usability	
Perceptive congruence	First derivative (slope or direction of change) is salient for people, but not accounted for.
Domain compatibility	The descriptions are abstract. Newcomers will not need to replace their usual vocabulary. However, in *negative exponential growth*, the label *negative* is redundant with a negative slope and can easily be confounded. Further, in the *decline* class, the two elementary non-linear curves with negative slope would be labeled *exponential decline* and *negative exponential decline*, which is even easier to become confounding. Additionally, *exponential decline* risks to be confounded with the technical term exponential decay with a negative slope contradicts the widespread definition of exponential decay.
Detachment from causes	The label *exponential* refers to mathematical functions representing the structure beneath the observable behavior pattern.

The following Figure 4 illustrates why the pattern names risk to be confusing for novices, who will certainly perceive the slope but not necessarily the curvature. Albeit systematic and consistent in its hierarchical ordering into classes representing slope and class members representing curvature, the visual clues make come the word "negative" to mind for the class decline, so there are two novice questions: why is one pattern called "negative" if it represents *growth*? And why is only one pattern in decline explicitly labeled "negative"? The *s-shaped* patterns in both classes are equivalent to two of the previous patterns in the same respective class, so they can be considered as redundant in the *growth* and *decline* classes. Also, the *growth-and-decline* and the *decline-and-growth* classes are compositions of various patterns already defined in the *growth* and *decline* classes.

Note that this will only be a risk if these pattern classes and its class members are alienated from their intended use and proposed to modeling novices. Yet, in the absence of a dedicated learning material for novices, and considering that these publications are freely available on the Internet, any novice in search of guidance can easily find, download and try to follow them.

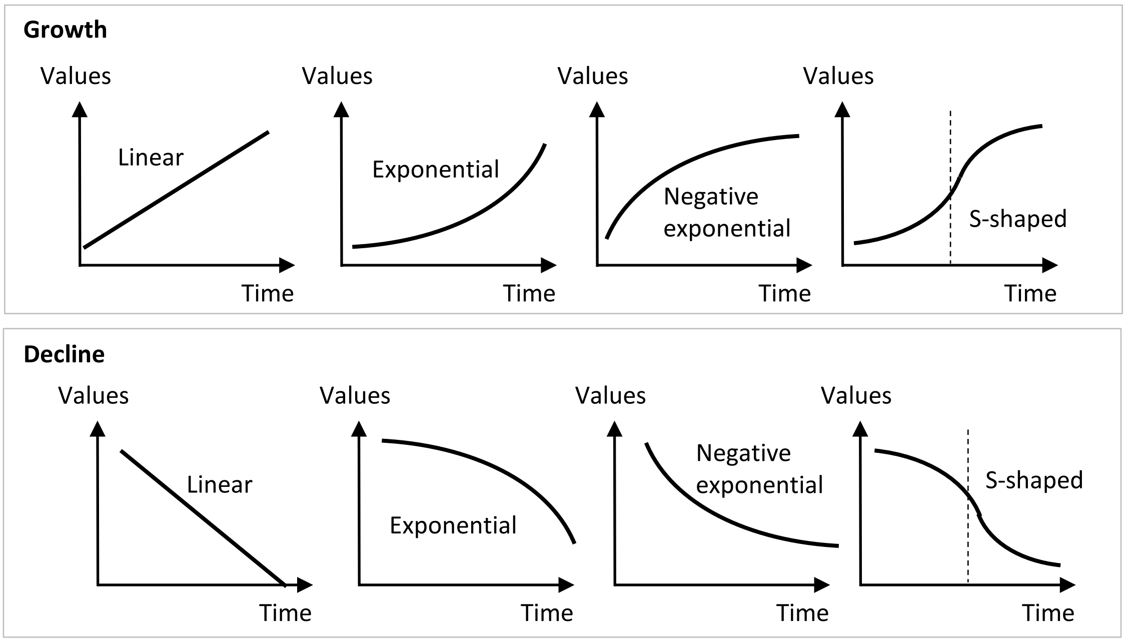

Figure 1: The basic dynamic patterns growth and decline used by Barlas and co-authors. The curves represent the cases labeled (a) through (c) in Fig. 2.1 in Barlas and Kanar (1999)

Sterman's fundamental behavior modes

In one of the most prominent textbooks, Sterman (2000) stated that the "most fundamental modes of behavior are exponential growth, goal seeking, and oscillation," (p. 108). *S-shaped growth*, *S-shaped growth with overshoot and oscillation*, and *overshoot and collapse* are referred to as "common" modes of behavior. Learners make their first contact with the behavior modes as curves in graphs with positive slope, which are represented in Figure 5:

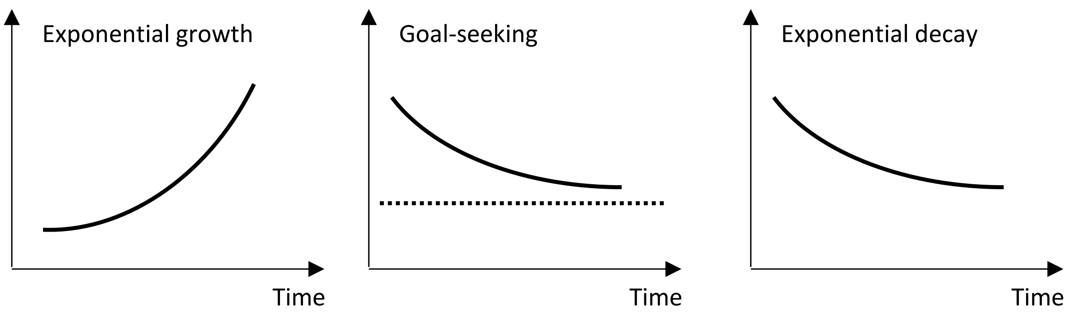

Figure 2: The fundmental and most common behavior modes in the business dynamics textbook

Each of these modes is linked to a "simple feedback structure." Later, the exponential function generates *exponential growth* (p. 265) and *exponential decay* (p. 274). Discussing accelerating change using instances of increase is a tradition; since Malthus published his "Essay on the Principle of Population" in the late 18th century, the mathematical representation of the phenomenon—the exponential function—has led to the label *exponential growth*. When the values of a variable decrease at a rate proportional to the current level of the variable, the label changes to *exponential decay*, but then change decelerates instead of accelerating.

Even if the textbook discusses virtually any shape of behavior, including collapse, readers must read far beyond the section where the fundamental modes are first announced. When judged through the criteria introduced above (see Table 5), the lack of modes for accelerating decrease and for decelerating increase is likely to decrease the matching consistency. With respect to usability, the lack of slopes reduces perceptive congruence and so does the fact that two modes describe a decelerating decrease. Additionally, the name label *goal-seeking* implies structure concepts, which collides with detachment from causes.

Table 5: The fundamental modes as seen by the quality criteria.

Criterion	Evaluation
Consistency	
Matching consistency	Covers accelerating increase and decelerating decrease, but not decelerating increase and accelerating decrease Decelerating decrease matches with two modes (*goal-seeking* and *exponential decay*).
Internal consistency	No contradiction detected
Usability	
Perceptive congruence	The direction of change (slope) is not accounted for, and the unilateral slopes in the graphs may induce to neglect the respective other slope. *Behaviors* with decelerating increase can be matched with two example graphs (each belonging to a different *mode*); the slope matches with one of them, whereas the change of slope matches with the other. The same holds for *behaviors* with accelerating decrease.
Domain compatibility	The name labels are congruent with the widely used terms *exponential growth* and *exponential decay*. The word "growth" will not always fit with concrete variables (e.g., the sea level). *Goal-seeking* presupposes some familiarity with control, and it may feel extraneous in situations where there is no explicit goal.
Detachment from causes	*Goal seeking* alludes to the underlying causal structure; this can be helpful to recover the link between a behavior mode and the causal structure driving it, but only if you already know the underlying set of causal structures. However, this should not be assumed of newcomers from other areas or individuals without prior mathematical training.

Behavior modes in Ford's system dynamics glossary

Approximately 20 years later, Ford (2019) published a glossary of SD terms and concepts, establishing *en passant* yet another set of behavior modes as parts of the glossary: *asymptotic growth/decay*, *exponential behavior*, *goal-seeking behavior*, *exponential growth (or collapse)*, and *exponential decay*. Even though it is not the purpose of a glossary to establish a taxonomy for classifying variable behaviors, the fact of including a series of behavior modes as part of the glossary makes them available for readers to use for that purpose. Novices are likely to use this glossary as an orientation aid. Table 6 summarizes the characteristic traits.

Table 6: Ford's glossary entries regarding behavior modes.

Label	Description (emphasis in quotes added by the author)
Exponential behavior	The "*change* in a stock variable is *proportional to the size of the variable itself*." (p. 373).
Exponential growth (or collapse	"[…] occurs when the *rate of increase or decrease* in a stock variable is *proportional to the size of the stock* at that point in time, so as to *accelerate its change*." (p. 373).

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Exponential decay	"[…] occurs when the *rate of increase or decrease* in a variable […] *is proportional to how far the stock is from its equilibrium*, so as to *slow down its rate of change*." (p. 373).
Goal-seeking behavior	"[…] the system moves towards an equilibrium or target condition. The *flow that changes the stock* value is typically modeled as a *fraction of the difference* between the equilibrium condition (or target) and the current condition. Therefore, the further the system is from the goal, the more it changes towards that goal and as it approaches the goal, the *increase or decrease slows*." (p. 373).
Asymptotic growth/decay	"[…] goal-seeking behavior produced by negative feedback. The control stock *moves towards the goal*, *slowing down* as it approaches the goal." (p. 369).

Two of the name labels contain the word "behavior," and one can safely interpret that *growth*, *decay*, and *collapse* are just particular types of behavior. The labels appear to imply a hierarchy based partly on the first derivative and partly on the second (as opposed to the case of the atomic behavior modes). Yet, a closer examination reveals some difficulties.

According to these definitions, *exponential growth or collapse* refers to accelerating change over time with a positive or a negative slope, which is equivalent to *exponential behavior* in the atomic behavior modes. However, now *exponential behavior* allows *exponential growth* and *exponential decay* (which is equivalent to Sterman's fundamental modes), but not *exponential collapse*. The new definition of *exponential behavior* does not specify if change over time increases or decreases. However, from the fact that the rate of increase or of decrease is proportional to the stock level, it follows that change accelerates for increases and decelerates for decreases.

This is equivalent to using the exponential function—a mathematical representation of causal structure—to replicate *exponential behavior*. However, labeling the behavior modes based on the mathematical representation of the causal structure beneath the behavior collides with the call for defining the behavior modes without reference to the causal structure. Further, the use of "exponential" is not consistent across the published definitions, which can confuse newcomers.

The behavior modes *asymptotic growth/decay* and *exponential decay* are identical, with only two minor differences: (a) "equilibrium" and "goal" are only identical if one supposes that an equilibrium may exist without an explicit goal, and that "goal" implies an explicit statement, and (b) decay always implies a negative slope. Both are synonymous to *goal-seeking* behavior (which implies the concept and existence of a goal, and the name label speaks of the causal structure instead of the behavior shape). In each case, change over time is slowing down (deceleration). Arguably, the term *asymptotic* is closer to the shape of behavior than *logarithmic* in the definitions of the atomic modes, which is semantically closer to the mathematical structure.

The assessment of the criteria shown in Table 7 suggests that both consistency and usability have weaknesses.

Table 7: The modes in the glossary as seen by the quality criteria.

Criterion	Evaluation
Consistency	
Matching consistency	Decelerating decrease behaviors match with *exponential decay*, *asymptotic decay*, *goal-seeking*, and *exponential behavior*. Decelerating increase behaviors match with *asymptotic growth* and *goal-seeking*.
Internal consistency	The hierarchical relationship between "behavior" and "growth" and "decay" and "collapse" is not made explicit.

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	"Exponential" in *exponential collapse* contradicts "exponential" in *exponential behavior*.
Usability	
Perceptive congruence	The first and the second derivative are accounted for. The name labels are congruent with the widely used terms *exponential growth* and *exponential decay*, except for *exponential collapse*.
Domain compatibility	The word "growth" will not always fit with concrete variables (e.g., the sea level). *Goal-seeking* presupposes some familiarity with control, and it may feel extraneous in situations where there is no explicit goal.
Detachment from causes	*Goal seeking* alludes to the underlying causal structure; this can be helpful to recover the link between a behavior mode and the causal structure driving it, but only if you already know the underlying set of causal structures. However, this should not be assumed of newcomers from other areas or individuals without prior mathematical training.

Critical appraisal of the three taxonomies

The taxonomies have some relevant differences between one another. First, there is diversity in the salient features accounted for, which is important for the *perceptive congruence* criterion. Referring to the salient features of behaviors to be classified, the "atomic" taxonomy uses only the second derivative in its descriptions and only positive slopes in its graph examples. The "basic dynamic patterns" use both slope and curvature, but they use name labels that risk confusion and have redundant patterns. The descriptions of the "fundamental" modes use the second derivative, and names and graphs limit themselves to unilateral slopes. The modes included in the "glossary" account for the first and second derivatives in names and descriptions.

Second, the use of the term *exponential* in the name labels of the modes is diverse. In the "atomic" modes, *exponential* is equivalent to "acceleration" for positive and negative slopes. The "fundamental" modes use *exponential* as equivalent to "proportional change" for positive and negative slopes, leading to acceleration for positive slopes and deceleration for negative ones. In the "glossary," *exponential* stands for "acceleration" in *exponential growth and collapse*, but it means "proportional change" for *exponential behavior*, *exponential decay*, and *exponential growth*.

Figure 4 recapitulates the matching consistency of each taxonomy. It shows a curve for each mode identified by name, description, and graph. The "fundamental modes" include two dotted curves because the taxonomy does not identify their shape. The "system dynamics glossary" has several modes for two of the shapes, violating the criterion.

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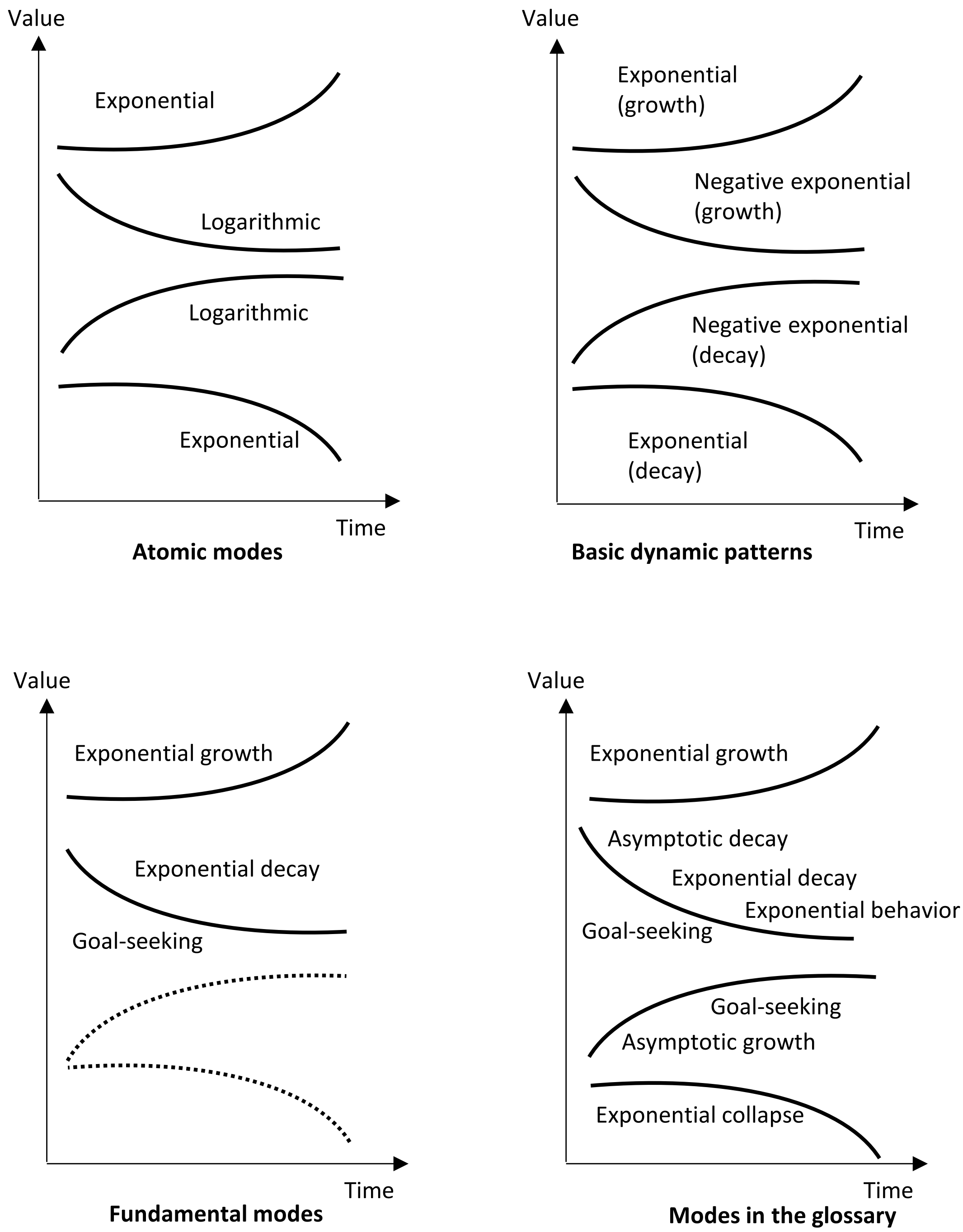

Figure 3: The matching consistency of each behavior mode taxonomy

Rounding up, consider which mode(s) the taxonomies would assign to the initial example of behavior (Figure 1). The gray curve in Figure 5 smooths the weekly ups and downs in the daily number of reported COVID cases and suggests three different behavior episodes. First, it increases at an accelerating rate during the first 4 weeks. Second, in the following 2 weeks, the increase slows down. Third, in the last week, it starts to decrease progressively. All three taxonomies would lead novices to classify the first episode *exponential growth*. The second episode is different; the "atomic" taxonomy suggests calling it *logarithmic*, whereas it is *asymptotic growth* or *goal-seeking* according to the "glossary." Our lecturing experience with Chilean freshmen over the past 18 years suggests that novices easily get used to identifying *exponential growth*, but they need considerable effort to learn

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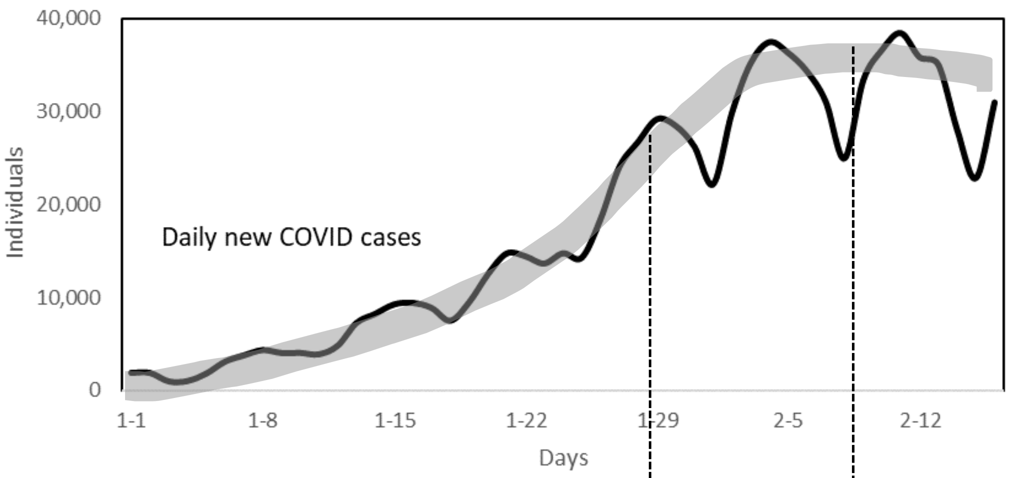
the modes for the second episode. The "fundamental" modes' description allows classifying this episode as *goal-seeking*, even if the graph given in the definitions does not show a curve with positive slope. The third episode is *exponential* in the "atomic" taxonomy and *exponential collapse* according to the "glossary." Again, practical lecturing experience suggests that learning this takes individuals without prior mathematical knowledge great effort, as also the experiments described below confirm. There is no classification for this shape in the taxonomy of "fundamental" modes.



Atomic	Exponential	Logarithmic	Exponential
Basic	Exponential	Exponential negative	Exponential
Fundamental	Exponential growth	Goal-seeking?	?
Glossary	Exponential growth	Asymptotic growth	Exponential collapse

Figure 4: Classification example

Mathematical functions as name givers

The predominance of the terms *exponential* and *logarithmic* may mislead novices for another reason, too; contrary to what many modeling newcomers might expect, the mathematical functions with these names can generate a wide range of behavioral shapes. There is no 1:1 relationship between these behavior shapes and the specific functions. To see this, consider the following curves generated by an exponential function, a power function, and a logarithmic function. The following instructions (using a Mathematica notebook) generate the curves displayed in Figure 6.

- Exponential: 1 * 1.1^n, 1 * 0.5^n, −1 * 1.1^n, −1 * 0.5^n, running n from 0 to 10
- Power: 1 * 1.1^x, 1 * 0.5^x, −1 * 1.1^x, −1 * 0.5^x, running x from 0 to 10
- Logarithmic: 1 * Log[x], −1 * Log[x], running x from 0 to 10

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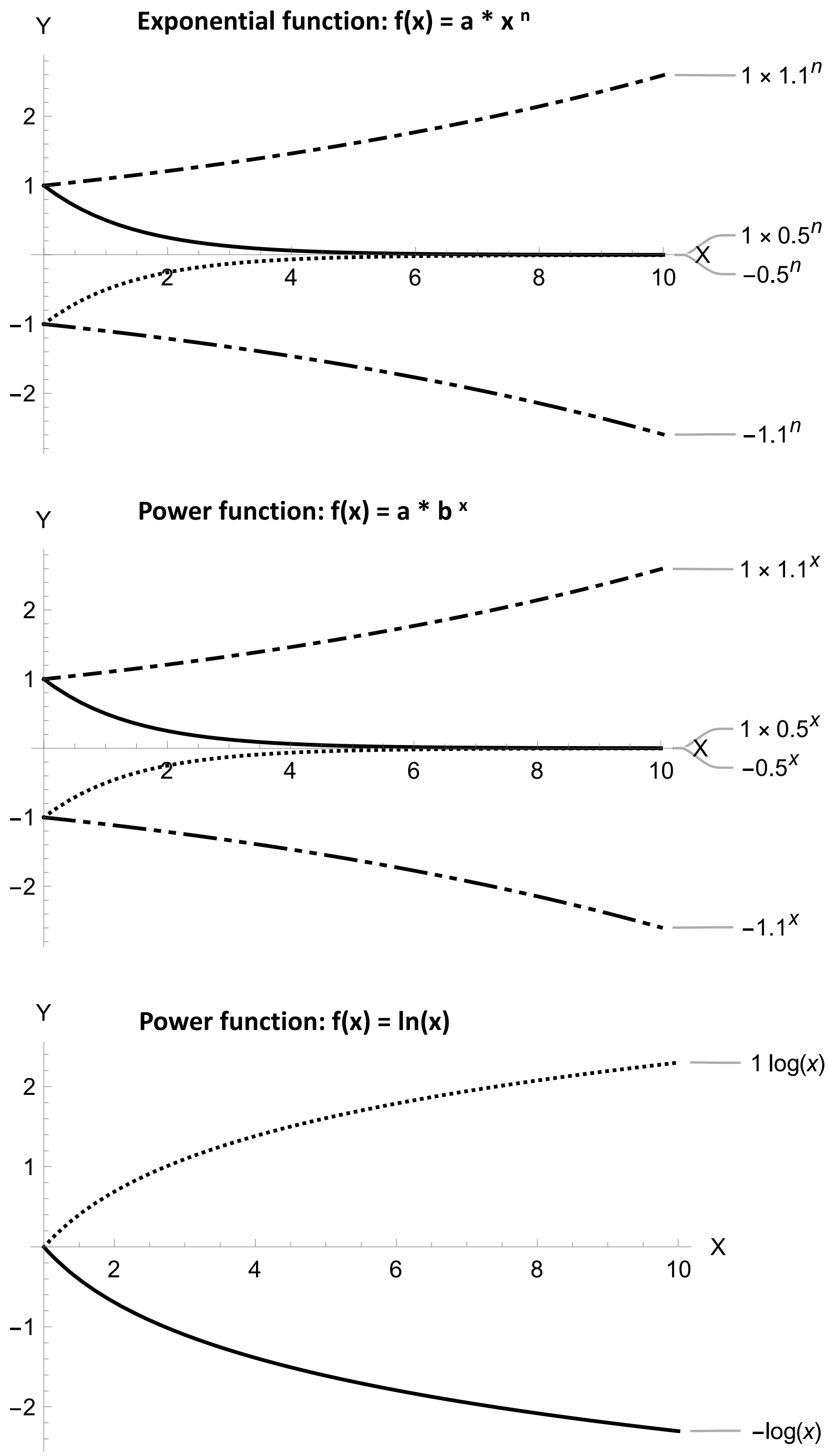

Figure 5: Different types of functions can generate very similar curves

Figure 6 shows a dashed-dotted curve when the exponential function and the power function apply a positive rate of change equal to 0.1 to an initial value of 1 and a dash-dot-dot curve when the initial value is −1. The reader may mentally simulate a situation where only the curve is visible (no hint at the function type). The black and the dotted curves represent what happens if one uses an exponent smaller than 1. Visibly, the logarithmic

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atomic pattern (asymptotic mode in the glossary) can also be generated by an exponential function or a power function.

By all means, the degree of abstraction of these curves is higher than the equations in an SD model. We use the X axis to represent time; however, the mathematical functions are independent of these considerations.

The point of showing these curves is that when you can only see the curve, there is no hint to the causal structure in the shape itself. Accordingly, it is debatable what the advantage of using structural terms in the names of behavior modes may be; we argue that calling accelerating change "exponential" is opaque for someone who is not used to the mathematical functions and implies a link to one mathematical formulation in particular. Turning to decelerating change, the label "logarithmic" can be even misleading.

A general set of elementary behavior modes

After identifying and analyzing the weak points of the prior definitions, this section proposes a taxonomy that is exclusively based on the directly observable features of a variable's behavior, does not impose new labels on practitioners of other areas, and that is internally free of ambiguities and contradictions. The term "elementary" avoids repeating "atomic" or "fundamental." It makes sense because these modes are the elements used to organize the overall behavior of variables. Other modes such as *S-shaped growth* or *overshoot and collapse* are referred to as the "compound" modes.

There are six elementary behavior modes of change, graphically represented in Figure 7. A seventh mode is "steady state," but it is trivial to recognize and not further discussed here. The taxonomy uses two features: (a) the direction of change and (b) the behavior of the speed of change (or curvature). Both features are salient in BOT graphs and easy to detect, even in tabular form. In the figure, dv refers to the change of value, and dt stands for the change of time from period to period. Subindices represent the points in time, so for example, $dv_{1,2}$ is the change of value observed during $dt_{1,2}$, that is, between moment 1 and moment 2. Then, the direction of change is the sign of $dv_{t,t+1}$ (remember that this is for humans looking at reported data, not for a computational model simulating continuous time). It is also easy to decide if the speed of change is constant, increasing or decreasing—independently of the direction of change:

If $dv_{t,t+1} = dv_{t+1,t+2}$, then constant.

If $\text{abs}(dv_{t,t+1}) < \text{abs}(dv_{t+1,t+2})$, then increasing.

If $\text{abs}(dv_{t,t+1}) > \text{abs}(dv_{t+1,t+2})$, then decreasing.

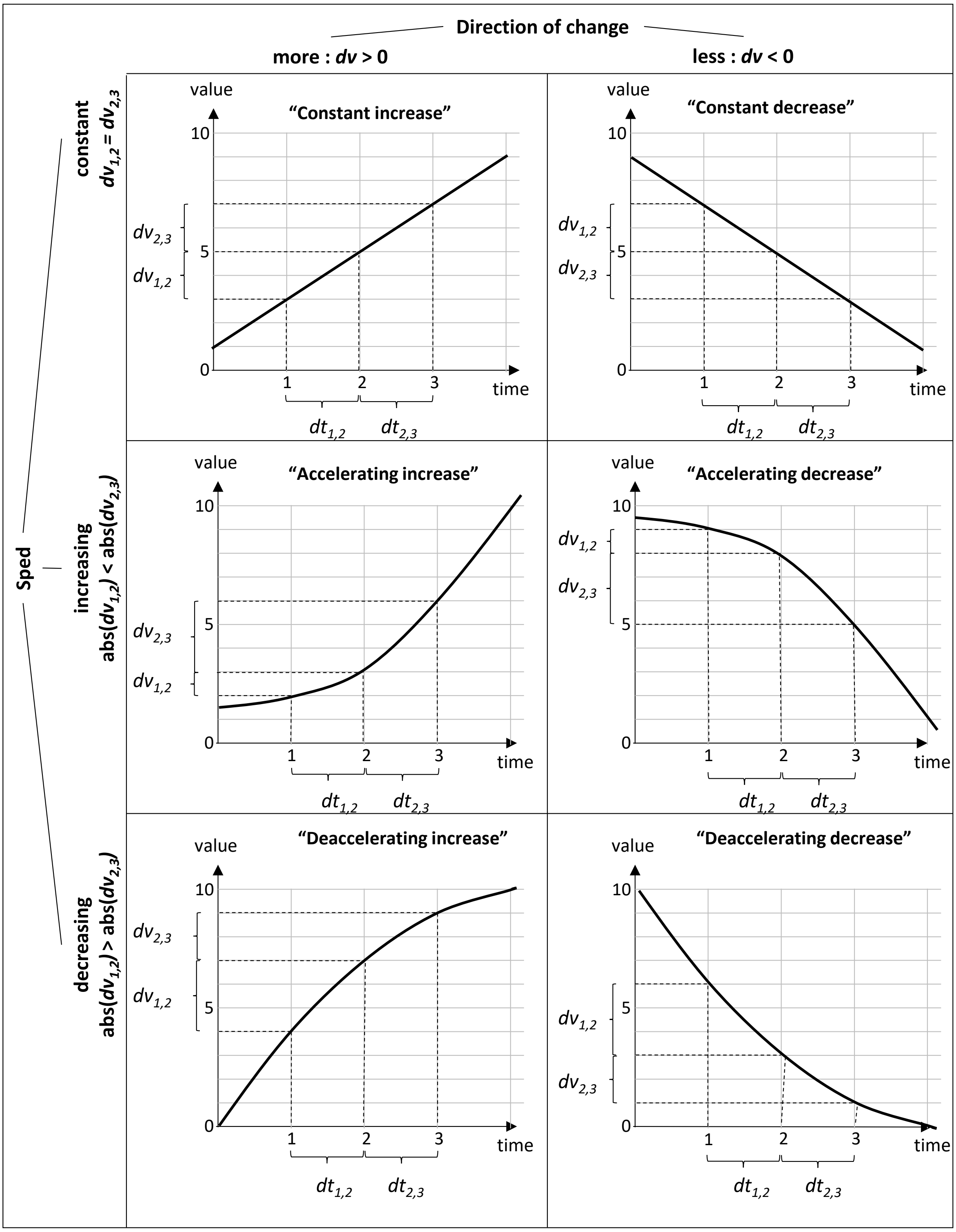

Figure 6: The elementary behavior modes with their generic names

With respect to *consistency*, the *matching consistency* criterion is satisfied; there is exactly one mode for each combination of direction and speed of change. The names of these modes are reminiscent of movement, figuratively answering the question "in which direction is this going, and at which speed?" The names, descriptions, and graphs are *internally consistent*. Turning to *usability*, accounting for slope (direction of change, first derivative) as well as for change of speed (second derivative) assures *perceptive congruence*. A list of synonyms (Table 8) assures *domain compatibility*, and *detachment from causes* is

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achieved by only referring to features of the behavioral shape to be classified: direction of change and speed of change.

Table 8: Synonyms of the elementary behavior modes.

Elementary behavior mode	Synonyms
Constant increase	Constant growth, linear growth
Accelerating increase	Exponential growth, explosive growth, expansion, recovery
Decelerating increase	Asymptotic growth, logarithmic growth, goal-seeking growth
Constant decrease	Constant decline, linear decline
Accelerating decrease	Collapse, exponential collapse
Decelerating decrease	Exponential decay, asymptotic decay, goal-seeking decline

Compliance with the criteria is higher than in the three taxonomies analyzed here, as summarized in Table 9.

Table 9: The elementary behavior modes as seen by the quality criteria.

Criterion	Evaluation
Consistency	
Matching consistency	There is a 1:1 relationship between each type of shape and the modes.
Internal consistency	No contradiction or ambiguities. The name label of each mode only refers to the visible features of the first and the second derivative, and there is no semantic collision between the words participating in the mode names.
Usability	
Perceptive congruence	It is expected that the first derivative is an intuitive feature even for absolute modeling novices; the second derivative may not be intuitive, but it is identified with precision in the mode descriptions. It remains to be seen which features actually are intuitive by empiric research.
Domain compatibility	The synonyms conserve domain compatibility.
Detachment from causes	The names and the descriptions of the modes do not contain elements implying or referring to causal structure.

Empirical experiments of identifying and classifying atomic behavior modes

Up to here, the article has discussed three established taxonomies and one proposed taxonomy devised by system dynamicists. However, how do naïve individuals (without technical prior knowledge) categorize curves displaying the behavior of variables? With two features, there are diverse possibilities: one can focus on the direction of change, or on the curvature, or on both features. Which features would naïve individuals pay attention to? A second question is which of the taxonomies is learnable with less effort?

Experimental design

To learn more about these questions and the difficulties modeling novices as naïve individuals have in identifying and classifying behavior modes, we ran a two-stage experiment, separated by a short break. In the first task, students had to group graphs together. They received a description of the importance of classifying behavior over time and then had to classify 12 graphs containing two cases of each elementary behavior mode as shown in Figure 7. The graphs were abstract, with only the label of the two axes (time and value). The two examples for each elementary mode differed only in the steepness of their curves. Experienced dynamicists will easily see that there are two instances of each

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elementary mode. However, as the numbering suggests, the examples come in an order that blurs the salience of the underlying modes; for instance, accelerating decrease appears in graph numbers 2 and 11, and decelerating increase is the mode in graph numbers 7 and 10.

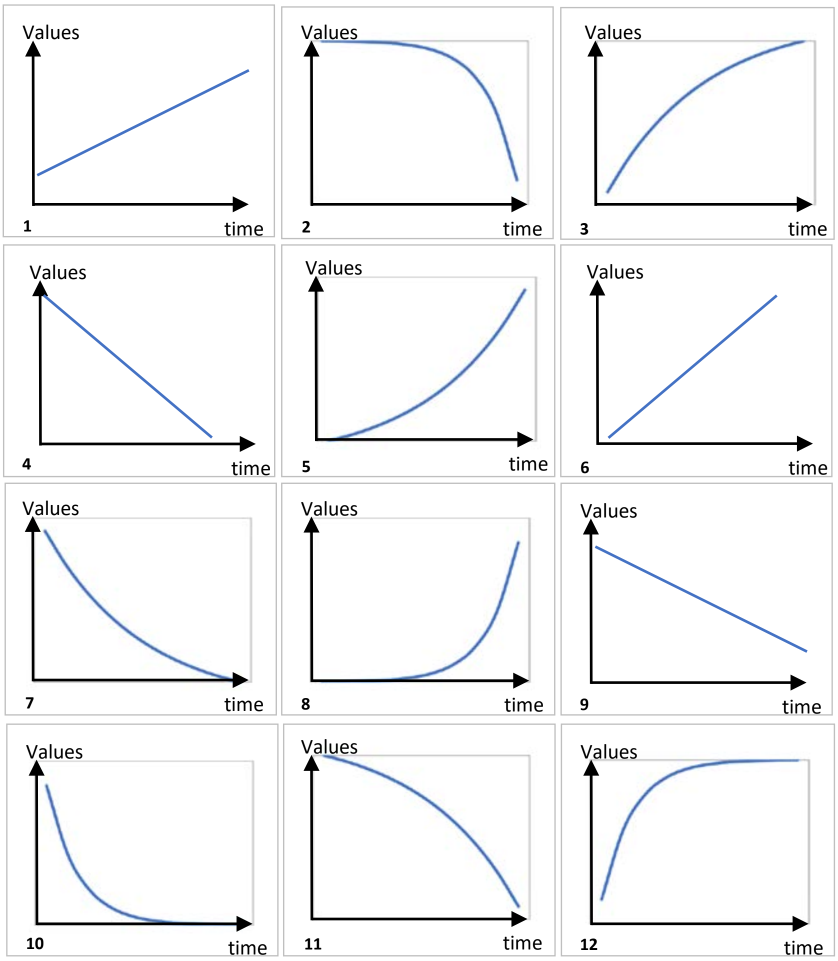

Figure 8: Twelve examples of behavior to be classified in task 1.

The students were free to form as many groups as they deemed appropriate and had to give each group a name and describe shortly the distinctive characteristics of each group. In the second task, they received three examples of accelerating increase labeled as "exponential growth" and three of decelerating increase labeled as "asymptotical growth." The task then was to draw a curve representing the opposite of each mode, which implies the modes accelerating decrease and decelerating decrease, respectively. The instructions of the experiment can be found in the supplementary material.

Participants were recruited from the pool of undergraduate students in business informatics at the University of Talca. Overall, 49 students in their first and second years of study with no prior knowledge of SD participated in the experiment.

Results

Of the 49 participants, less than half correctly classified six groups, suggesting that their categorization accounted for both defining features of the elementary modes (the direction of change and the change of speed or curvature), as shown in Table 10. However, actual recognition was lower than that: from the 24 students who correctly identified six groups, only 19 assigned them correctly. Furthermore, four students assigned graphs to more than one groups. For individuals who focus on only one of the defining features, it

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would have been logical to classify the 12 graphs into two groups; yet, only one student did so. This means that half of the students cannot have grouped the 12 graphs according to the defining features of the elementary modes.

Table 10: Results from classification exercise.

No. of groups	No. of students	Percentage
2	1	2.0
3	10	20.4
4	11	22.4
5	3	6.1
6	24	49.0

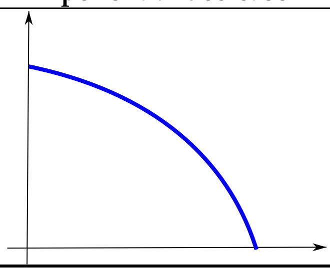
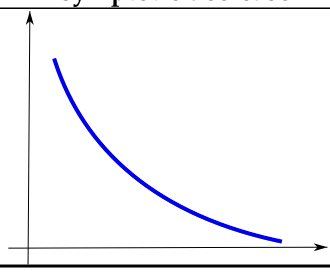
This interpretation solidifies when considering that there was a great variability in the naming of the groups: from mere numbering to phantasy names and actually employing some mathematical description. The mathematical description ranged from "steady rise," "steady drop," "rising graph," "partial descending graph" to "linear graph," "constants," "exponential graph," or "logarithmic graph," and some included references to "wave" or "fluctuation." Aside from non-characterizing labeling, one could identify a greater number of students who actually tried to express with their labeling the characteristics of the group; however, this only resulted in correct labeling in some cases, and not necessarily for the students who rightly identified six groups.

The characterization of the group revealed that only very few students could describe their groups accurately. It was easiest for the group with linear growth, where a substantial share described it correctly. For the group with exponential growth, some correctly identified and described the observed patterns, but for the remaining groups, the potential for an accurate description was very low. Overall, the students did not achieve matching or internal consistency.

We also found several judgmental expressions in the group's description, for example, "subtle," "not possible to stabilize," "controlled growth," "drastic fall," or "success." There has also been one incident where the labeling of the y-axis with "v" (for value or "*valor*" in Spanish) has been mistaken for sales ("*ventas*" in Spanish), together with some corresponding interpretation for the groups.

In the second task, we asked the students to draw one curve for exponential decay and one for asymptotic decay. Table 11 shows the summary for the two graphs, together with the respective frequency. Note that some students basically used the same behavior mode for both cases.

Table 11: Results from drawing task.

Exponential decrease[1]	#	Asymptotic decrease	#
	23		19

[1] The English translation here is somewhat different from the logic of the selected Spanish wording. While exponential growth ("*crecimiento exponencial*") is unambiguous, the opposite ("*decrecimiento exponencial*") could be translated as either exponential collapse or exponential decay, which are, of course, very different phenomena. While this fact could have contributed to some of the confusion in the experiment, it only reinforces the central point of the paper.

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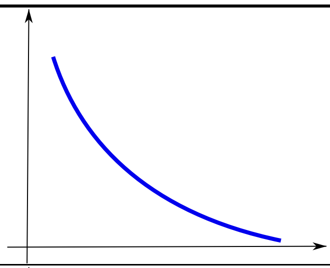
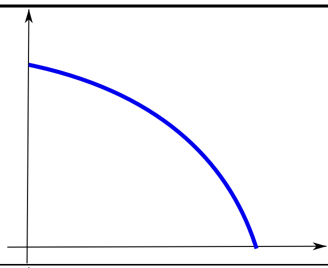
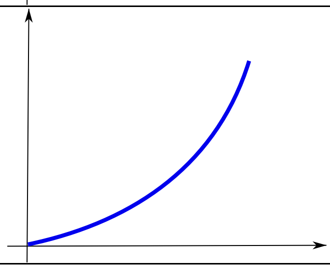
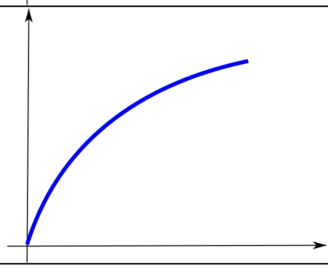
	18		18
	6		5
Other	2	Other	7
Total	49	Total	49

Discussion of results

The findings appear to be in line with reported difficulties of students' recognition of exponential growth (Ellis *et al.*, 2012). Furthermore, as Thompson and Carlson (2017) stated, students might need to have concepts of function in mind to think about integration, differentiation, and the organic relationship between them in efficient ways. One might speculate whether students need to make some approximations of differentiation before they can sort the graphs into coherent groups, or whether there are yet unknown mechanisms at work. What remains furthermore unclear is how students envision quantities or variables varying (Thompson and Carlson, 2017). It appears to be the case that detecting "correct" patterns also requires covariational reasoning (Ferrari-Escolá, Martínez-Sierra, and Méndez-Guevara, 2016) on the part of the students.

Regarding the second task, Paoletti (2020) argued that students may assign different meanings to graphical and analytical representations of inverse functions. Specifically, students may make an "in-the-moment functional accommodation to their inverse function meanings in the graphical context" (Paoletti, 2020). In our example, however, it is not necessarily true that the students recognize the possibility to inverse a function, or that the "two meanings" of the graphical and verbal representation led to assignment errors; still, the high failure rate in the second task and the naming of the groups in the first task suggest that the mapping between verbal and graphical representations is a difficult task for the students.

Presmeg and Nenduradu (2005) supported this finding by showing problems in transferring algebraic and graphical representations for an exponential function. Difficulties in understanding exponential growth are well-documented, and even teachers appear to struggle in transferring mathematical knowledge as in analysis of exponential growth to recognize this phenomenon in nature or to generalize rules (Ellis *et al.*, 2016).

While there are no studies explicitly treating the problem of verbal versus graphical representation of a function, there are known differences in brain activities in accessing arithmetic facts, visual number form, and the verbal system (Dehaene and Cohen, 1995). Mathematical intuition, which might be at play in a crude form at least in the second experiment, may emerge from the interplay of linguistic competence and visuospatial representations (Dehaene *et al.*, 1999). If this is the case, an ex-ante knowledge of verbal and graphical representation of common behavior modes cannot be expected of participants, if they have not learned it beforehand and internalizing mathematical ability is characterized by a shift in activation in different cerebral areas (Zamarian, Ischebeck, and Delazer, 2009). This means that a highly consistent and usable scheme for identifying elementary behavior modes is even more relevant.

Discussion

At the merely conceptual level, the proposed taxonomy of elementary behavior modes has some advantages over the three previous taxonomies:

1. It covers all possible elementary combinations of the first and second derivatives.
2. There is one and only one mode for every possible combination.
3. The name labels are consistent with the combined derivatives and free of semantic links into the realm of causal structure or mathematical representations.
4. The synonyms make it tolerant of any discipline or application domain with a proper taxonomy.

However, is it worth the effort to revise the taxonomies already in use in SD? Although being from a small sample, our data suggest that at least students without some prior mathematical competence intuitively classify curves in ways that do not facilitate model conceptualization, and that the typical labels of behavior modes can indeed increase the difficulty of learning the required categories. Insofar as freshmen and second-year students represent the wider group of mathematically naïve individuals, there appears to be a need for reconsideration and improvement.

SD is not the only discipline with an established set of behavior modes for classifying curves. Researchers and practitioners in different areas deal with different problems and different context, looking at reality through different conceptual lenses. Using area-specific name labels for typical shapes of behavior allows reducing cognitive load and is therefore rational. The fact that the causal structures are isomorph and that the same mathematical formulations are used is secondary for the predominant work situations.

One may categorize behavior shapes into modes according to diverse criteria: the direction of change (slope, first derivative), change of speed (second derivative), or both. Use of the second derivative is typical in mathematics and revealing for the terms used. The sign of the second derivative can be positive, zero, or negative, but is called "concavity" in mathematics. If the second derivative is positive, the numerical representation of the slope is increasing—becoming more positive or less negative; this is called "concave up," and "concave down" refers to a curve becoming less positive or more negative.

In the case of economics, the business cycle is a concatenation of modes comprising four phases, and one can use the behavior modes to classify them, as shown in Table 12.

Table 1: Behavior modes of the business cycle

Phase	Modes			
	Atomic	Fundamental	Glossary	Elementary
Expansion	Logarithmic	Goal-seeking	Asymptotic growth, goal-seeking	Decelerating increase
Recession	Exponential	?	Exponential collapse	Accelerating decrease
Depression	Logarithmic	Goal-seeking	Asymptotic degrowth, goal-seeking, exponential behavior	Decelerating decrease
Recovery	Exponential	Exponential growth	Exponential growth, exponential behavior	Accelerating increase

The "atomic" taxonomy focuses on the acceleration behavior and leaves the direction of change out of consideration. This orients the search for generative causal structures; but at the same time, individuals interested in economics may feel a cognitive dissonance because there is no difference between increases and decreases in economic production. Economists may also wonder why the peak and the through of the business cycle would

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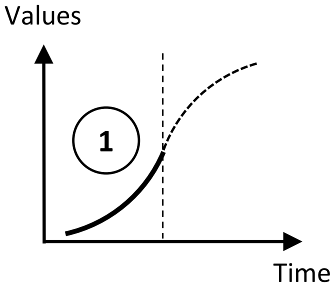
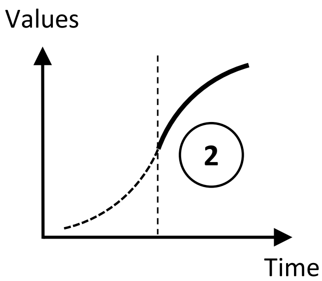
be goals, to speak of *goal-seeking* behavior. The definitions in the glossary are ambiguous; different modes match with the shapes and can be used to detect relevant patterns.

Another case are the diverse uses of *logistic growth* in economics, marketing, innovation, population dynamics, epidemiology, and others. The *S-shaped growth* curve can be decomposed in episodes, which can then be classified by the three sets of definitions, as shown in Table 13.

Table 2: Behavior modes of logistic growth

Episode	Modes			
	Atomic	**Fundamental**	**Glossary**	**Elementary**
	Exponential	Exponential growth	Exponential growth, exponential behavior	Accelerating increase
	Logarithmic	Goal-seeking	Asymptotic growth, goal-seeking, exponential behavior	Decelerating increase

Modelers interested in marketing, who know the Bass model, as well as public health modelers who know the SIR model, can doubtlessly fare well with the "atomic" or the "fundamental" modes. They may find it bewildering that the market saturation reveals an unstated goal, or that the goal of a virus would be to make the entire surviving population resistant. Again, the definitions in the "glossary" are ambiguous, as in the case of the business cycle.

Even if a single discipline such as SD adopts a taxonomy of behavior modes based on general criteria such as consistency and usability, it would be pointless to call all other disciplines to adopt that taxonomy. Rather, the SD field can teach modeling novices a taxonomy consisting of modes that are explicitly linked to the modes established in other disciplines. For instance, learning the elementary models may even facilitate learning the relevant mathematical structures or learning about the business cycle, as exemplified in Table 14.

Table 14: Elementary behavior modes with synonyms.

System dynamics	Other disciplines	
	Mathematics	**Economics**
Accelerating increase	Exponential growth	Recovery
Decelerating increase	Logarithmic growth	Expansion
Accelerating decrease	?	Recession
Decelerating decrease	Exponential decay	Depression

At this stage, further research to answer several empirical questions will help to advance the discussion.

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1. Which visual features do novice modelers use when they categorize behavior graphs when the variables shown are abstract (have no conceptual link into previously known application domains)?
2. Which visual features do newcomers from other disciplines (economics, management, public policy, health, ecology, etc.) use when they categorize behavior graphs when the variables shown are concrete (belong to a domain where their prior knowledge will guide their classification)?
3. Does the use of other taxonomies impair the recognition of behavior modes in transfer tasks? For example, suppose individuals are randomly assigned to one of two conditions, either the "elementary" taxonomy or one of the other taxonomies. They receive a set of examples with only positively sloped curves, but curvature varying between positive and negative. One possible task is then "draw a curve for each of the modes when the slope is negative," and another possible task is handing out examples with negatively sloped curves and prompt "give the corresponding names." Would individuals using the "elementary" taxonomy draw correct curves and give the correct name more frequently and/or in less time?

Conclusion

SD is characterized by examining the relationship between structure and behavior. Establishing this relationship requires practitioners to identify elementary behavior modes in time series, as these behavior modes can be directly linked to stock and flow structure. Establishing this link is important both in the early stages of model building and in the model's validation. Hence, this identification is one important piece in the overall modeling process.

However, the elementary behavior modes come in different names, stemming from their underlying mathematical function or from naming traditions within each field. The field of SD is no exception in this respect, with different published taxonomies for the behavior modes. In this article, we discussed the taxonomies of Ford (1999), Sterman (2000). (Barlas and Kanar, 1999) and Ford (2019) and highlighted similarities and differences. Examination of these prior taxonomies according to the criteria of consistency and usability has revealed some inconsistencies and usability limitations, as far as the needs of novices are concerned. While these inconsistencies such as differences in naming conventions may be of little concern for experienced modelers since they already know what is meant, an experiment with novices suggests that these limitations make a difference for those without such prior knowledge. We discovered that both the classification and naming of elementary behavior modes pose significant challenges for college students without prior knowledge of SD. Only about half of the students were able to correctly identify groups with identical elementary behavior modes, but without being able to properly discuss their characteristic features. Furthermore, a majority of them could not draw a graph of behavior modes corresponding to given information. It appears therefore plausible to conclude that differences in taxonomy can pose a significant hurdle for newcomers to SD.

We propose a new taxonomy for the elementary behavior modes, which is oriented at their mathematical properties of the first and second derivatives, but uses name labels that are semantically independent from mathematical concepts and only refer to visual clues. While persons with a profound mathematical knowledge or experts in SD might not find much novelty in this taxonomy, the new taxonomy is targeted at novices in SD without strong prior mathematical knowledge and teachers/university professors for introductory courses in SD. The proposed taxonomy can help both because of its simplicity and systematic application scheme and can be applied in fields with different naming traditions.

Several limitations need to be mentioned. First, the data from one experiment allow only cautious statements regarding novice difficulties in learning about behavior classification. Additional data about frequent classification errors or learning hurdles—especially with the proposed taxonomy—are called for. Second, proposing a taxonomy is effectless unless it becomes part of learning materials. This is an invitation for instructional designers and lecturers to develop such materials and collect evidence on their effects for novices.

We close by underlining that pattern recognition in SD remains a manual task, as it is an essential part of model conceptualization and building. Rather than intending to overthrow existing taxonomies, we hope that the proposed taxonomy can minimize some of the confusion novices in SD have in getting to know the central concepts of the field; easing the early learning work for beginners is an important first step toward high-quality modeling.

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