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Article

On Duality Principles and Related Convex Dual Formulations Suitable for Local Non-Convex Variational Optimization

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Abstract: This article develops duality principles and related convex dual and primal dual formulations suitable for the local optimization of non-convex primal formulations for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a Ginzburg-Landau type equation. Finally, we emphasize the novelty here is that both the dual and primal dual formulations developed are convex for a primal formulation which is originally non-convex.

Keywords: Convex dual variational formulation; duality principle for non-convex primal local optimization; Ginzburg-Landau type equation

MSC: 49N15

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1. Introduction

In this article we establish a duality principle and a related convex dual formulation suitable for the local optimization of the primal formulation for a large class of models in non-convex optimization.

The main duality principle is applied to the Ginzburg-Landau system in superconductivity in the absence of a magnetic field.

Such results are based on the works of J.J. Telega and W.R. Bielski [2,3,13,14] and on a D.C. optimization approach developed in Toland [15].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [5–7,9,12]. Finally, similar models on the superconductivity physics may be found in [4,11].

Remark 1.1. *It is worth highlighting, we may generically denote*

$$\int_\Omega [(-\gamma\nabla^2 + KI_d)^{-1}v^*]v^*\, dx$$

simply by

$$\int_\Omega \frac{(v^*)^2}{-\gamma\nabla^2 + K}\, dx,$$

where I_d denotes a concerning identity operator.

Other similar notations may be used along this text as their indicated meaning are sufficiently clear.

Finally, ∇^2 denotes the Laplace operator and for real constants $K_2 > 0$ and $K_1 > 0$, the notation $K_2 \gg K_1$ means that $K_2 > 0$ is much larger than $K_1 > 0$.

Now we present some basic definitions and statements.

Definition 1.2. *Let V be a Banach space. We define the topological dual space of V, denoted by V', as the set of all continuous and linear functionals defined on V.*

We assume V' may be represented through another Banach space denoted by V^* and a bilinear form

$$\langle \cdot\, ,\, \cdot \rangle_V : V \times V^* \to \mathbb{R}.$$

More specifically, for each $f \in V'$, we suppose there exists a unique $u^* \in V^*$ such that

$$f(u) = \langle u, u^* \rangle_V,\ \forall u \in V.$$

Moreover, we define the norm of f, denoted by

$$\|f\|_{V^*}$$

by

$$\|f\|_{V^*} = \sup\{|\langle u, u^* \rangle_V \ :\ u \in V \text{ and } \|u\|_V \leq 1\} \equiv \|u^*\|_{V^*}.$$

For an open, bounded and connected set $\Omega \subset \mathbb{R}^N$ and $Y = Y^* = L^2(\Omega)$ we recall that

$$\langle u, u^* \rangle_{L^2} = \int_\Omega u\, u^*\, dx.$$

More specifically, for each continuous and linear functional $f : Y \to \mathbb{R}$ there exists a unique $u^* \in Y^* = L^2(\Omega)$ such that

$$f(u) = \int_\Omega u\, u^*\, dx,\ \forall u \in Y = L^2(\Omega).$$

Definition 1.3 (Polar functional)**.** *Let V be a Banach space and let $F : V \to \mathbb{R}$ be a functional.*

We define the polar functional of F, denoted by $F^* : V^* \to \mathbb{R}$, by

$$F^*(u^*) = \sup_{u \in V}\{\langle u, u^* \rangle_{L^2} - F(u)\},\ \forall u^* \in V^*.$$

Another important definition refers to the Legendre transform one and respective relevant propriety, which are summarized in the next theorem.

Theorem 1.4 (Legendre transform theorem)**.** *Let V be a Banach space and let $F : V \to \mathbb{R}$ be a twice continuously Fréchet differentiable functional.*

Let $u^* \in V^*$. Assume there exists a unique $\hat{u} \in V$ such that

$$u^* = \frac{\partial F(\hat{u})}{\partial u}.$$

Suppose also

$$\det\left\{\frac{\partial^2 F(u)}{\partial u^2}\right\} \neq 0,$$

in a neighborhood of $\hat{u}$.

Under such hypotheses, defining the Legendre tranform of F at u^* by $F_L^*(u^*)$ where

$$F_L^*(u^*) = \langle \hat{u}, u^* \rangle_V - F(\hat{u})$$

we have that

$$\hat{u} = \frac{\partial F_L^*(u^*)}{\partial u^*}.$$

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Remark 1.5. *Concerning such a last definition, observe that if F is convex on V, then the extremal condition*

$$u^* = \frac{\partial F(\hat{u})}{\partial u},$$

corresponds to globally maximize

$$H(u) = \langle u, u^* \rangle_V - F(u)$$

on V, so that, in such a case,

$$F^*(u^*) = H(\hat{u}) = \langle \hat{u}, u^* \rangle_V - F(\hat{u}) = F_L^*(u^*).$$

Summarizing, if F is convex, under the hypotheses of the last theorem, the polar functional $F^*(u^*)$ coincides with the Legendre transform of F on V^* already denoted by F_L^*, that is,

$$F^*(u) = F_L^*(u^*), \; \forall u^* \in V^*.$$

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx \\ &+ \frac{\alpha}{2}\int_\Omega (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}. \end{aligned} \tag{1}$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functionals $F_1 : V \times Y \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^* \rangle_{L^2} \\ &- \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2v_0^* u - f)^2 \, dx + \frac{K_2}{2}\int_\Omega (\nabla^2 u)^2 \, dx \\ &- \langle u, f \rangle_{L^2} - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx \\ &- \beta \int_\Omega v_0^* \, dx, \end{aligned} \tag{2}$$

and

$$F_2(u) = \frac{K_2}{2}\int_\Omega (\nabla^2 u)^2 \, dx.$$

At this point we assume a finite dimensional version for this concerning model. For example, we may define a new domain for the primal functional considering the projection of V on the space spanned by the first N (in general N=10, is enough) eigen-vectors of the Laplace operator, corresponding to the first N eigen-values. On this new not relabeled finite dimensional space V, we recall that the Laplace operator is bounded, so that we assume the following restriction,

$$-\gamma\nabla^2 + 2v_0^* \leq \mathbf{0}.$$

Hence, we denote

$$B^* = \{v_0^* \in Y^* \; : \; \|2v_0^*\|_\infty < K/2 \text{ and } -\gamma\nabla^2 + 2v_0^* \leq \mathbf{0}\},$$

for an appropriate real constant $K > 0$.

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We define also $F_1^* : Y^* \times B^* \to \mathbb{R}$ and $F_2^* : Y^* \to \mathbb{R}$, by

$$\begin{aligned}
&F_1^*(v_2^*, v_0^*) \\
=\ & \sup_{u \in V} \{\langle u, v_2^* \rangle_{L^2} - F_1(u, v_0^*)\} \\
=\ & \frac{1}{2} \int_\Omega \frac{(v_2^* + f - K_1(-\gamma\nabla^2 + 2v_0^*)f)^2}{K_2\nabla^4 - \gamma\nabla^2 + 2v_0^* - K_1(-\gamma\nabla^2 + 2v_0^*)^2}\, dx \\
& + \frac{1}{2\alpha} \int_\Omega (v_0^*)^2\, dx + \beta \int_\Omega v_0^*\, dx, + \frac{K_1}{2} \int_\Omega f^2\, dx \qquad (3)
\end{aligned}$$

and,

$$\begin{aligned}
F_2^*(v_2^*) &= \sup_{u \in V} \{\langle u, v_2^* \rangle_{L^2} - F_2(u)\} \\
&= \frac{1}{2K_2} \int_\Omega \frac{(v_2^*)^2}{\nabla^4}\, dx. \qquad (4)
\end{aligned}$$

Furthermore, we define

$$D^* = \{v_2^* \in Y^* \ :\ \|v_2^*\|_\infty \leq 5K_2/4\}$$

and $J_1^* : D^* \times B^* \to \mathbb{R}$, by

$$J_1^*(v_2^*, v_0^*) = -F_1^*(v_2^*, v_0^*) + F_2^*(v_2^*).$$

Assuming $0 < \alpha \ll 1$ (through a re-scaling, if necessary) and

$$K_2 \gg K_1 \gg K \gg \max\{\|f\|_\infty, \alpha, \beta, \gamma, 1\}$$

by directly computing $\delta^2 J_1^*(v_2^*, v_0^*)$ we may easily obtain that for such specified real constants, J_1^* in concave in (v_2^*, v_0^*) on $D^* \times B^*$.

2. The main duality principle and a concerning convex (in fact concave) dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

Theorem 2.1. *Let $(\hat{v}_2^*, \hat{v}_0^*) \in D^* \times B^*$ be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V$ be such that

$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*}.$$

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

and

$$\begin{aligned}
J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{1}{2K_2} \int_\Omega \left(\frac{\hat{v}_2^*}{-\nabla^2} - K_2(-\nabla^2 u) \right)^2 dx \right\} \\
&= \sup_{(v_2^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_0^*) \\
&= J_1^*(\hat{v}_2^*, \hat{v}_0^*). \qquad (5)
\end{aligned}$$

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Proof. Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_0^*) = \mathbf{0}$ so that, since J_1^* is concave in (v_2^*, v_0^*) on $D^* \times B^*$, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_0^*) = \sup_{(v_2^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_0^*).$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, v_0^*)}{\partial v_2^*} + u_0 = \mathbf{0}$$

and

$$\hat{v}_2^* - K_2 \nabla^4 u_0 = \mathbf{0}.$$

Observe now that denoting

$$H(v_2^*, v_0^*, u) = \langle u, v_2^* \rangle_{L^2} - F_1(u, v_0^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_2^*, \hat{v}_0^*) = H(\hat{v}_2^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}. \end{aligned} \tag{6}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_2^*} = \hat{u}.$$

Also, denoting

$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

from

$$\frac{\partial H(\hat{v}_2{}^*, \hat{v}_0^*, u_0)}{\partial u} = \mathbf{0},$$

we have

$$-(-\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f - K_1(-\gamma \nabla^2 + 2\hat{v}_0^*) A(u_0, \hat{v}_0^*) - \hat{v}_2^* + K_2 \nabla^4 u_0) = \mathbf{0},$$

so that

$$A(u_0, \hat{v}_0^*) - K_1(-\gamma \nabla^2 + 2\hat{v}_0^*) A(u_0, \hat{v}_0^*) = \mathbf{0}. \tag{7}$$

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From such results, we may infer that

$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* - f = 0, \text{ in } \Omega.$$

Moreover, from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$-K_1 A(u_0, \hat{v}_0^*) 2u_0 - \frac{\hat{v}_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

From such last results we get

$$-\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta) u_0 - f = \mathbf{0},$$

and thus

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_2^*, \hat{v}_0^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_1(u_0, \hat{v}_0^*),$$

$$F_2^*(\hat{v}_2^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0),$$

so that

$$\begin{aligned} & J_1^*(\hat{v}_2^*, \hat{v}_0^*) \\ = & -F_1^*(\hat{v}_2^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) \\ = & F_1(u_0, \hat{v}_0^*) - F_2(u_0) \\ = & J(u_0). \end{aligned} \tag{8}$$

Finally, observe that

$$J_1^*(v_2^*, v_0^*) \leq -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_0^*) + F_2^*(v_2^*),$$

$\forall u \in V,\ v_2^* \in D^*, v_0^* \in B^*$.

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Thus, we may obtain

$$\begin{aligned}
& J_1^*(\hat{v}_2^*, \hat{v}_0^*) \\
\leq & -\langle u, \hat{v}_2^* \rangle_{L^2} + \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \\
& -\frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx + F_2(u) + F_2^*(\hat{v}_2^*) - \langle u, f \rangle_{L^2} \\
& -\frac{1}{2\alpha} \int_\Omega (\hat{v}_0^*)^2 \, dx - \beta \int_\Omega \hat{v}_0^* \, dx \\
\leq & -\langle u, \hat{v}_2^* \rangle_{L^2} + \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \\
& + F_2(u) + F_2^*(\hat{v}_2^*) - \langle u, f \rangle_{L^2} \\
& -\frac{1}{2\alpha} \int_\Omega (\hat{v}_0^*)^2 \, dx - \beta \int_\Omega \hat{v}_0^* \, dx \\
\leq & \sup_{v_0^* \in Y^*} \left\{ -\langle u, \hat{v}_2^* \rangle_{L^2} + \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^* \rangle_{L^2} \right. \\
& + F_2(u) + F_2^*(\hat{v}_2^*) - \langle u, f \rangle_{L^2} \\
& \left. -\frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx - \beta \int_\Omega v_0^* \, dx \right\} \\
= & J(u) + F_2(u) - \langle u, \hat{v}_2^* \rangle_{L^2} + F_2^*(\hat{v}_2^*), \; \forall u \in V. \qquad (9)
\end{aligned}$$

Summarizing, we have got

$$J_1^*(\hat{v}_2^*, \hat{v}_0^*) \leq J(u) + F_2(u) - \langle u, \hat{v}_2^* \rangle_{L^2} + F_2^*(\hat{v}_2^*), \; \forall u \in V. \qquad (10)$$

Joining the pieces, from a concerning convexity in u, we have got

$$\begin{aligned}
J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{1}{2K_2} \int_\Omega \left(\frac{\hat{v}_2^*}{-\nabla^2} - K_2(-\nabla^2 u) \right)^2 \, dx \right\} \\
&= \sup_{(v_2^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_0^*) \\
&= J_1^*(\hat{v}_2^*, \hat{v}_0^*). \qquad (11)
\end{aligned}$$

The proof is complete.

□

3. A convex primal dual formulation for a local optimization of the primal one

In this section we develop a convex primal dual formulation corresponding to a non-convex primal formulation.

We start by describing the primal formulation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$\begin{aligned}
J(u) &= \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx \\
&+ \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}. \qquad (12)
\end{aligned}$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functional $J_1^*: V \times [Y^*]^2 \to \mathbb{R}$, by

$$\begin{aligned} J_1^*(u, v_3^*, v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^* \rangle_{L^2} \\ &+ \frac{K_1}{2}\int_\Omega (-\gamma \nabla^2 u + 2v_3^* u - f)^2 \, dx + \frac{K_1}{2}\int_\Omega (v_3^* - \alpha(u^2-\beta))^2 \, dx \\ &- \langle u, f \rangle_{L^2} - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx \\ &- \beta \int_\Omega v_0^* \, dx. \end{aligned} \tag{13}$$

We define also

$$B^* = \{v_0^* \in Y^* \ : \ \|2v_0^*\|_\infty < K/2\},$$

for an appropriate real constant $K > 0$.

Furthermore, we define

$$D^* = \{v_3^* \in Y^* \ : \ \|v_3^*\|_\infty \leq K_2\}$$

$$A^+ = \{u \in V \ : \ u\, f \geq 0, \text{ a.e. in } \Omega\},$$

$$V_2 = \{u \in V \ : \ \|u\|_\infty \leq K_3\}$$

for an appropriate real constant $K_3 > 0$ and

$$V_1 = A^+ \cap V_2.$$

Now observe that denoting $\varphi_1 = v_3^* - \alpha(u^2 - \beta)$, we have

$$\begin{aligned} \frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial u^2} &= K_1(-\gamma\nabla^2 + 2v_3^*)^2 + 4K_1\alpha^2 u^2 \\ &\quad -2K_1\alpha\varphi_1 - \gamma\nabla^2 + 2v_0^* \end{aligned} \tag{14}$$

and

$$\frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial (v_3^*)^2} = K_1 + 4K_1 u^2. \tag{15}$$

Denoting $\varphi = -\gamma\nabla^2 u + 2v_0^* u - f$ we have also that

$$\frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial u \partial v_3^*} = K_1(-2\gamma\nabla^2 u + 2v_3^* u) + 2K_1\varphi - 2K_1\alpha u. \tag{16}$$

In such a case, we obtain

$$\begin{aligned} \det\left\{\frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial u \partial v_3^*}\right\} &= \frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial (v_3^*)^2}\frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial u^2} - \left(\frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial v_3^* \partial u}\right)^2 \\ &= K_1^2(-\gamma\nabla^2 + 2v_3^* + 4\alpha u^2)^2 \\ &\quad +(-\gamma\nabla^2 + 2v_0^*)\mathcal{O}(K_1) \\ &\quad -4K_1^2\varphi^2 - 4K_1^2\varphi\left[(-\gamma\nabla^2 u + 2v_0^* u) - 2\alpha u\right] \\ &\quad -2K_1^2\alpha\, \varphi_1(1 + 4u^2). \end{aligned} \tag{17}$$

Observe that at a critical point

$$\varphi = \mathbf{0}$$

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and

$$\varphi_1 = 0.$$

From such results we may infer that

$$\det\left\{\frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial u \partial v_3^*}\right\} > \mathbf{0}$$

around any critical point.

With such results in mind, we may prove the following theorem.

Theorem 3.1. *Let $(u_0, \hat{v}_3^*, \hat{v}_0^*) \in V_1 \times D^* \times B^*$ be such that*

$$\delta J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}.$$

Under such hypotheses, we have

$$\delta J(u_0) = -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0}$$

and there exists $r > 0$ such that

$$\begin{aligned} J(u_0) &= \sup_{v_0^* \in B^*}\left\{\inf_{(u, v_3^*) \in B_r(u_0, \hat{v}_3^*)} J_1^*(u, v_3^*, v_0^*)\right\} \\ &= J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*). \end{aligned} \tag{18}$$

Proof. The proof that

$$\delta J(u_0) = \mathbf{0}$$

and

$$J(u_0) = J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*)$$

may be done similarly as in the previous sections.

Observe that, as previously obtained, there exists $r > 0$ such that

$$\det\left\{\frac{\partial^2 J_1^*(u, v_3^*, \hat{v}_0^*)}{\partial u \partial v_3^*}\right\} > \mathbf{0},\ \forall (u, v_3^*) \in B_r(u_0, \hat{v}_3^*)$$

and

$$\frac{\partial^2 J_1^*(u_0, \hat{v}_3^*, v_0^*)}{\partial (v_0^*)^2} < \mathbf{0}, \forall v_0^* \in B^*.$$

Since for a sufficiently large $K_1 > 0$ we have

$$\frac{\partial^2 J_1^*(u, v_3^*, \hat{v}_0^*)}{\partial u^2} > \mathbf{0},\ \text{in } B_r(u_0, \hat{v}_3^*),$$

from these last results and the standard Saddle point theorem, we have

$$J(u_0) = J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*) = \sup_{v_0^* \in B^*}\left\{\inf_{(u, v_3^*) \in B_r(u_0, \hat{v}_3^*)} J_1^*(u, v_3^*, v_0^*)\right\}.$$

The proof is complete. □

4. A duality principle for a related relaxed formulation concerning the vectorial approach in the calculus of variations

In this section we develop a duality principle for a related vectorial model in the calculus of variations.

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega = \Gamma$.

For $1 < p < +\infty$, consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = G(\nabla u) + F(u) - \langle u, f\rangle_{L^2},$$

where

$$V = \left\{u \in W^{1,p}(\Omega;\mathbb{R}^N) \ : \ u = u_0 \text{ in } \partial\Omega\right\}$$

and $f \in L^2(\Omega;\mathbb{R}^N)$.

We assume $G : Y \to \mathbb{R}$ and $F : V \to \mathbb{R}$ are Fréchet differentiable and F is also convex.

Also

$$G(\nabla u) = \int_\Omega g(\nabla u)\, dx,$$

where $g : \mathbb{R}^{N\times n}$ it is supposed to be Fréchet differentiable. Here we have denoted $Y = L^p(\Omega;\mathbb{R}^{N\times n})$.

We define also $J_1 : V \times Y_1 \to \mathbb{R}$ by

$$J_1(u,\phi) = G_1(\nabla u + \nabla_y \phi) + F(u) - \langle u, f\rangle_{L^2},$$

where

$$Y_1 = W^{1,p}(\Omega\times\Omega;\mathbb{R}^N)$$

and

$$G_1(\nabla u + \nabla_y\phi) = \frac{1}{|\Omega|}\int_\Omega\int_\Omega g(\nabla u(x) + \nabla_y\phi(x,y))\, dx\, dy.$$

Moreover, we define the relaxed functional $J_2 : V \to \mathbb{R}$ by

$$J_2(u) = \inf_{\phi\in V_0} J_1(u,\phi),$$

where

$$V_0 = \{\phi \in Y_1 \ : \ \phi(x,y) = 0, \ \text{ in } \Omega\times\partial\Omega\}.$$

Now observe that

$$\begin{aligned}
J_1(u,\phi) &= G_1(\nabla u + \nabla_y\phi) + F(u) - \langle u, f\rangle_{L^2} \\
&= -\frac{1}{|\Omega|}\int_\Omega\int_\Omega v^*(x,y)\cdot(\nabla u + \nabla_y\phi(x,y))\, dy\, dx + G_1(\nabla u + \nabla_y\phi) \\
&\quad + \frac{1}{|\Omega|}\int_\Omega\int_\Omega v^*(x,y)\cdot(\nabla u + \nabla_y\phi(x,y))\, dy\, dx + F(u) - \langle u, f\rangle_{L^2} \\
&\geq \inf_{v\in Y_2}\left\{-\frac{1}{|\Omega|}\int_\Omega\int_\Omega v^*(x,y)\cdot v(x,y)\, dy\, dx + G_1(v)\right\} \\
&\quad + \inf_{(v,\phi)\in V\times V_0}\left\{\frac{1}{|\Omega|}\int_\Omega\int_\Omega v^*(x,y)\cdot(\nabla u + \nabla_y\phi(x,y))\, dy\, dx + F(u) - \langle u, f\rangle_{L^2}\right\} \\
&= -G_1^*(v^*) - F^*\left(\operatorname{div}_x\left(\frac{1}{|\Omega|}\int_\Omega v^*(x,y)\, dy\right) + f\right) \\
&\quad + \frac{1}{|\Omega|}\int_{\partial\Omega}\left(\int_\Omega v^*(x,y)\, dy\right)\cdot u_0\, d\Gamma, \qquad (19)
\end{aligned}$$

$\forall (u, \phi) \in V \times V_0, v^* \in A^*$, where

$$A^* = \{v^* \in Y_2^* \; : \; \text{div}_y v^*(x,y) = 0, \text{ in } \Omega\}.$$

Here we have denoted

$$G_1^*(v^*) = \sup_{v \in Y_2} \left\{ \frac{1}{|\Omega|} \int_\Omega \int_\Omega v^*(x,y) \cdot v(x,y)\, dy\, dx - G_1(v) \right\},$$

where $Y_2 = L^p(\Omega \times \Omega; \mathbb{R}^{N \times n})$, $Y_2^* = L^q(\Omega \times \Omega; \mathbb{R}^{N \times n})$, and where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Furthermore,

$$\begin{aligned} & F^*\left(\text{div}_x\left(\frac{1}{|\Omega|}\int_\Omega v^*(x,y)\,dy\right) + f\right) - \frac{1}{|\Omega|}\int_{\partial\Omega}\left(\int_\Omega v^*(x,y)\,dy\right)\cdot u_0\,d\Gamma \\ = & \sup_{(v,\phi)\in V\times V_0}\left\{-\frac{1}{|\Omega|}\int_\Omega\int_\Omega v^*(x,y)\cdot(\nabla u + \nabla_y\phi(x,y))\,dy\,dx - F(u) + \langle u, f\rangle_{L^2}\right\}, \end{aligned} \quad (20)$$

Therefore, denoting $J_3^* : Y_2^* \to \mathbb{R}$ by

$$J_3^*(v^*) = -G_1^*(v^*) - F^*\left(\text{div}_x\left(\int_\Omega v^*(x,y)\,dy\right) + f\right) + \frac{1}{|\Omega|}\int_{\partial\Omega}\Omega\left(\int_\Omega v^*(x,y)\,dy\right)\cdot u_0\,d\Gamma,$$

we have got

$$\inf_{u \in V} J_2(u) \geq \sup_{v^* \in A^*} J_3^*(v^*).$$

Finally, we highlight such a dual functional J_3^* is convex (in fact concave).

5. Conclusion

In this article we have developed convex dual and primal dual variational formulations suitable for the local optimization of non-convex primal formulations.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principles here presented are applied to a Ginzburg-Landau type equation. In a future research, we intend to extend such results for some models of plates and shells and other models in the elasticity theory.

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