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# A duality principle and a related convex dual formulation suitable for local non-convex variational optimization

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**Abstract**

This article develops a duality principle and a related convex dual formulation suitable for the local optimization of a non-convex primal formulation for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a Ginzburg-Landau type equation.

**Key words:** Convex dual variational formulation, duality principle for non-convex local primal optimization, Ginzburg-Landau type equation

**MSC** 49N15

## 1 Introduction

In this article we establish a duality principle and a related convex dual formulation suitable for the local optimization of the primal formulation for a large class of models in non-convex optimization.

The main duality principle is applied to the Ginzburg-Landau system in superconductivity in the absence of a magnetic field.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 13, 14] and on a D.C. optimization approach developed in Toland [15].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [9, 5, 6, 7, 12]. Finally, similar models on the superconductivity physics may be found in [4, 11].

**Remark 1.1.** *It is worth highlighting, we may generically denote*

$$\int_\Omega [(-\gamma \nabla^2 + K I_d)^{-1} v^*] v^* \, dx$$

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*simply by*

$$\int_\Omega \frac{(v^*)^2}{-\gamma\nabla^2+K}\,dx,$$

*where $I_d$ denotes a concerning identity operator.*

*Other similar notations may be used along this text as their indicated meaning are sufficiently clear.*

*Finally, $\nabla^2$ denotes the Laplace operator and for real constants $K_2>0$ and $K_1>0$, the notation $K_2 \gg K_1$ means that $K_2>0$ is much larger than $K_1>0$.*

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J: V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx \\ &+\frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\,dx - \langle u,f\rangle_{L^2}. \end{aligned} \tag{1}$$

Here $\gamma>0$, $\alpha>0$, $\beta>0$ and $f \in L^2(\Omega)\cap L^\infty(\Omega)$.

Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functionals $F_1 : V \times Y \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $G : V \times Y \to \mathbb{R}$ by

$$\begin{aligned} F_1(u,v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx - \frac{K}{2}\int_\Omega u^2\,dx \\ &+\frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2v_0^* u - f)^2\,dx + \frac{K_2}{2}\int_\Omega u^2\,dx, \end{aligned} \tag{2}$$

$$F_2(u) = \frac{K_2}{2}\int_\Omega u^2\,dx$$

and

$$G(u,v) = \frac{\alpha}{2}\int_\Omega (u^2-\beta+v)^2\,dx + \frac{K}{2}\int_\Omega u^2\,dx - \langle u,f\rangle_{L^2}.$$

We define also $F_1^* : [Y^*]^3 \to \mathbb{R}$, $F_2^* : Y^* \to \mathbb{R}$, and $G^* : [Y^*]^2 \to \mathbb{R}$, by

$$\begin{aligned} &F_1^*(v_2^*,v_1^*,v_0^*) \\ =\ & \sup_{u\in V}\{\langle u, v_1^*+v_2^*\rangle_{L^2} - F_1(u,v_0^*)\} \\ =\ & \int_\Omega \frac{-K_1 f(-\gamma\nabla^2+K+K_2)f + (v_1^*+v_2^*)^2 - 2K_1 f(-\gamma\nabla^2+2v_0^*)(v_1^*+v_2^*)}{2[K_2+K-\gamma\nabla^2+K_1(-\gamma\nabla^2+2v_0^*)^2]}\,dx \end{aligned} \tag{3}$$

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u\in V}\{\langle u,v_2^*\rangle_{L^2} - F_2(u)\} \\ &= \frac{1}{2K_2}\int_\Omega (v_2^*)^2\,dx \end{aligned} \tag{4}$$

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and

$$\begin{aligned} G^*(v_1^*, v_0^*) &= \sup_{(u,v)\in V\times Y}\{-\langle u, v_1^*\rangle_{L^2} + \langle v, v_0^*\rangle_{L^2} - G(u,v)\} \\ &= \frac{1}{2}\int_\Omega \frac{(v_1^*-f)^2}{2v_0^*+K}\,dx + \frac{1}{2\alpha}\int_\Omega (v_0^*)^2\,dx \\ &\quad +\beta\int_\Omega v_0^*\,dx \end{aligned} \tag{5}$$

if $v_0^* \in B^*$ where

$$B^* = \left\{v_0^* \in Y^* \;:\; \|2v_0^*\|_\infty < K/8 \text{ and } -\gamma\nabla^2 + 2v_0^* < -\varepsilon I_d\right\},$$

for a small parameter $0 < \varepsilon \ll 1$.

Furthermore, we define

$$D^* = \{v_1^* \in Y^* \;:\; \|v_1^*\|_\infty \leq (3/2)K\}$$

and $J_1^* : Y^* \times D^* \times B^* \to \mathbb{R}$, by

$$J_1^*(v_2^*, v_1^*, v_0^*) = -F_1^*(v_2^*, v_1^*, v_0^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*).$$

Assuming

$$K_2 \gg K_1 \gg K \gg \max\{\|f\|_\infty, \alpha, \beta, \gamma, 1/\varepsilon^2\}$$

by directly computing $\delta^2 J_1^*(v_2^*, v_1^*, v_0^*)$ we may obtain that for such specified real constants, $J_1^*$ in convex in $v_2^*$ and it is concave in $(v_1^*, v_0^*)$ on $Y^* \times D^* \times B^*$.

# 2 The main duality principle and a concerning convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

**Theorem 2.1.** *Let $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \in Y^* \times D^* \times B^*$ be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$$

*and $u_0 \in V$ be such that*

$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*}.$$

*Under such hypotheses, we have*

$$\delta J(u_0) = \mathbf{0},$$

*and*

$$\begin{aligned} J(u_0) &= \inf_{u\in V}\left\{J(u) + \frac{K_1}{2}\int_\Omega(-\gamma\nabla^2 u + 2\hat{v}_0^* u - f)^2\,dx\right\} \\ &= \inf_{v_2^*\in Y^*}\left\{\sup_{(v_1^*,v_0^*)\in D^*\times B^*} J_1^*(v_2^*, v_1^*, v_0^*)\right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \end{aligned} \tag{6}$$

*Proof.* Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$ so that, since $J_1^*$ is convex in $v_2^*$ and concave in $(v_1^*, v_0^*)$ on $Y^* \times D^* \times B^*$, from the Min-Max theorem, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, v_0^*)}{\partial v_2^*} + u_0 = \mathbf{0}$$

and

$$\hat{v}_2^* - K_2 u_0 = \mathbf{0}.$$

Observe now that denoting

$$H(v_2^*, v_1^*, v_0^*, u) = \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_0^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = H(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}. \end{aligned} \tag{7}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \hat{u}.$$

Also, denoting

$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

from

$$\frac{\partial H(\hat{v}_1^*, \hat{v_2}^*, \hat{v}_0^*, u_0)}{\partial u} = \mathbf{0},$$

we have

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$$-\hat{v}_1^* + Ku_0 + \gamma\nabla^2 u_0 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) - \hat{v}_2^* + K_2 u_0 = \mathbf{0},$$

so that

$$-\hat{v}_1^* + Ku_0 + \gamma\nabla^2 u_0 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = \mathbf{0}. \tag{8}$$

From such results, we may infer that

$$\begin{aligned} & \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} \\ = \ & \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial v_1^*} \\ & + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_1^*} \\ = \ & \hat{u} \\ = \ & u_0. \end{aligned} \tag{9}$$

Now observe that from the variation of $J_1^*$ in $v_1^*$, we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0}$$

so that

$$-u_0 - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0}$$

that is

$$-u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = \mathbf{0}.$$

From this and (8), we may infer that

$$\hat{v}_1^* = -\gamma\nabla^2 u_0 - Ku_0 - K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = -(2\hat{v}_0^* + K)u_0 + f,$$

so that

$$-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f - K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = 0.$$

From this and the concerning boundary conditions, since

$$A(u_0, v_0^*) = -\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

we may obtain

$$-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f = A(u_0, \hat{v}_0^*) = 0.$$

Moreover, from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$A(u_0, \hat{v}_0^*) 2u_0 - \frac{\hat{v}_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

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From such last results we get

$$-\gamma\nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0},$$

and thus

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \langle u_0, \hat{v}_2^* + \hat{v}_1^*\rangle_{L^2} - F_1(u_0, \hat{v}_0^*),$$

$$F_2^*(\hat{v}_2^*) = \langle u_0, \hat{v}_2^*\rangle_{L^2} - F_2(u_0),$$

$$G^*(\hat{v}_1^*, \hat{v}_0^*) = -\langle u_0, \hat{v}_1^*\rangle_{L^2} + \langle 0, \hat{v}_0^*\rangle_{L^2} - G(u_0, \mathbf{0}),$$

so that

$$\begin{aligned} & J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ = & -F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{v}_1^*, \hat{v}_0^*) \\ = & F_1(u_0, \hat{v}_0^*) - F_2(u_0) + G(u_0, \mathbf{0}) \\ = & J(u_0). \end{aligned} \tag{10}$$

Finally, observe that

$$J_1^*(v_2^*, v_1^*, v_0^*) \leq -\langle u, v_2^*\rangle_{L^2} + F_1(u, v_0^*) + F_2^*(v_2^*) + G(u, \mathbf{0}),$$

$\forall u \in V,\ v_2^* \in Y^*,\ v_1^* \in D^*, v_0^* \in B^*$.

Thus, we may obtain

$$\begin{aligned} & \inf_{v_2^* \in Y^*} J_1^*(v_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ \leq & \inf_{v_2^* \in Y^*} \{-\langle u, v_2^*\rangle_{L^2} + F_1(u, \hat{v}_0^*) + F_2^*(v_2^*) + G(u, \mathbf{0})\} \\ = & F_1(u, \hat{v}_0^*) - F_2(u) + G(u, \mathbf{0}) \\ = & J(u) + \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2\hat{v}_0^* u - f)^2\, dx,\ \forall u \in V. \end{aligned} \tag{11}$$

From this and (11), we obtain

$$\begin{aligned} & J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ = & \inf_{v_2^* \in Y^*}\left\{\sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*)\right\} \\ \leq & \inf_{u \in V}\left\{J(u) + \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2\hat{v}_0^* u - f)^2\, dx\right\}. \end{aligned} \tag{12}$$

Joining the pieces, from a concerning convexity in $u$, we have got

$$\begin{aligned} J(u_0) &= \inf_{u \in V}\left\{J(u) + \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2\hat{v}_0^* u - f)^2\, dx\right\} \\ &= \inf_{v_2^* \in Y^*}\left\{\sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*)\right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \end{aligned} \tag{13}$$

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The proof is complete.

□

**Remark 2.2.** *We could have also defined*

$$B^* = \left\{v_0^* \in Y^* \ : \ \|2v_0^*\|_\infty < K/8 \text{ and } -\gamma\nabla^2 + 2v_0^* > \varepsilon I_d\right\},$$

*for a small parameter* $0 < \varepsilon \ll 1$. *This corresponds to* $-\gamma\nabla^2 + 2v_0^*$ *be positive definite, whereas the previous case corresponds to* $-\gamma\nabla^2 + 2v_0^*$ *be negative definite.*

# 3 One more duality principle and a concerning convex dual variational formulation

In this section we establish another duality principle and related convex dual formulation. Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx \\ &+\frac{\alpha}{2}\int_\Omega (u^2 - \beta)^2 \, dx - \langle u, f\rangle_{L^2}. \end{aligned} \tag{14}$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $G : V \times Y \to \mathbb{R}$ by

$$F_1(u) = \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx - \langle u, f\rangle_{L^2},$$

$$F_2(u) = \frac{K}{2}\int_\Omega u^2 \, dx$$

and

$$G(u, v, v_3^*) = \frac{\alpha}{2}\int_\Omega (u^2 - \beta + v)^2 \, dx + \frac{K_2}{2}\int_\Omega u^2 \, dx + \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2v_3^* - h_1)^2 \, dx,$$

for appropriate $\gamma_1 > 0$ and $h_1 \in L^2(\Omega)$ to be specified.

We define also $F_1^* : Y^* \to \mathbb{R}$, $F_2^* : Y^* \to \mathbb{R}$, and $G^* : [Y^*]^4 \to \mathbb{R}$, by

$$\begin{aligned} F_1^*(v_1^*) &= \sup_{u \in V}\{\langle u, v_1^*\rangle_{L^2} - F_1(u)\} \\ &= \frac{1}{2}\int_\Omega \frac{(v_1^*)^2}{-\gamma\nabla^2} \, dx, \end{aligned} \tag{15}$$

$$\begin{aligned} F_2^*(z^*) &= \sup_{u \in V}\{\langle u, z^*\rangle_{L^2} - F_2(u)\} \\ &= \frac{1}{2K}\int_\Omega (z^*)^2 \, dx \end{aligned} \tag{16}$$

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and

$$\begin{aligned} & G^*(v_3^*, v_1^*, v_0^*, z^*) \\ = & \sup_{(u,v)\in V\times Y} \{\langle u, -v_1^*\rangle_{L^2} + \langle v, v_0^*\rangle_{L^2} - G(u,v)\} \\ = & \frac{-1}{2}\int_\Omega \frac{-h_1^2 K_1(K+2v_0^*) + 2h_1K_1(-\gamma_1\nabla^2 + 2v_3^*)(-v_1^*+z^*) + (-v_1^*+z^*)^2}{K+2v_0^* + K_1(-\gamma_1\nabla^2+2v_3^*)^2}\,dx \quad (17) \end{aligned}$$

if $v_0^* \in B^*$ where

$$B^* = \{v_0^* \in Y^* \ : \ \|2v_0^*\|_\infty < K/8\}.$$

Furthermore, we define

$$D^* = \{v_1^* \in Y^* \ : \ \|v_1^*\|_\infty \leq (3/2)K\}$$

and $J_1^* : Y^* \times D^* \times B^* \to \mathbb{R}$, by

$$J_1^*(v_3^*, v_1^*, v_0^*, z^*) = -F_1^*(v_1^*) + F_2^*(z^*) - G^*(v_3^*, v_1^*, v_0^*, z^*).$$

where we also define

$$C^* = \{v_3^* \in Y^* \ : \ -\gamma_1\nabla^2 + 2v_3^* \geq K_3 I_d\}.$$

Observe that we may choose $\gamma_1 > 0$ and $h_1 \in L^2(\Omega)$ so that such a last constraint is satisfied by a critical point.

Moreover, assuming

$$K_1 \gg K \gg \max\{K_3, \|f\|_\infty, \alpha, \beta, \gamma, \gamma_1\}$$

by directly computing $\delta^2 J_1^*(v_3^*, v_1^*, v_0^*, z^*)$ we may obtain that for such specified real constants, $J_1^*$ in convex in $z^*$ and it is concave in $(v_3^*, v_1^*, v_0^*)$ on $C^* \times D^* \times B^* \times Y^*$.

## 3.1 The main duality principle and a related convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

**Theorem 3.1.** *Let* $(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \in C^* \times D^* \times B^* \times Y^*$ *be such that*

$$\delta J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \mathbf{0}$$

*and* $u_0 \in V$ *be such that*

$$u_0 = \frac{\partial F_2^*(\hat{z}^*)}{\partial z^*}.$$

*Under such hypotheses, we have*

$$\delta J(u_0) = \mathbf{0},$$

$$-\gamma_1\nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1 = \mathbf{0},$$

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*and*

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 \, dx \right\} \\ &= \inf_{z^* \in Y^*} \left\{ \sup_{(v_3^*, v_1^*, v_0^*) \in C^* \times D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*, z^*) \right\} \\ &= J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*). \end{aligned} \tag{18}$$

*Proof.* Observe that $\delta J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \mathbf{0}$ so that, since $J_1^*$ is convex in $z^*$ and concave in $(v_3^*, v_1^*, v_0^*)$ on $C^* \times D^* \times B^* \times Y^*$, from the Min-Max theorem, we obtain

$$J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \inf_{z^* \in Y^*} \left\{ \sup_{(v_3^*, v_1^*, v_0^*) \in C^* \times D^* \times B^*} J_1^*(v_3^*, v_1^*, v_0^*, z^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial z^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{z}^*)}{\partial z^*} = u_0$$

we have

$$-\frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial z^*} + u_0 = \mathbf{0}$$

and

$$\hat{z}^* = K u_0.$$

Observe now that

$$\begin{aligned} & G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ = & \sup_{(u,v) \in V \times Y} \{ \langle u, v_1^* + z^* \rangle_{L^2} + \langle v, v_0^* \rangle_{L^2} - G(u,v) \} \\ = & \sup_{(u,v) \in V \times Y} \left\{ \langle u, v_1^* + z^* \rangle_{L^2} + \langle v, v_0^* \rangle_{L^2} - \frac{\alpha}{2} \int_\Omega (u^2 - \beta + v)^2 \, dx \right. \\ & \left. - \frac{K}{2} \int_u^2 dx - \frac{K_1}{2} \int_\Omega (-\gamma_1 \nabla^2 u + 2 v_3^* u - h_1)^2 \, dx \right\}. \end{aligned} \tag{19}$$

Denoting $w = u^2 - \beta + v$ so that $v = w - u^2 + \beta$, we obtain

$$\begin{aligned} & G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ = & \sup_{(u,w) \in V \times Y} \left\{ \langle u, -v_1^* + z^* \rangle_{L^2} + \langle -u^2 + \beta + w, v_0^* \rangle_{L^2} - \frac{\alpha}{2} \int_\Omega w^2 \, dx - \frac{K}{2} \int_\Omega u^2 \, dx \right. \\ & \left. - \frac{K_1}{2} \int_\Omega (-\gamma_1 \nabla^2 u + 2 v_3^* u - h_1)^2 \, dx \right\} \\ = & \sup_{u \in V} \left\{ \langle u, -v_1^* + z^* \rangle_{L^2} + \langle -u^2 + \beta, v_0^* \rangle_{L^2} - \frac{K}{2} \int_\Omega u^2 \, dx \right. \\ & \left. + \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx - \frac{K_1}{2} \int_\Omega (-\gamma_1 \nabla^2 u + 2 v_3^* u - h_1)^2 \, dx \right\}. \end{aligned} \tag{20}$$

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Denoting

$$H(v_3^*, v_1^*, v_0^*, z^*, u) = \langle u, v_1^* + z^* \rangle_{L^2} + \langle -u^2 + \beta, v_0^* \rangle_{L^2} - \frac{K}{2} \int_\Omega u^2 \, dx + \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx - \frac{K_1}{2} \int_\Omega (-\gamma_1 \nabla^2 u + 2v_3^* u - h_1)^2 \, dx, \quad (21)$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial z^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial z^*} \\ &+ \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial z^*} \\ &= \hat{u}. \end{aligned} \quad (22)$$

Summarizing, we have got

$$u_0 = \frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial z^*} = \hat{u}.$$

Furthermore,

$$\begin{aligned} \frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial v_1^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial v_1^*} \\ &+ \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_1^*} \\ &= -\hat{u} \\ &= -u_0. \end{aligned} \quad (23)$$

Also,

$$\begin{aligned} \frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial v_0^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial v_0^*} \\ &+ \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_0^*} \\ &= \hat{u}^2 \\ &= u_0^2. \end{aligned} \quad (24)$$

Finally,

$$\begin{aligned} \frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial v_3^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial v_3^*} \\ &+ \frac{\partial H(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_3^*} \\ &= -K_1(-\gamma_1 \nabla^2 \hat{u} + 2\hat{v}_3^* \hat{u} - h_1) 2\hat{u}. \end{aligned} \quad (25)$$

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Thus, from this last result and from the variation of $J_1^*$ in $v_3^*$, we have

$$-\frac{\partial G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial v_3^*} = K_1(-\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1) 2u_0 = \mathbf{0}.$$

Hence, since $u_0 \neq 0$, a.e. in $\Omega$, we have got

$$-\gamma_1 \nabla^2 u_0 + 2\hat{v}_3^* u_0 - h_1 = \mathbf{0}.$$

Moreover, from the variation of $J_1^*$ in $v_0^*$, we obtain

$$-\frac{v_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

From the variation of $J^*$ in $v_1^*$ we get

$$-\frac{\hat{v}_1^* + f}{-\gamma \nabla^2} + u_0 = \mathbf{0}$$

so that

$$\hat{v}_1^* = -\gamma \nabla^2 u_0 - f.$$

On the other hand from these last results and from

$$\frac{\partial H(\hat{v}_3, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, u_0)}{\partial u} = \mathbf{0},$$

we get

$$-\hat{v}_1^* + \hat{z}^* - K u_0 - 2v_0^* u_0 = \mathbf{0},$$

so that

$$\hat{v}_1^* = -\gamma \nabla^2 u_0 - f = -2v_0^* u_0 = -2\alpha(u_0^2 - \beta) u_0,$$

that is,

$$-\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta) u_0 - f = \mathbf{0}.$$

From this, we may infer that

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_1^*) = \langle u_0, \hat{v}_1^* \rangle_{L^2} - F_1(u_0),$$

$$F_2^*(\hat{z}^*) = \langle u_0, \hat{z}^* \rangle_{L^2} - F_2(u_0),$$

$$G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \langle u_0, -\hat{v}_1^* + \hat{z}^* \rangle_{L^2} + \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, \mathbf{0}, \hat{v}_3^*),$$

so that

$$\begin{aligned} & J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ = & -F_1^*(\hat{v}_1^*) + F_2^*(\hat{z}^*) - G^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ = & F_1(u_0) - F_2(u_0) + G(u_0, \mathbf{0}, \hat{v}_3^*) \\ = & J(u_0). \end{aligned} \tag{26}$$

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Finally, observe that

$$J_1^*(v_3^*, v_1^*, v_0^*, z^*) \leq -\langle u, z^* \rangle_{L^2} + F_1(u) + F_2^*(z^*) + G(u, \mathbf{0}, \hat{v}_3^*),$$

$\forall u \in V,\ z^* \in Y^*,\ v_3^* \in C^*,\ v_1^* \in D^*, v_0^* \in B^*.$

Thus, we may obtain

$$\begin{aligned} & \inf_{z^* \in Y^*} J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, z^*) \\ \leq & \inf_{z^* \in Y^*} \{-\langle u, z^* \rangle_{L^2} + F_1(u) + F_2^*(z^*) + G(u, \mathbf{0}, \hat{v}_3^*)\} \\ = & F_1(u) - F_2(u) + G(u, \mathbf{0}, \hat{v}_3^*) \\ = & J(u) + \frac{K_1}{2} \int_\Omega (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 \, dx, \ \forall u \in V. \end{aligned} \quad (27)$$

From this, we obtain

$$\begin{aligned} & J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ = & \inf_{z^* \in Y^*} \left\{ \sup_{(v_3^*, v_1^*, v_0^*) \in C^* \times D^* \times B^*} J_1^*(v_3^*, v_1^*, v_0^*, z^*) \right\} \\ \leq & \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h)^2 \, dx \right\}. \end{aligned} \quad (28)$$

Joining the pieces, from a concerning convexity in $u$, we have got

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (-\gamma_1 \nabla^2 u + 2\hat{v}_3^* u - h_1)^2 \, dx \right\} \\ &= \inf_{z^* \in Y^*} \left\{ \sup_{(v_3^*, v_1^*, v_0^*) \in C^* \times D^* \times B^*} J_1^*(v_3^*, v_1^*, v_0^*, z^*) \right\} \\ &= J_1^*(\hat{v}_3^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*). \end{aligned} \quad (29)$$

The proof is complete.

□

# 4 Conclusion

In this article we have developed a convex dual variational formulation suitable for the local optimization of a non-convex primal formulation.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principle here presented is applied to a Ginzburg-Landau type equation. In a future research, we intend to extend such results for some models of plates and shells and other models in the elasticity theory.

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