

# On duality principles and related convex dual formulations suitable for local non-convex variational optimization

Fabio Silva Botelho

Department of Mathematics

Federal University of Santa Catarina

Florianópolis - SC, Brazil

## Abstract

This article develops duality principles and related convex dual formulations suitable for the local optimization of non-convex primal formulations for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a Ginzburg-Landau type equation.

**Key words:** Convex dual variational formulation, duality principle for non-convex local primal optimization, Ginzburg-Landau type equation

MSC 49N15

## 1 Introduction

In this article we establish a duality principle and a related convex dual formulation suitable for the local optimization of the primal formulation for a large class of models in non-convex optimization.

The main duality principle is applied to the Ginzburg-Landau system in superconductivity in the absence of a magnetic field.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 13, 14] and on a D.C. optimization approach developed in Toland [15].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [9, 5, 6, 7, 12]. Finally, similar models on the superconductivity physics may be found in [4, 11].

**Remark 1.1.** *It is worth highlighting, we may generically denote*

$$\int_{\Omega} [(-\gamma \nabla^2 + KI_d)^{-1} v^*] v^* dx$$

simply by

$$\int_{\Omega} \frac{(v^*)^2}{-\gamma \nabla^2 + K} dx,$$

where  $I_d$  denotes a concerning identity operator.

Other similar notations may be used along this text as their indicated meaning are sufficiently clear.

Finally,  $\nabla^2$  denotes the Laplace operator and for real constants  $K_2 > 0$  and  $K_1 > 0$ , the notation  $K_2 \gg K_1$  means that  $K_2 > 0$  is much larger than  $K_1 > 0$ .

At this point we start to describe the primal and dual variational formulations.

Let  $\Omega \subset \mathbb{R}^3$  be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by  $\partial\Omega$ .

For the primal formulation, consider a functional  $J : V \rightarrow \mathbb{R}$  where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx \\ &\quad + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}. \end{aligned} \quad (1)$$

Here  $\gamma > 0$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $f \in L^2(\Omega) \cap L^\infty(\Omega)$ .

Moreover,  $V = W_0^{1,2}(\Omega)$  and we denote  $Y = Y^* = L^2(\Omega)$ .

Define the functionals  $F_1 : V \times Y \rightarrow \mathbb{R}$ ,  $F_2 : V \rightarrow \mathbb{R}$  and  $G : V \times Y \rightarrow \mathbb{R}$  by

$$\begin{aligned} F_1(u, v_0^*) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx - \frac{K}{2} \int_{\Omega} u^2 \, dx \\ &\quad + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2v_0^* u - f)^2 \, dx + \frac{K_2}{2} \int_{\Omega} u^2 \, dx, \end{aligned} \quad (2)$$

$$F_2(u) = \frac{K_2}{2} \int_{\Omega} u^2 \, dx$$

and

$$G(u, v) = \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta + v)^2 \, dx + \frac{K}{2} \int_{\Omega} u^2 \, dx - \langle u, f \rangle_{L^2}.$$

We define also  $F_1^* : [Y^*]^3 \rightarrow \mathbb{R}$ ,  $F_2^* : Y^* \rightarrow \mathbb{R}$ , and  $G^* : [Y^*]^2 \rightarrow \mathbb{R}$ , by

$$\begin{aligned} &F_1^*(v_2^*, v_1^*, v_0^*) \\ &= \sup_{u \in V} \{ \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_0^*) \} \\ &= \int_{\Omega} \frac{-K_1 f(-\gamma \nabla^2 + K + K_2) f + (v_1^* + v_2^*)^2 - 2K_1 f(-\gamma \nabla^2 + 2v_0^*)(v_1^* + v_2^*)}{2[K_2 + K - \gamma \nabla^2 + K_1(-\gamma \nabla^2 + 2v_0^*)^2]} \, dx \end{aligned} \quad (3)$$

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u \in V} \{ \langle u, v_2^* \rangle_{L^2} - F_2(u) \} \\ &= \frac{1}{2K_2} \int_{\Omega} (v_2^*)^2 \, dx \end{aligned} \quad (4)$$

and

$$\begin{aligned}
 G^*(v_1^*, v_0^*) &= \sup_{(u,v) \in V \times Y} \{-\langle u, v_1^* \rangle_{L^2} + \langle v, v_0^* \rangle_{L^2} - G(u, v)\} \\
 &= \frac{1}{2} \int_{\Omega} \frac{(v_1^* - f)^2}{2v_0^* + K} dx + \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 dx \\
 &\quad + \beta \int_{\Omega} v_0^* dx
 \end{aligned} \tag{5}$$

if  $v_0^* \in B^*$  where

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_{\infty} < K/8 \text{ and } -\gamma \nabla^2 + 2v_0^* < -\varepsilon I_d\},$$

for a small parameter  $0 < \varepsilon \ll 1$ .

Furthermore, we define

$$D^* = \{v_1^* \in Y^* : \|v_1^*\|_{\infty} \leq (3/2)K\}$$

and  $J_1^* : Y^* \times D^* \times B^* \rightarrow \mathbb{R}$ , by

$$J_1^*(v_2^*, v_1^*, v_0^*) = -F_1^*(v_2^*, v_1^*, v_0^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*).$$

Assuming

$$K_2 \gg K_1 \gg K \gg \max\{\|f\|_{\infty}, \alpha, \beta, \gamma, 1/\varepsilon^2\}$$

by directly computing  $\delta^2 J_1^*(v_2^*, v_1^*, v_0^*)$  we may obtain that for such specified real constants,  $J_1^*$  in convex in  $v_2^*$  and it is concave in  $(v_1^*, v_0^*)$  on  $Y^* \times D^* \times B^*$ .

## 2 The main duality principle and a concerning convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

**Theorem 2.1.** *Let  $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \in Y^* \times D^* \times B^*$  be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$$

*and  $u_0 \in V$  be such that*

$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*}.$$

*Under such hypotheses, we have*

$$\delta J(u_0) = \mathbf{0},$$

*and*

$$\begin{aligned}
 J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx \right\} \\
 &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\
 &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*).
 \end{aligned} \tag{6}$$

*Proof.* Observe that  $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$  so that, since  $J_1^*$  is convex in  $v_2^*$  and concave in  $(v_1^*, v_0^*)$  on  $Y^* \times D^* \times B^*$ , from the Min-Max theorem, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, v_0^*)}{\partial v_2^*} + u_0 = \mathbf{0}$$

and

$$\hat{v}_2^* - K_2 u_0 = \mathbf{0}.$$

Observe now that denoting

$$H(v_2^*, v_1^*, v_0^*, u) = \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_0^*),$$

there exists  $\hat{u} \in V$  such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = H(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}. \end{aligned} \tag{7}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \hat{u}.$$

Also, denoting

$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

from

$$\frac{\partial H(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, u_0)}{\partial u} = \mathbf{0},$$

we have

$$-\hat{v}_1^* + Ku_0 + \gamma \nabla^2 u_0 + K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) - \hat{v}_2^* + K_2 u_0 = \mathbf{0},$$

so that

$$-\hat{v}_1^* + Ku_0 + \gamma \nabla^2 u_0 + K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = \mathbf{0}. \quad (8)$$

From such results, we may infer that

$$\begin{aligned} & \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} \\ &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial v_1^*} \\ & \quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_1^*} \\ &= \hat{u} \\ &= u_0. \end{aligned} \quad (9)$$

Now observe that from the variation of  $J_1^*$  in  $v_1^*$ , we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0}$$

so that

$$-u_0 - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0}$$

that is

$$-u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = \mathbf{0}.$$

From this and (8), we may infer that

$$\hat{v}_1^* = -\gamma \nabla^2 u_0 - Ku_0 - K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = -(2\hat{v}_0^* + K)u_0 + f,$$

so that

$$-\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f - K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = 0.$$

From this and the concerning boundary conditions, since

$$A(u_0, v_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

we may obtain

$$-\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f = A(u_0, \hat{v}_0^*) = 0.$$

Moreover, from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$A(u_0, \hat{v}_0^*)2u_0 - \frac{\hat{v}_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

From such last results we get

$$-\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0},$$

and thus

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$\begin{aligned} F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) &= \langle u_0, \hat{v}_2^* + \hat{v}_1^* \rangle_{L^2} - F_1(u_0, \hat{v}_0^*), \\ F_2^*(\hat{v}_2^*) &= \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0), \\ G^*(\hat{v}_1^*, \hat{v}_0^*) &= -\langle u_0, \hat{v}_1^* \rangle_{L^2} + \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, \mathbf{0}), \end{aligned}$$

so that

$$\begin{aligned} &J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ &= -F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{v}_1^*, \hat{v}_0^*) \\ &= F_1(u_0, \hat{v}_0^*) - F_2(u_0) + G(u_0, \mathbf{0}) \\ &= J(u_0). \end{aligned} \tag{10}$$

Finally, observe that

$$J_1^*(v_2^*, v_1^*, v_0^*) \leq -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_0^*) + F_2^*(v_2^*) + G(u, \mathbf{0}),$$

$$\forall u \in V, v_2^* \in Y^*, v_1^* \in D^*, v_0^* \in B^*.$$

Thus, we may obtain

$$\begin{aligned} &\inf_{v_2^* \in Y^*} J_1^*(v_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ &\leq \inf_{v_2^* \in Y^*} \{-\langle u, v_2^* \rangle_{L^2} + F_1(u, \hat{v}_0^*) + F_2^*(v_2^*) + G(u, \mathbf{0})\} \\ &= F_1(u, \hat{v}_0^*) - F_2(u) + G(u, \mathbf{0}) \\ &= J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx, \quad \forall u \in V. \end{aligned} \tag{11}$$

From this and (11), we obtain

$$\begin{aligned} &J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\ &\leq \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx \right\}. \end{aligned} \tag{12}$$

Joining the pieces, from a concerning convexity in  $u$ , we have got

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx \right\} \\ &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \end{aligned} \tag{13}$$

The proof is complete. □

**Remark 2.2.** We could have also defined

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_\infty < K/8 \text{ and } -\gamma \nabla^2 + 2v_0^* > \varepsilon I_d\},$$

for a small parameter  $0 < \varepsilon \ll 1$ . This corresponds to  $-\gamma \nabla^2 + 2v_0^*$  be positive definite, whereas the previous case corresponds to  $-\gamma \nabla^2 + 2v_0^*$  be negative definite.

### 3 One more duality principle and a concerning convex dual variational formulation

In this section we establish a second duality principle and related convex dual formulation. Let  $\Omega \subset \mathbb{R}^3$  be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by  $\partial\Omega$ .

For the primal formulation, consider a functional  $J : V \rightarrow \mathbb{R}$  where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx \\ &\quad + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}. \end{aligned} \quad (14)$$

Here  $\gamma > 0$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $f \in L^2(\Omega) \cap L^\infty(\Omega)$ .

Moreover,  $V = W_0^{1,2}(\Omega)$  and we denote  $Y = Y^* = L^2(\Omega)$ .

Define the functionals  $F_1 : V \times Y \rightarrow \mathbb{R}$ ,  $F_2 : V \rightarrow \mathbb{R}$  and  $G : V \times Y \rightarrow \mathbb{R}$  by

$$\begin{aligned} F_1(u, v_3^*) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx - \frac{K}{2} \int_{\Omega} u^2 \, dx \\ &\quad + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2v_3^* u - h_1)^2 \, dx + \frac{K_2}{2} \int_{\Omega} u^2 \, dx, \end{aligned} \quad (15)$$

$$F_2(u) = \frac{K_2}{2} \int_{\Omega} u^2 \, dx$$

and

$$G(u, v) = \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta + v)^2 \, dx + \frac{K}{2} \int_{\Omega} u^2 \, dx - \langle u, f \rangle_{L^2},$$

for appropriate  $\gamma_1 > 0$  and  $h_1 \in L^2(\Omega)$  to be specified.

We define also  $F_1^* : [Y^*]^3 \rightarrow \mathbb{R}$ ,  $F_2^* : Y^* \rightarrow \mathbb{R}$ , and  $G^* : [Y^*]^2 \rightarrow \mathbb{R}$ , by

$$\begin{aligned} &F_1^*(v_2^*, v_1^*, v_3^*) \\ &= \sup_{u \in V} \{ \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_3^*) \} \\ &= \frac{1}{2} \int_{\Omega} \frac{-h_1(-\gamma \nabla^2 - K + K_2)h_1 + (v_1^* + v_2^*)^2 + 2K_1 h_1(-\gamma_1 \nabla^2 + 2v_3^*)(v_1^* + v_2^*)}{-\gamma \nabla - K + K_1(-\gamma_1 \nabla^2 + 2v_3^*)^2 + K_2} \, dx, \end{aligned}$$

$$\begin{aligned}
F_2^*(v_2^*) &= \sup_{u \in V} \{ \langle u, v_2^* \rangle_{L^2} - F_2(u) \} \\
&= \frac{1}{2K_2} \int_{\Omega} (v_2^*)^2 dx
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&G^*(v_1^*, v_0^*) \\
&= \sup_{(u,v) \in V \times Y} \{ \langle u, -v_1^* \rangle_{L^2} + \langle v, v_0^* \rangle_{L^2} - G(u, v) \} \\
&= \frac{1}{2} \int_{\Omega} \frac{(v_1^* - f)^2}{2v_0^* + K} dx + \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 dx + \beta \int_{\Omega} v_0^* dx
\end{aligned} \tag{17}$$

if  $v_0^* \in B^*$  where

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_{\infty} < K/8\}.$$

Furthermore, we define

$$D^* = \{v_1^* \in Y^* : \|v_1^*\|_{\infty} \leq (3/2)K\}$$

and  $J_1^* : Y^* \times D^* \times B^* \times C^* \rightarrow \mathbb{R}$ , by

$$J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) = -F_1^*(v_2^*, v_1^*, v_3^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*).$$

where

$$C^* = \{v_3^* \in Y^* : -\gamma_1 \nabla^2 + 2v_3^* \geq K_3 I_d\}.$$

Observe that we may choose  $\gamma_1 > 0$  and  $h_1 \in L^2(\Omega)$  so that such a last constraint is satisfied by a critical point.

Moreover, assuming

$$K_1 \gg 1$$

and

$$K_2 \gg K_1 K_3^2 \gg K \gg \max\{1, \|f\|_{\infty}, \alpha, \beta, \gamma, \gamma_1\}$$

by directly computing  $\delta^2 J_1^*(v_2^*, v_1^*, v_0^*, v_3^*)$  we may obtain that for such specified real constants,  $J_1^*$  is convex in  $v_2^*$  and it is concave in  $(v_1^*, v_0^*, v_3^*)$  on  $Y^* \times D^* \times B^* \times C^*$ .

### 3.1 The main duality principle and a related convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

**Theorem 3.1.** *Let  $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) \in Y^* \times D^* \times B^* \times C^*$  be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) = \mathbf{0}$$

*and  $u_0 \in V$  be such that*

$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*}$$



where we assume

$$u_0 \neq 0, \text{ a.e. in } \Omega.$$

Under such hypotheses, we have

$$\begin{aligned} \delta J(u_0) &= \mathbf{0}, \\ -\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1 &= \mathbf{0}, \end{aligned}$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 dx \right\} \\ &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*, v_3^*) \in D^* \times B^* \times C^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*). \end{aligned} \quad (18)$$

*Proof.* Observe that  $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) = \mathbf{0}$  so that, since  $J_1^*$  is convex in  $v_2^*$  and concave in  $(v_1^*, v_0^*, v_3^*)$  on  $Y^* \times D^* \times B^* \times C^*$ , from the Min-Max theorem, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) = \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*, v_3^*) \in D^* \times B^* \times C^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_2^*} + u_0 = \mathbf{0}$$

and

$$\hat{v}_2^* = K_2 u_0.$$

Observe now that

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*) = \sup_{(u, v) \in V \times Y} \{ \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_3^*) \}.$$

Denoting

$$H(v_2^*, v_1^*, v_3^*, u) = \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_3^*),$$

there exists  $\hat{u} \in V$  such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*) = H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u}),$$

so that

$$\begin{aligned}\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}.\end{aligned}\tag{19}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_2^*} = \hat{u}.$$

Furthermore,

$$\begin{aligned}\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_1^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial v_1^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_1^*} \\ &= \hat{u} \\ &= u_0.\end{aligned}\tag{20}$$

From this and the variation of  $J_1^*$  in  $v_1^*$ , we obtain

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_1^*} - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0},$$

so that

$$-u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = \mathbf{0}.$$

Hence

$$\hat{v}_1^* = -(2\hat{v}_0^* + K)u_0 + f.$$

Thus, from these last results and from the variation of  $J_1^*$  in  $v_3^*$ , we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_3^*} = K_1(-\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1)2u_0 = \mathbf{0}.$$

Hence, since  $u_0 \neq 0$ , a.e. in  $\Omega$ , we have got

$$-\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1 = \mathbf{0}.$$

Moreover, from the variation of  $J_1^*$  in  $v_0^*$ , we obtain

$$-\frac{v_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

Also from

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial u} = \mathbf{0},$$

so that

$$-\hat{v}_1^* - \gamma \nabla^2 u_0 - K u_0 - \hat{v}_2^* + K_2 u_0 = \mathbf{0},$$

that is

$$\hat{v}_1^* = -\gamma \nabla^2 u_0 + K u_0.$$

Thus,

$$\hat{v}_1^* = -\gamma \nabla^2 u_0 - K u_0 = -(2\hat{v}_0^* + K)u_0,$$

so that

$$\begin{aligned} -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f &= \mathbf{0}. \\ -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f &= \mathbf{0}. \end{aligned}$$

From this, we may infer that

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$\begin{aligned} F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*) &= \langle u_0, \hat{v}_1^* + \hat{v}_2^* \rangle_{L^2} - F_1(u_0, \hat{v}_3^*), \\ F_2^*(\hat{v}_2^*) &= \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0), \\ G^*(\hat{v}_1^*, \hat{v}_0^*) &= \langle u_0, -\hat{v}_1^* \rangle_{L^2} + \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, \mathbf{0}), \end{aligned}$$

so that

$$\begin{aligned} &J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) \\ &= -F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{v}_1^*, \hat{v}_0^*) \\ &= F_1(u_0, \hat{v}_3^*) - F_2(u_0) + G(u_0, \mathbf{0}) \\ &= J(u_0). \end{aligned} \tag{21}$$

Finally, observe that

$$J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \leq -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_3^*) + F_2^*(v_2^*) + G(u, \mathbf{0}),$$

$$\forall u \in V, v_2^* \in Y^*, v_1^* \in D^*, v_0^* \in B^*, v_3^* \in C^*, .$$

Thus, we may obtain

$$\begin{aligned} &\inf_{v_2^* \in Y^*} J_1^*(v_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) \\ &\leq \inf_{v_2^* \in Y^*} \{ -\langle u, v_2^* \rangle_{L^2} + F_1(u, \hat{v}_3^*) + F_2^*(v_2^*) + G(u, \mathbf{0}) \} \\ &= F_1(u, \hat{v}_3^*) - F_2(u) + G(u, \mathbf{0}) \\ &= J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 dx, \quad \forall u \in V. \end{aligned} \tag{22}$$

From this, we obtain

$$\begin{aligned} &J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) \\ &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*, v_3^*) \in D^* \times B^* \times C^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\} \\ &\leq \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h)^2 dx \right\}. \end{aligned} \tag{23}$$

Joining the pieces, from a concerning convexity in  $u$ , we have got

$$\begin{aligned}
 J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 dx \right\} \\
 &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*, v_3^*) \in D^* \times B^* \times C^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\} \\
 &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*).
 \end{aligned} \tag{24}$$

The proof is complete. □

## 4 Conclusion

In this article we have developed convex dual variational formulations suitable for the local optimization of non-convex primal formulations.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principles here presented are applied to a Ginzburg-Landau type equation. In a future research, we intend to extend such results for some models of plates and shells and other models in the elasticity theory.

## References

- [1] R.A. Adams and J.F. Fournier, Sobolev Spaces, 2nd edn. (Elsevier, New York, 2003).
- [2] W.R. Bielski, A. Galka, J.J. Telega, The Complementary Energy Principle and Duality for Geometrically Nonlinear Elastic Shells. I. Simple case of moderate rotations around a tangent to the middle surface. Bulletin of the Polish Academy of Sciences, Technical Sciences, Vol. 38, No. 7-9, 1988.
- [3] W.R. Bielski and J.J. Telega, A Contribution to Contact Problems for a Class of Solids and Structures, Arch. Mech., 37, 4-5, pp. 303-320, Warszawa 1985.
- [4] J.F. Annet, Superconductivity, Superfluids and Condensates, 2nd edn. ( Oxford Master Series in Condensed Matter Physics, Oxford University Press, Reprint, 2010)
- [5] F.S. Botelho, Functional Analysis, Calculus of Variations and Numerical Methods in Physics and Engineering, CRC Taylor and Francis, Florida, 2020.
- [6] F.S. Botelho, *Variational Convex Analysis*, Ph.D. thesis, Virginia Tech, Blacksburg, VA -USA, (2009).
- [7] F. Botelho, *Topics on Functional Analysis, Calculus of Variations and Duality*, Academic Publications, Sofia, (2011).
- [8] F. Botelho, *Existence of solution for the Ginzburg-Landau system, a related optimal control problem and its computation by the generalized method of lines*, Applied Mathematics and Computation, 218, 11976-11989, (2012).

- [9] F. Botelho, *Functional Analysis and Applied Optimization in Banach Spaces*, Springer Switzerland, 2014.
- [10] J.C. Strikwerda, *Finite Difference Schemes and Partial Differential Equations*, SIAM, second edition (Philadelphia, 2004).
- [11] L.D. Landau and E.M. Lifschits, *Course of Theoretical Physics, Vol. 5- Statistical Physics, part 1.* (Butterworth-Heinemann, Elsevier, reprint 2008).
- [12] R.T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, (1970).
- [13] J.J. Telega, *On the complementary energy principle in non-linear elasticity. Part I: Von Karman plates and three dimensional solids*, C.R. Acad. Sci. Paris, Serie II, 308, 1193-1198; Part II: Linear elastic solid and non-convex boundary condition. Minimax approach, *ibid*, pp. 1313-1317 (1989)
- [14] A.Galka and J.J.Telega *Duality and the complementary energy principle for a class of geometrically non-linear structures. Part I. Five parameter shell model; Part II. Anomalous dual variational principles for compressed elastic beams*, Arch. Mech. 47 (1995) 677-698, 699-724.
- [15] J.F. Toland, A duality principle for non-convex optimisation and the calculus of variations, Arch. Rat. Mech. Anal., **71**, No. 1 (1979), 41-61.