

# On duality principles and related convex dual formulations suitable for local non-convex variational optimization

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## Abstract

This article develops duality principles and related convex dual formulations suitable for the local optimization of non-convex primal formulations for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a Ginzburg-Landau type equation.

**Key words:** Convex dual variational formulation, duality principle for non-convex local primal optimization, Ginzburg-Landau type equation

MSC 49N15

## 1 Introduction

In this article we establish a duality principle and a related convex dual formulation suitable for the local optimization of the primal formulation for a large class of models in non-convex optimization.

The main duality principle is applied to the Ginzburg-Landau system in superconductivity in the absence of a magnetic field.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 13, 14] and on a D.C. optimization approach developed in Toland [15].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [9, 5, 6, 7, 12]. Finally, similar models on the superconductivity physics may be found in [4, 11].

**Remark 1.1.** *It is worth highlighting, we may generically denote*

$$\int_{\Omega} [(-\gamma \nabla^2 + KI_d)^{-1} v^*] v^* dx$$

simply by

$$\int_{\Omega} \frac{(v^*)^2}{-\gamma \nabla^2 + K} dx,$$

where  $I_d$  denotes a concerning identity operator.

Other similar notations may be used along this text as their indicated meaning are sufficiently clear.

Finally,  $\nabla^2$  denotes the Laplace operator and for real constants  $K_2 > 0$  and  $K_1 > 0$ , the notation  $K_2 \gg K_1$  means that  $K_2 > 0$  is much larger than  $K_1 > 0$ .

At this point we start to describe the primal and dual variational formulations.

Let  $\Omega \subset \mathbb{R}^3$  be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by  $\partial\Omega$ .

For the primal formulation, consider a functional  $J : V \rightarrow \mathbb{R}$  where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx \\ &\quad + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}. \end{aligned} \quad (1)$$

Here  $\gamma > 0$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $f \in L^2(\Omega) \cap L^\infty(\Omega)$ .

Moreover,  $V = W_0^{1,2}(\Omega)$  and we denote  $Y = Y^* = L^2(\Omega)$ .

Define the functionals  $F_1 : V \times Y \rightarrow \mathbb{R}$ ,  $F_2 : V \rightarrow \mathbb{R}$  and  $G : V \times Y \rightarrow \mathbb{R}$  by

$$\begin{aligned} F_1(u, v_0^*) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx - \frac{K}{2} \int_{\Omega} u^2 \, dx \\ &\quad + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2v_0^* u - f)^2 \, dx + \frac{K_2}{2} \int_{\Omega} u^2 \, dx, \end{aligned} \quad (2)$$

$$F_2(u) = \frac{K_2}{2} \int_{\Omega} u^2 \, dx$$

and

$$G(u, v) = \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta + v)^2 \, dx + \frac{K}{2} \int_{\Omega} u^2 \, dx - \langle u, f \rangle_{L^2}.$$

We define also  $F_1^* : [Y^*]^3 \rightarrow \mathbb{R}$ ,  $F_2^* : Y^* \rightarrow \mathbb{R}$ , and  $G^* : [Y^*]^2 \rightarrow \mathbb{R}$ , by

$$\begin{aligned} &F_1^*(v_2^*, v_1^*, v_0^*) \\ &= \sup_{u \in V} \{ \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_0^*) \} \\ &= \int_{\Omega} \frac{-K_1 f(-\gamma \nabla^2 + K + K_2) f + (v_1^* + v_2^*)^2 - 2K_1 f(-\gamma \nabla^2 + 2v_0^*)(v_1^* + v_2^*)}{2[K_2 + K - \gamma \nabla^2 + K_1(-\gamma \nabla^2 + 2v_0^*)^2]} \, dx \end{aligned} \quad (3)$$

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u \in V} \{ \langle u, v_2^* \rangle_{L^2} - F_2(u) \} \\ &= \frac{1}{2K_2} \int_{\Omega} (v_2^*)^2 \, dx \end{aligned} \quad (4)$$

and

$$\begin{aligned}
 G^*(v_1^*, v_0^*) &= \sup_{(u,v) \in V \times Y} \{-\langle u, v_1^* \rangle_{L^2} + \langle v, v_0^* \rangle_{L^2} - G(u, v)\} \\
 &= \frac{1}{2} \int_{\Omega} \frac{(v_1^* - f)^2}{2v_0^* + K} dx + \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 dx \\
 &\quad + \beta \int_{\Omega} v_0^* dx
 \end{aligned} \tag{5}$$

if  $v_0^* \in B^*$  where

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_{\infty} < K/8 \text{ and } -\gamma \nabla^2 + 2v_0^* < -\varepsilon I_d\},$$

for a small parameter  $0 < \varepsilon \ll 1$ .

Furthermore, we define

$$D^* = \{v_1^* \in Y^* : \|v_1^*\|_{\infty} \leq (3/2)K\}$$

and  $J_1^* : Y^* \times D^* \times B^* \rightarrow \mathbb{R}$ , by

$$J_1^*(v_2^*, v_1^*, v_0^*) = -F_1^*(v_2^*, v_1^*, v_0^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*).$$

Assuming

$$K_2 \gg K_1 \gg K \gg \max\{\|f\|_{\infty}, \alpha, \beta, \gamma, 1/\varepsilon^2\}$$

by directly computing  $\delta^2 J_1^*(v_2^*, v_1^*, v_0^*)$  we may obtain that for such specified real constants,  $J_1^*$  in convex in  $v_2^*$  and it is concave in  $(v_1^*, v_0^*)$  on  $Y^* \times D^* \times B^*$ .

## 2 The main duality principle and a concerning convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

**Theorem 2.1.** *Let  $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \in Y^* \times D^* \times B^*$  be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$$

*and  $u_0 \in V$  be such that*

$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*}.$$

*Under such hypotheses, we have*

$$\delta J(u_0) = \mathbf{0},$$

*and*

$$\begin{aligned}
 J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx \right\} \\
 &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\
 &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*).
 \end{aligned} \tag{6}$$

*Proof.* Observe that  $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$  so that, since  $J_1^*$  is convex in  $v_2^*$  and concave in  $(v_1^*, v_0^*)$  on  $Y^* \times D^* \times B^*$ , from the Min-Max theorem, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, v_0^*)}{\partial v_2^*} + u_0 = \mathbf{0}$$

and

$$\hat{v}_2^* - K_2 u_0 = \mathbf{0}.$$

Observe now that denoting

$$H(v_2^*, v_1^*, v_0^*, u) = \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_0^*),$$

there exists  $\hat{u} \in V$  such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = H(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}. \end{aligned} \tag{7}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \hat{u}.$$

Also, denoting

$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

from

$$\frac{\partial H(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, u_0)}{\partial u} = \mathbf{0},$$

we have

$$-\hat{v}_1^* + Ku_0 + \gamma \nabla^2 u_0 + K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) - \hat{v}_2^* + K_2 u_0 = \mathbf{0},$$

so that

$$-\hat{v}_1^* + Ku_0 + \gamma \nabla^2 u_0 + K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = \mathbf{0}. \quad (8)$$

From such results, we may infer that

$$\begin{aligned} & \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} \\ &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial v_1^*} \\ & \quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_1^*} \\ &= \hat{u} \\ &= u_0. \end{aligned} \quad (9)$$

Now observe that from the variation of  $J_1^*$  in  $v_1^*$ , we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0}$$

so that

$$-u_0 - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0}$$

that is

$$-u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = \mathbf{0}.$$

From this and (8), we may infer that

$$\hat{v}_1^* = -\gamma \nabla^2 u_0 - Ku_0 - K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = -(2\hat{v}_0^* + K)u_0 + f,$$

so that

$$-\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f - K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = 0.$$

From this and the concerning boundary conditions, since

$$A(u_0, v_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

we may obtain

$$-\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f = A(u_0, \hat{v}_0^*) = 0.$$

Moreover, from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$A(u_0, \hat{v}_0^*)2u_0 - \frac{\hat{v}_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

From such last results we get

$$-\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0},$$

and thus

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \langle u_0, \hat{v}_2^* + \hat{v}_1^* \rangle_{L^2} - F_1(u_0, \hat{v}_0^*),$$

$$F_2^*(\hat{v}_2^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0),$$

$$G^*(\hat{v}_1^*, \hat{v}_0^*) = -\langle u_0, \hat{v}_1^* \rangle_{L^2} + \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, \mathbf{0}),$$

so that

$$\begin{aligned} & J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ &= -F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{v}_1^*, \hat{v}_0^*) \\ &= F_1(u_0, \hat{v}_0^*) - F_2(u_0) + G(u_0, \mathbf{0}) \\ &= J(u_0). \end{aligned} \tag{10}$$

Finally, observe that

$$J_1^*(v_2^*, v_1^*, v_0^*) \leq -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_0^*) + F_2^*(v_2^*) + G(u, \mathbf{0}),$$

$$\forall u \in V, v_2^* \in Y^*, v_1^* \in D^*, v_0^* \in B^*.$$

Thus, we may obtain

$$\begin{aligned} & \inf_{v_2^* \in Y^*} J_1^*(v_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ & \leq \inf_{v_2^* \in Y^*} \{ -\langle u, v_2^* \rangle_{L^2} + F_1(u, \hat{v}_0^*) + F_2^*(v_2^*) + G(u, \mathbf{0}) \} \\ & = F_1(u, \hat{v}_0^*) - F_2(u) + G(u, \mathbf{0}) \\ & = J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx, \quad \forall u \in V. \end{aligned} \tag{11}$$

From this and (11), we obtain

$$\begin{aligned} & J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\ & \leq \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx \right\}. \end{aligned} \tag{12}$$

Joining the pieces, from a concerning convexity in  $u$ , we have got

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx \right\} \\ &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \end{aligned} \tag{13}$$

The proof is complete. □

**Remark 2.2.** *We could have also defined*

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_\infty < K/8 \text{ and } -\gamma \nabla^2 + 2v_0^* > \varepsilon I_d\},$$

for a small parameter  $0 < \varepsilon \ll 1$ . This corresponds to  $-\gamma \nabla^2 + 2v_0^*$  be positive definite, whereas the previous case corresponds to  $-\gamma \nabla^2 + 2v_0^*$  be negative definite.

### 3 One more duality principle and a concerning convex dual variational formulation

In this section we establish another duality principle and related convex dual formulation. Let  $\Omega \subset \mathbb{R}^3$  be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by  $\partial\Omega$ .

For the primal formulation, consider a functional  $J : V \rightarrow \mathbb{R}$  where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx \\ &\quad + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}. \end{aligned} \quad (14)$$

Here  $\gamma > 0$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $f \in L^2(\Omega) \cap L^\infty(\Omega)$ .

Moreover,  $V = W_0^{1,2}(\Omega)$  and we denote  $Y = Y^* = L^2(\Omega)$ .

Define the functionals  $F_1 : V \rightarrow \mathbb{R}$ ,  $F_2 : V \rightarrow \mathbb{R}$  and  $G : V \times Y \rightarrow \mathbb{R}$  by

$$F_1(u) = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx - \langle u, f \rangle_{L^2},$$

$$F_2(u) = \frac{K}{2} \int_{\Omega} u^2 \, dx$$

and

$$G(u, v, v_3^*) = \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta + v)^2 \, dx + \frac{K_2}{2} \int_{\Omega} u^2 \, dx + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2v_3^* u - h_1)^2 \, dx,$$

for appropriate  $\gamma_1 > 0$  and  $h_1 \in L^2(\Omega)$  to be specified.

We define also  $F_1^* : Y^* \rightarrow \mathbb{R}$ ,  $F_2^* : Y^* \rightarrow \mathbb{R}$ , and  $G^* : [Y^*]^4 \rightarrow \mathbb{R}$ , by

$$\begin{aligned} F_1^*(v_1^*) &= \sup_{u \in V} \{ \langle u, v_1^* \rangle_{L^2} - F_1(u) \} \\ &= \frac{1}{2} \int_{\Omega} \frac{(v_1^* + f)^2}{-\gamma \nabla^2} \, dx, \end{aligned} \quad (15)$$

$$\begin{aligned} F_2^*(z^*) &= \sup_{u \in V} \{ \langle u, z^* \rangle_{L^2} - F_2(u) \} \\ &= \frac{1}{2K} \int_{\Omega} (z^*)^2 \, dx \end{aligned} \quad (16)$$

and

$$\begin{aligned}
& G^*(v_3^*, v_1^*, v_0^*, z^*) \\
&= \sup_{(u,v) \in V \times Y} \{ \langle u, -v_1^* \rangle_{L^2} + \langle v, v_0^* \rangle_{L^2} - G(u, v) \} \\
&= \frac{-1}{2} \int_{\Omega} \frac{-h_1^2 K_1 (K + 2v_0^*) + 2h_1 K_1 (-\gamma_1 \nabla^2 + 2v_3^*) (-v_1^* + z^*) + (-v_1^* + z^*)^2}{K + 2v_0^* + K_1 (-\gamma_1 \nabla^2 + 2v_3^*)^2} dx \quad (17)
\end{aligned}$$

if  $v_0^* \in B^*$  where

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_{\infty} < K/8\}.$$

Furthermore, we define

$$D^* = \{v_1^* \in Y^* : \|v_1^*\|_{\infty} \leq (3/2)K\}$$

and  $J_1^* : Y^* \times D^* \times B^* \rightarrow \mathbb{R}$ , by

$$J_1^*(v_3^*, v_1^*, v_0^*, z^*) = -F_1^*(v_1^*) + F_2^*(z^*) - G^*(v_3^*, v_1^*, v_0^*, z^*).$$

where we also define

$$C^* = \{v_3^* \in Y^* : -\gamma_1 \nabla^2 + 2v_3^* \geq K_3 I_d\}.$$

Observe that we may choose  $\gamma_1 > 0$  and  $h_1 \in L^2(\Omega)$  so that such a last constraint is satisfied by a critical point.

Moreover, assuming

$$K_1 \gg K \gg \max\{K_3, \|f\|_{\infty}, \alpha, \beta, \gamma, \gamma_1\}$$

by directly computing  $\delta^2 J_1^*(v_3^*, v_1^*, v_0^*, z^*)$  we may obtain that for such specified real constants,  $J_1^*$  is convex in  $z^*$  and it is concave in  $(v_3^*, v_1^*, v_0^*)$  on  $C^* \times D^* \times B^* \times Y^*$ .

### 3.1 The main duality principle and a related convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

**Theorem 3.1.** *Let  $(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \in C^* \times D^* \times B^* \times Y^*$  be such that*

$$\delta J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \mathbf{0}$$

*and  $u_0 \in V$  be such that*

$$u_0 = \frac{\partial F_2^*(\hat{z}^*)}{\partial z^*}.$$

*Under such hypotheses, we have*

$$\begin{aligned}
& \delta J(u_0) = \mathbf{0}, \\
& -\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1 = \mathbf{0},
\end{aligned}$$

and

$$\begin{aligned}
 J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 dx \right\} \\
 &= \inf_{z^* \in Y^*} \left\{ \sup_{(v_3^*, v_1^*, v_0^*) \in C^* \times D^* \times B^*} J_1^*(v_3^*, v_1^*, v_0^*, z^*) \right\} \\
 &= J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*).
 \end{aligned} \tag{18}$$

*Proof.* Observe that  $\delta J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \mathbf{0}$  so that, since  $J_1^*$  is convex in  $z^*$  and concave in  $(v_3^*, v_1^*, v_0^*)$  on  $C^* \times D^* \times B^* \times Y^*$ , from the Min-Max theorem, we obtain

$$J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \inf_{z^* \in Y^*} \left\{ \sup_{(v_3^*, v_1^*, v_0^*) \in C^* \times D^* \times B^*} J_1^*(v_3^*, v_1^*, v_0^*, z^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial z^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{z}^*)}{\partial z^*} = u_0$$

we have

$$-\frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial z^*} + u_0 = \mathbf{0}$$

and

$$\hat{z}^* = K u_0.$$

Observe now that

$$\begin{aligned}
 &G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\
 &= \sup_{(u, v) \in V \times Y} \{ \langle u, v_1^* + z^* \rangle_{L^2} + \langle v, v_0^* \rangle_{L^2} - G(u, v) \} \\
 &= \sup_{(u, v) \in V \times Y} \left\{ \langle u, v_1^* + z^* \rangle_{L^2} + \langle v, v_0^* \rangle_{L^2} - \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta + v)^2 dx \right. \\
 &\quad \left. - \frac{K}{2} \int_{\Omega} dx - \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2v_3^* u - h_1)^2 dx \right\}.
 \end{aligned} \tag{19}$$

Denoting  $w = u^2 - \beta + v$  so that  $v = w - u^2 + \beta$ , we obtain

$$\begin{aligned}
 &G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\
 &= \sup_{(u, w) \in V \times Y} \left\{ \langle u, -v_1^* + z^* \rangle_{L^2} + \langle -u^2 + \beta + w, v_0^* \rangle_{L^2} - \frac{\alpha}{2} \int_{\Omega} w^2 dx - \frac{K}{2} \int_{\Omega} u^2 dx \right. \\
 &\quad \left. - \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2v_3^* u - h_1)^2 dx \right\} \\
 &= \sup_{u \in V} \left\{ \langle u, -v_1^* + z^* \rangle_{L^2} + \langle -u^2 + \beta, v_0^* \rangle_{L^2} - \frac{K}{2} \int_{\Omega} u^2 dx \right. \\
 &\quad \left. + \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 dx - \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2v_3^* u - h_1)^2 dx \right\}.
 \end{aligned} \tag{20}$$

Denoting

$$\begin{aligned} H(v_3^*, v_1^*, v_0^*, z^*, u) &= \langle u, v_1^* + z^* \rangle_{L^2} + \langle -u^2 + \beta, v_0^* \rangle_{L^2} - \frac{K}{2} \int_{\Omega} u^2 \, dx \\ &\quad + \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx - \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2v_3^* u - h_1)^2 \, dx, \end{aligned} \quad (21)$$

there exists  $\hat{u} \in V$  such that

$$\frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial z^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial z^*} \\ &\quad + \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial z^*} \\ &= \hat{u}. \end{aligned} \quad (22)$$

Summarizing, we have got

$$u_0 = \frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial z^*} = \hat{u}.$$

Furthermore,

$$\begin{aligned} \frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial v_1^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial v_1^*} \\ &\quad + \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_1^*} \\ &= -\hat{u} \\ &= -u_0. \end{aligned} \quad (23)$$

Also,

$$\begin{aligned} \frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial v_0^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial v_0^*} \\ &\quad + \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_0^*} \\ &= \hat{u}^2 \\ &= u_0^2. \end{aligned} \quad (24)$$

Finally,

$$\begin{aligned} \frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial v_3^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial v_3^*} \\ &\quad + \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_3^*} \\ &= -K_1(-\gamma_1 \nabla^2 \hat{u} + 2\hat{v}_3^* \hat{u} - h_1)2\hat{u}. \end{aligned} \quad (25)$$

Thus, from this last result and from the variation of  $J_1^*$  in  $v_3^*$ , we have

$$-\frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial v_3^*} = K_1(-\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1)2u_0 = \mathbf{0}.$$

Hence, since  $u_0 \neq 0$ , a.e. in  $\Omega$ , we have got

$$-\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1 = \mathbf{0}.$$

Moreover, from the variation of  $J_1^*$  in  $v_0^*$ , we obtain

$$-\frac{v_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

From the variation of  $J^*$  in  $v_1^*$  we get

$$-\frac{\hat{v}_1^* + f}{-\gamma \nabla^2} + u_0 = \mathbf{0}$$

so that

$$\hat{v}_1^* = -\gamma \nabla^2 u_0 - f.$$

On the other hand from these last results and from

$$\frac{\partial H(\hat{v}_3, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, u_0)}{\partial u} = \mathbf{0},$$

we get

$$-\hat{v}_1^* + \hat{z}^* - K u_0 - 2v_0^* u_0 = \mathbf{0},$$

so that

$$\hat{v}_1^* = -\gamma \nabla^2 u_0 - f = -2v_0^* u_0 = -2\alpha(u_0^2 - \beta)u_0,$$

that is,

$$-\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0}.$$

From this, we may infer that

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_1^*) = \langle u_0, \hat{v}_1^* \rangle_{L^2} - F_1(u_0),$$

$$F_2^*(\hat{z}^*) = \langle u_0, \hat{z}^* \rangle_{L^2} - F_2(u_0),$$

$$G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \langle u_0, -\hat{v}_1^* + \hat{z}^* \rangle_{L^2} + \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, \mathbf{0}, \hat{v}_3^*),$$

so that

$$\begin{aligned} & J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ &= -F_1^*(\hat{v}_1^*) + F_2^*(\hat{z}^*) - G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ &= F_1(u_0) - F_2(u_0) + G(u_0, \mathbf{0}, \hat{v}_3^*) \\ &= J(u_0). \end{aligned} \tag{26}$$

Finally, observe that

$$J_1^*(v_3^*, v_1^*, v_0^*, z^*) \leq -\langle u, z^* \rangle_{L^2} + F_1(u) + F_2^*(z^*) + G(u, \mathbf{0}, \hat{v}_3^*),$$

$$\forall u \in V, z^* \in Y^*, v_3^* \in C^*, v_1^* \in D^*, v_0^* \in B^*.$$

Thus, we may obtain

$$\begin{aligned} & \inf_{z^* \in Y^*} J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, z^*) \\ & \leq \inf_{z^* \in Y^*} \{-\langle u, z^* \rangle_{L^2} + F_1(u) + F_2^*(z^*) + G(u, \mathbf{0}, \hat{v}_3^*)\} \\ & = F_1(u) - F_2(u) + G(u, \mathbf{0}, \hat{v}_3^*) \\ & = J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 dx, \quad \forall u \in V. \end{aligned} \quad (27)$$

From this, we obtain

$$\begin{aligned} & J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ & = \inf_{z^* \in Y^*} \left\{ \sup_{(v_3^*, v_1^*, v_0^*) \in C^* \times D^* \times B^*} J_1^*(v_3^*, v_1^*, v_0^*, z^*) \right\} \\ & \leq \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h)^2 dx \right\}. \end{aligned} \quad (28)$$

Joining the pieces, from a concerning convexity in  $u$ , we have got

$$\begin{aligned} J(u_0) & = \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 dx \right\} \\ & = \inf_{z^* \in Y^*} \left\{ \sup_{(v_3^*, v_1^*, v_0^*) \in C^* \times D^* \times B^*} J_1^*(v_3^*, v_1^*, v_0^*, z^*) \right\} \\ & = J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*). \end{aligned} \quad (29)$$

The proof is complete.  $\square$

## 4 A third duality principle and a concerning convex dual variational formulation

In this section we establish a third duality principle and related convex dual formulation. Let  $\Omega \subset \mathbb{R}^3$  be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by  $\partial\Omega$ .

For the primal formulation, consider a functional  $J : V \rightarrow \mathbb{R}$  where

$$\begin{aligned} J(u) & = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u dx \\ & \quad + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 dx - \langle u, f \rangle_{L^2}. \end{aligned} \quad (30)$$

Here  $\gamma > 0$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $f \in L^2(\Omega) \cap L^\infty(\Omega)$ .

Moreover,  $V = W_0^{1,2}(\Omega)$  and we denote  $Y = Y^* = L^2(\Omega)$ .

Define the functionals  $F_1 : V \times Y \rightarrow \mathbb{R}$ ,  $F_2 : V \rightarrow \mathbb{R}$  and  $G : V \times Y \rightarrow \mathbb{R}$  by

$$\begin{aligned} F_1(u, v_3^*) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx - \frac{K}{2} \int_{\Omega} u^2 \, dx \\ &\quad + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2v_3^* u - h_1)^2 \, dx + \frac{K_2}{2} \int_{\Omega} u^2 \, dx, \\ F_2(u) &= \frac{K_2}{2} \int_{\Omega} u^2 \, dx \end{aligned} \quad (31)$$

and

$$G(u, v) = \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta + v)^2 \, dx + \frac{K}{2} \int_{\Omega} u^2 \, dx - \langle u, f \rangle_{L^2},$$

for appropriate  $\gamma_1 > 0$  and  $h_1 \in L^2(\Omega)$  to be specified.

We define also  $F_1^* : [Y^*]^3 \rightarrow \mathbb{R}$ ,  $F_2^* : Y^* \rightarrow \mathbb{R}$ , and  $G^* : [Y^*]^2 \rightarrow \mathbb{R}$ , by

$$\begin{aligned} &F_1^*(v_2^*, v_1^*, v_3^*) \\ &= \sup_{u \in V} \{ \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_3^*) \} \\ &= \frac{1}{2} \int_{\Omega} \frac{-h_1(-\gamma \nabla^2 - K + K_2)h_1 + (v_1^* + v_2^*)^2 + 2K_1 h_1(-\gamma_1 \nabla^2 + 2v_3^*)(v_1^* + v_2^*)}{-\gamma \nabla - K + K_1(-\gamma_1 \nabla^2 + 2v_3^*)^2 + K_2} \, dx, \\ F_2^*(v_2^*) &= \sup_{u \in V} \{ \langle u, v_2^* \rangle_{L^2} - F_2(u) \} \\ &= \frac{1}{2K_2} \int_{\Omega} (v_2^*)^2 \, dx \end{aligned} \quad (32)$$

and

$$\begin{aligned} &G^*(v_1^*, v_0^*) \\ &= \sup_{(u,v) \in V \times Y} \{ \langle u, -v_1^* \rangle_{L^2} + \langle v, v_0^* \rangle_{L^2} - G(u, v) \} \\ &= \frac{1}{2} \int_{\Omega} \frac{(v_1^* - f)^2}{2v_0^* + K} \, dx + \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx + \beta \int_{\Omega} v_0^* \, dx \end{aligned} \quad (33)$$

if  $v_0^* \in B^*$  where

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_{\infty} < K/8\}.$$

Furthermore, we define

$$D^* = \{v_1^* \in Y^* : \|v_1^*\|_{\infty} \leq (3/2)K\}$$

and  $J_1^* : Y^* \times D^* \times B^* \times C^* \rightarrow \mathbb{R}$ , by

$$J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) = -F_1^*(v_2^*, v_1^*, v_3^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*).$$

where

$$C^* = \{v_3^* \in Y^* : -\gamma_1 \nabla^2 + 2v_3^* \geq K_3 I_d\}.$$

Observe that we may choose  $\gamma_1 > 0$  and  $h_1 \in L^2(\Omega)$  so that such a last constraint is satisfied by a critical point.

Moreover, assuming

$$K_1 \gg 1$$

and

$$K_2 \gg K_1 K_3^2 \gg K \gg \max\{1, \|f\|_\infty, \alpha, \beta, \gamma, \gamma_1\}$$

by directly computing  $\delta^2 J_1^*(v_2^*, v_1^*, v_0^*, v_3^*)$  we may obtain that for such specified real constants,  $J_1^*$  is convex in  $v_2^*$  and it is concave in  $(v_1^*, v_0^*, v_3^*)$  on  $Y^* \times D^* \times B^* \times C^*$ .

## 4.1 The main duality principle and a related convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

**Theorem 4.1.** *Let  $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) \in Y^* \times D^* \times B^* \times C^*$  be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) = \mathbf{0}$$

*and  $u_0 \in V$  be such that*

$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*}$$

*where we assume*

$$u_0 \neq 0, \text{ a.e. in } \Omega.$$

*Under such hypotheses, we have*

$$\delta J(u_0) = \mathbf{0},$$

$$-\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1 = \mathbf{0},$$

*and*

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 dx \right\} \\ &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*, v_3^*) \in D^* \times B^* \times C^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*). \end{aligned} \tag{34}$$

*Proof.* Observe that  $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) = \mathbf{0}$  so that, since  $J_1^*$  is convex in  $v_2^*$  and concave in  $(v_1^*, v_0^*, v_3^*)$  on  $Y^* \times D^* \times B^* \times C^*$ , from the Min-Max theorem, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) = \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*, v_3^*) \in D^* \times B^* \times C^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_2^*} + u_0 = \mathbf{0}$$

and

$$\hat{v}_2^* = K_2 u_0.$$

Observe now that

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*) = \sup_{(u,v) \in V \times Y} \{ \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_3^*) \}.$$

Denoting

$$H(v_2^*, v_1^*, v_3^*, u) = \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_3^*),$$

there exists  $\hat{u} \in V$  such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*) = H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}. \end{aligned} \tag{35}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_2^*} = \hat{u}.$$

Furthermore,

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_1^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial v_1^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_1^*} \\ &= \hat{u} \\ &= u_0. \end{aligned} \tag{36}$$

From this and the variation of  $J_1^*$  in  $v_1^*$ , we obtain

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_1^*} - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0},$$

so that

$$-u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = \mathbf{0}.$$

Hence

$$\hat{v}_1^* = -(2\hat{v}_0^* + K)u_0 + f.$$

Thus, from these last results and from the variation of  $J_1^*$  in  $v_3^*$ , we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*)}{\partial v_3^*} = K_1(-\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1)2u_0 = \mathbf{0}.$$

Hence, since  $u_0 \neq 0$ , a.e. in  $\Omega$ , we have got

$$-\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1 = \mathbf{0}.$$

Moreover, from the variation of  $J_1^*$  in  $v_0^*$ , we obtain

$$-\frac{v_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

Also from

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*, \hat{u})}{\partial u} = \mathbf{0},$$

so that

$$-\hat{v}_1^* - \gamma \nabla^2 u_0 - K u_0 - \hat{v}_2^* + K_2 u_0 = \mathbf{0},$$

that is

$$\hat{v}_1^* = -\gamma \nabla^2 u_0 + K u_0.$$

Thus,

$$\hat{v}_1^* = -\gamma \nabla^2 u_0 - K u_0 = -(2\hat{v}_0^* + K)u_0,$$

so that

$$\begin{aligned} -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f &= \mathbf{0}. \\ -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f &= \mathbf{0}. \end{aligned}$$

From this, we may infer that

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$\begin{aligned} F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*) &= \langle u_0, \hat{v}_1^* + \hat{v}_2^* \rangle_{L^2} - F_1(u_0, \hat{v}_3^*), \\ F_2^*(\hat{v}_2^*) &= \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0), \\ G^*(\hat{v}_1^*, \hat{v}_0^*) &= \langle u_0, -\hat{v}_1^* \rangle_{L^2} + \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, \mathbf{0}), \end{aligned}$$

so that

$$\begin{aligned}
& J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) \\
&= -F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_3^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{v}_1^*, \hat{v}_0^*) \\
&= F_1(u_0, \hat{v}_3^*) - F_2(u_0) + G(u_0, \mathbf{0}) \\
&= J(u_0).
\end{aligned} \tag{37}$$

Finally, observe that

$$\begin{aligned}
& J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \leq -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_3^*) + F_2^*(v_2^*) + G(u, \mathbf{0}), \\
& \forall u \in V, v_2^* \in Y^*, v_1^* \in D^*, v_0^* \in B^*, v_3^* \in C^*, .
\end{aligned}$$

Thus, we may obtain

$$\begin{aligned}
& \inf_{v_2^* \in Y^*} J_1^*(v_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) \\
& \leq \inf_{v_2^* \in Y^*} \{-\langle u, v_2^* \rangle_{L^2} + F_1(u, \hat{v}_3^*) + F_2^*(v_2^*) + G(u, \mathbf{0})\} \\
& = F_1(u, \hat{v}_3^*) - F_2(u) + G(u, \mathbf{0}) \\
& = J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 dx, \forall u \in V.
\end{aligned} \tag{38}$$

From this, we obtain

$$\begin{aligned}
& J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) \\
&= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*, v_3^*) \in D^* \times B^* \times C^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\} \\
& \leq \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h)^2 dx \right\}.
\end{aligned} \tag{39}$$

Joining the pieces, from a concerning convexity in  $u$ , we have got

$$\begin{aligned}
J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 dx \right\} \\
&= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*, v_3^*) \in D^* \times B^* \times C^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\} \\
&= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*).
\end{aligned} \tag{40}$$

The proof is complete.  $\square$

## 5 Conclusion

In this article we have developed convex dual variational formulations suitable for the local optimization of non-convex primal formulations.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principles here presented are applied to a Ginzburg-Landau type equation. In a future research, we intend to extend such results for some models of plates and shells and other models in the elasticity theory.

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