On duality principles and related convex dual formulations suitable for local and global non-convex variational optimization

Fabio Silva Botelho
Department of Mathematics
Federal University of Santa Catarina
Florianópolis - SC, Brazil

Abstract

This article develops duality principles and related convex dual formulations suitable for the local and global optimization of non-convex primal formulations for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a Ginzburg-Landau type equation.

Key words: Convex dual variational formulation, duality principle for non-convex local primal optimization, Ginzburg-Landau type equation

MSC 49N15

1 Introduction

In this article we establish a duality principle and a related convex dual formulation suitable for the local optimization of the primal formulation for a large class of models in non-convex optimization.

The main duality principle is applied to the Ginzburg-Landau system in superconductivity in the absence of a magnetic field.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 13, 14] and on a D.C. optimization approach developed in Toland [15].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [9, 5, 6, 7, 12]. Finally, similar models on the superconductivity physics may be found in [4, 11].

Remark 1.1. *It is worth highlighting, we may generically denote*

$$\int_\Omega [(-\gamma\nabla^2 + KI_d)^{-1}v^*]v^*\,dx$$

simply by

$$\int_\Omega \frac{(v^*)^2}{-\gamma\nabla^2+K}\,dx,$$

where I_d denotes a concerning identity operator.

Other similar notations may be used along this text as their indicated meaning are sufficiently clear.

Finally, ∇^2 denotes the Laplace operator and for real constants $K_2>0$ and $K_1>0$, the notation $K_2 \gg K_1$ means that $K_2>0$ is much larger than $K_1>0$.

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx \\ &+\frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\,dx - \langle u,f\rangle_{L^2}. \end{aligned} \tag{1}$$

Here $\gamma>0$, $\alpha>0$, $\beta>0$ and $f\in L^2(\Omega)\cap L^\infty(\Omega)$.

Moreover, $V=W_0^{1,2}(\Omega)$ and we denote $Y=Y^*=L^2(\Omega)$.

At this point we define $F_1 : V\times Y \to \mathbb{R}$, $F_2 : V\to\mathbb{R}$ and $G : V\times Y\to\mathbb{R}$ by

$$\begin{aligned} F_1(u,v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx - \frac{K}{2}\int_\Omega u^2\,dx \\ &+\frac{K_1}{2}\int_\Omega(-\gamma\nabla^2 u+2v_0^*u-f)^2\,dx+\frac{K_2}{2}\int_\Omega u^2\,dx, \end{aligned} \tag{2}$$

$$F_2(u)=\frac{K_2}{2}\int_\Omega u^2\,dx+\langle u,f\rangle_{L^2},$$

and

$$G(u,v)=\frac{\alpha}{2}\int_\Omega (u^2-\beta+v)^2\,dx+\frac{K}{2}\int_\Omega u^2\,dx.$$

We define also

$$J_1(u,v_0^*)=F_1(u,v_0^*)-F_2(u)+G(u,0),$$

$$J(u)=\frac{\gamma}{2}\int_\Omega\nabla u\cdot\nabla u\,dx+\frac{\alpha}{2}\int_\Omega(u^2-\beta)^2\,dx-\langle u,f\rangle_{L^2},$$

and $F_1^* : [Y^*]^3\to\mathbb{R}$, $F_2^* : Y^*\to\mathbb{R}$, and $G^* : [Y^*]^2\to\mathbb{R}$, by

$$\begin{aligned} & F_1^*(v_2^*,v_1^*,v_0^*) \\ =\ & \sup_{u\in V}\{\langle u, v_1^*+v_2^*\rangle_{L^2}-F_1(u,v_0^*)\} \\ =\ & \frac{1}{2}\int_\Omega \frac{\left(v_1^*+v_2^*+K_1(-\gamma\nabla^2+2v_0^*)f\right)^2}{(-\gamma\nabla^2-K+K_2+K_1(-\gamma\nabla^2+2v_0^*)^2)}\,dx \\ & -\frac{K_1}{2}\int_\Omega f^2\,dx, \end{aligned} \tag{3}$$

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u \in V}\{\langle u, v_2^*\rangle_{L^2} - F_2(u)\} \\ &= \frac{1}{2K_2}\int_\Omega (v_2^* - f)^2\,dx, \end{aligned} \tag{4}$$

and

$$\begin{aligned} G^*(v_1^*, v_0^*) &= \sup_{(u,v)\in V\times Y}\{\langle u, v_1^*\rangle_{L^2} - \langle v, v_0^*\rangle_{L^2} - G(u,v)\} \\ &= \frac{1}{2}\int_\Omega \frac{(v_1^*)^2}{2v_0^* + K}\,dx + \frac{1}{2\alpha}\int_\Omega (v_0^*)^2\,dx \\ &\quad +\beta\int_\Omega v_0^*\,dx \end{aligned} \tag{5}$$

if $v_0^* \in B^*$ where

$$B^* = \{v_0^* \in Y^* \;:\; \|v_0^*\|_\infty \leq K/2\}.$$

Define also

$$V_2 = \{u \in V \;:\; \|u\|_\infty \leq K_3\},$$

$$A^+ = \{u \in V \;:\; u\,f \geq 0 \text{ a.e. in } \Omega\},$$

$$V_1 = V_2 \cap A^+,$$

$$B_2^* = \{v_0^* \in Y^* \;:\; -\gamma\nabla^2 - K + K_1(-\gamma\nabla^2 + 2v_0^*)^2 > \mathbf{0}\},$$

$$D_3^* = \{(v_1^*, v_2^*) \in Y^* \times Y^* \;:\; -1/\alpha + 4K_1[u(v_1^*, v_2^*, v_0^*)^2] + 100/K_2 \leq \mathbf{0}, \forall v_0^* \in B^*\},$$

where

$$u(v_2^*, v_0^*) = \frac{\varphi_1}{\varphi},$$

$$\varphi_1 = (v_1^* + v_2^* + K_1(-\gamma\nabla^2 + 2v_0^*)f)$$

and

$$\varphi = (-\gamma\nabla^2 - K + K_1(-\gamma\nabla^2 + 2v_0^*)^2 + K_2),$$

$$D^* = \{v_2^* \in Y^*\,;\; \|v_2^*\|_\infty < K_4\}$$

$$E^* = \{v_1^* \in Y^* \;:\; \|v_1^*\|_\infty \leq K_5\},$$

for some $K_3, K_4, K_5 > 0$ to be specified,

Finally, we also define $J_1^* : [Y^*]^2 \times B^* \to \mathbb{R}$,

$$J_1^*(v_2^*, v_1^*, v_0^*) = -F_1^*(v_2^*, v_1^*, v_0^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*).$$

Assume now $K_1 = 1/[4(\alpha+\varepsilon)K_3^2]$,

$$K_2 \gg K_1 \gg \max\{K_3, K_4, K_5, 1, \gamma, \alpha, \beta\}.$$

Observe that, by direct computation, we may obtain

$$\frac{\partial^2 J_1^*(v_2^*, v_1^*, v_0^*)}{\partial (v_0^*)^2} = -\frac{1}{\alpha} + 4K_1 u(v^*)^2 + \mathcal{O}(1/K_2) < \mathbf{0},$$

for $v_0^* \in B_3^*$.

Considering such statements and definitions, we may prove the following theorem.

Theorem 1.2. *Let $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \in ((D^* \times E^*) \cap D_3^*) \times (B_2^* \cap B^*)$ be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V_1$, where

$$u_0 = \frac{\hat{v}_1^* + \hat{v}_2^* + K_1(-\gamma \nabla^2 + 2v_0^*) f}{K_2 - K - \gamma \nabla^2 + K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)^2}.$$

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

so that

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx \right\} \\ &= \inf_{v_2^* \in D^*} \left\{ \sup_{(v_1^*, v_0^*) \in E^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \end{aligned} \tag{6}$$

Proof. Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$ so that, since $(\hat{v}_2^*, \hat{v}_1^*) \in D_3^*, \hat{v}_0^* \in B_2^*$ and J_1^* is quadratic in v_2^*, we may infer that

$$\begin{aligned} J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) &= \inf_{v_2^* \in Y^*} J_1^*(v_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ &= \sup_{(v_1^*, v_0^*) \in E^* \times B^*} J_1^*(\hat{v}_2^*, v_1^*, v_0^*). \end{aligned} \tag{7}$$

Therefore, from a standard saddle point theorem, we have that

$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in E^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

we have

$$-u_0 + \frac{\hat{v}_2^*}{K_2} = 0,$$

and thus

$$\hat{v}_2^* = K_2 u_0.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0},$$

we obtain

$$-u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = 0,$$

and thus

$$\hat{v}_1^* = -2\hat{v}_0^* u_0 - K u_0 + f.$$

Finally, denoting

$$D = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$-2Du_0 + u_0^2 - \frac{\hat{v}_0^*}{\alpha} - \beta = 0,$$

so that

$$\hat{v}_0^* = \alpha(u_0^2 - \beta - 2Du_0). \tag{8}$$

Observe now that

$$\hat{v}_1^* + \hat{v}_2^* + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)f = (K_2 - K - \gamma\nabla^2 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)^2)u_0$$

so that

$$\begin{aligned} & K_2 u_0 - 2\hat{v}_0 u_0 - K u_0 + f \\ = \; & K_2 u_0 - K u_0 - \gamma \nabla^2 u_0 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)(-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f). \end{aligned} \tag{9}$$

The solution for this last system of equations (8) and (9) is obtained through the relations

$$\hat{v}_0^* = \alpha(u_0^2 - \beta)$$

and

$$-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f = D = 0,$$

so that

$$\delta J(u_0) = -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = 0$$

and

$$\delta \left\{ J(u_0) + \frac{K_1}{2} \int_\Omega (-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f)^2 \, dx \right\} = 0.$$

Moreover, from the Legendre transform properties

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \langle u_0, \hat{v}_2^* + \hat{v}_1^* \rangle_{L^2} - F_1(u_0, \hat{v}_0^*),$$

$$F_2^*(\hat{v}_2^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0),$$

$$G^*(\hat{v}_1^*, \hat{v}_0^*) = -\langle u_0, \hat{v}_1^* \rangle_{L^2} - \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, 0),$$

so that

$$
\begin{aligned}
J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) &= -F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{v}_1^*, \hat{v}_0^*) \\
&= F_1(u_0, \hat{v}_0^*) - F_2(u_0) + G(u_0, 0) \\
&= J(u_0). \qquad (10)
\end{aligned}
$$

Observe now that

$$
\begin{aligned}
J(u_0) &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\
&\leq \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx - \frac{K}{2}\int_\Omega u^2 \, dx \\
&+\frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + \hat{v}_0^* u - f)^2 \, dx + \langle u, \hat{v}_1^* \langle_{L^2} - \langle u, f\rangle_{L^2} \\
&-\frac{1}{2}\int_\Omega \frac{(v_1^*)^2}{2v_0^* + K} \, dx - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx - \beta \int_\Omega v_0^* \, dx \\
&\leq \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx - \langle u, f\rangle_{L^2} \\
&+\frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + \hat{v}_0^* u - f)^2 \, dx \\
&+ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} \left\{ +\langle u, \hat{v}_1^*\rangle_{L^2} - \frac{1}{2}\int_\Omega \frac{(v_1^*)^2}{2v_0^* + K} \, dx \right. \\
&\left. -\frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx - \beta\int_\Omega v_0^* \, dx \right\} \\
&= J(u) + \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx, \qquad (11)
\end{aligned}
$$

$\forall u \in V_1$.

Hence, we have got

$$
J(u_0) = \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx \right\}.
$$

Joining the pieces, we have got

$$
\begin{aligned}
J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx \right\} \\
&= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in E^* \times (B^* \cap B_r(\hat{v}_0^*))} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\
&= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \qquad (12)
\end{aligned}
$$

The proof is complete.

□

2 Another duality principle suitable for a local optimization of the primal formulation

In this section we develop a second duality principle which the dual formulation is concave.

We start by describing the primal formulation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \rightarrow \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx \\ &+ \frac{\alpha}{2}\int_\Omega (u^2-\beta)^2 \, dx - \langle u, f\rangle_{L^2}. \end{aligned} \quad (13)$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functionals $F_1 : V \times Y \rightarrow \mathbb{R}$, $F_2 : V \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^*\rangle_{L^2} \\ &- \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2v_0^* u - f)^2 \, dx + \frac{K_2}{2}\int_\Omega (\nabla^2 u)^2 \, dx \\ &- \langle u, f\rangle_{L^2} - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx \\ &- \beta\int_\Omega v_0^* \, dx, \end{aligned} \quad (14)$$

and

$$F_2(u) = \frac{K_2}{2}\int_\Omega (\nabla^2 u)^2 \, dx.$$

We define also $F_1^* : [Y^*]^2 \rightarrow \mathbb{R}$ and $F_2^* : Y^* \rightarrow \mathbb{R}$, by

$$\begin{aligned} & F_1^*(v_2^*, v_0^*) \\ = & \sup_{u \in V}\{\langle u, v_2^*\rangle_{L^2} - F_1(u, v_0^*)\} \\ = & \frac{1}{2}\int_\Omega \frac{(v_2^* + f - K_1(-\gamma\nabla^2 + 2v_0^*)f)^2}{K_2\nabla^4 - \gamma\nabla^2 + 2v_0^* - K_1(-\gamma\nabla^2 + 2v_0^*)^2} \, dx \\ & + \frac{K_1}{2}\int_\Omega f^2 \, dx \end{aligned} \quad (15)$$

and,

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u \in V}\{\langle u, v_2^*\rangle_{L^2} - F_2(u)\} \\ &= \frac{1}{2K_2}\int_\Omega \frac{(v_2^*)^2}{\nabla^4} \, dx. \end{aligned} \quad (16)$$

Here we denote

$$B^* = \{v_0^* \in Y^* \ : \ \|2v_0^*\|_\infty < K/2\},$$

for an appropriate real constant $K > 0$.

Furthermore, we define

$$D^* = \{v_2^* \in Y^* \ : \ \|v_2^*\|_\infty \leq 5K_2/4\}$$

and $J_1^*: D^* \times B^* \to \mathbb{R}$, by

$$J_1^*(v_2^*, v_0^*) = -F_1^*(v_2^*, v_0^*) + F_2^*(v_2^*).$$

Assuming $0 < \alpha \ll 1$ (through a re-scaling, if necessary) and

$$K_2 \gg K_1 \gg K \gg \max\{\|f\|_\infty, \alpha, \beta, \gamma, 1\}$$

by directly computing $\delta^2 J_1^*(v_2^*, v_0^*)$ we may easily obtain that for such specified real constants, J_1^* in concave in (v_2^*, v_0^*) on $D^* \times B^*$.

2.1 The main duality principle and a concerning convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

Theorem 2.1. *Let $(\hat{v}_2^*, \hat{v}_0^*) \in D^* \times B^*$ be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V$ be such that

$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*}.$$

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V}\left\{J(u) + \frac{1}{2K_2}\int_\Omega \left(\frac{\hat{v}_2^*}{-\nabla^2} - K_2(-\nabla^2 u)\right)^2 dx\right\} \\ &= \sup_{(v_2^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_0^*) \\ &= J_1^*(\hat{v}_2^*, \hat{v}_0^*). \end{aligned} \quad (17)$$

Proof. Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_0^*) = \mathbf{0}$ so that, since J_1^* is concave in (v_2^*, v_0^*) on $D^* \times B^*$, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_0^*) = \sup_{(v_2^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_0^*).$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, v_0^*)}{\partial v_2^*} + u_0 = \mathbf{0}$$

and

$$\hat{v}_2^* - K_2 \nabla^4 u_0 = \mathbf{0}.$$

Observe now that denoting

$$H(v_2^*, v_0^*, u) = \langle u, v_2^* \rangle_{L^2} - F_1(u, v_0^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_2^*, \hat{v}_0^*) = H(\hat{v}_2^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}. \end{aligned} \tag{18}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_2^*} = \hat{u}.$$

Also, denoting

$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

from

$$\frac{\partial H(\hat{v_2}^*, \hat{v}_0^*, u_0)}{\partial u} = \mathbf{0},$$

we have

$$-(-\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f - K_1(-\gamma \nabla^2 + 2\hat{v}_0^*) A(u_0, \hat{v}_0^*) - \hat{v}_2^* + K_2 \nabla^4 u_0) = \mathbf{0},$$

so that

$$A(u_0, \hat{v}_0^*) - K_1(-\gamma \nabla^2 + 2\hat{v}_0^*) A(u_0, \hat{v}_0^*) = \mathbf{0}. \tag{19}$$

From such results, we may infer that

$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* - f = 0, \text{ in } \Omega.$$

Moreover, from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$-K_1 A(u_0, \hat{v}_0^*) 2u_0 - \frac{\hat{v}_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

From such last results we get

$$-\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0},$$

and thus

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_2^*, \hat{v}_0^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_1(u_0, \hat{v}_0^*),$$

$$F_2^*(\hat{v}_2^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0),$$

so that

$$\begin{aligned}
& J_1^*(\hat{v}_2^*, \hat{v}_0^*) \\
= & -F_1^*(\hat{v}_2^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) \\
= & F_1(u_0, \hat{v}_0^*) - F_2(u_0) \\
= & J(u_0).
\end{aligned} \tag{20}$$

Finally, observe that

$$J_1^*(v_2^*, v_0^*) \leq -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_0^*) + F_2^*(v_2^*),$$

$\forall u \in V,\ v_2^* \in D^*, v_0^* \in B^*$.

Thus, we may obtain

$$\begin{aligned}
& J_1^*(\hat{v}_2^*, \hat{v}_0^*) \\
\leq & -\langle u, \hat{v}_2^* \rangle_{L^2} + \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \\
& -\frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx + F_2(u) + F_2^*(\hat{v}_2^*) - \langle u, f \rangle_{L^2} \\
& -\frac{1}{2\alpha} \int_\Omega (\hat{v}_0^*)^2 \, dx - \beta \int_\Omega \hat{v}_0^* \, dx \\
\leq & -\langle u, \hat{v}_2^* \rangle_{L^2} + \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \\
& +F_2(u) + F_2^*(\hat{v}_2^*) - \langle u, f \rangle_{L^2} \\
& -\frac{1}{2\alpha} \int_\Omega (\hat{v}_0^*)^2 \, dx - \beta \int_\Omega \hat{v}_0^* \, dx \\
\leq & \sup_{v_0^* \in Y^*} \left\{ -\langle u, \hat{v}_2^* \rangle_{L^2} + \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \right. \\
& +F_2(u) + F_2^*(\hat{v}_2^*) - \langle u, f \rangle_{L^2} \\
& \left. -\frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx - \beta \int_\Omega v_0^* \, dx \right\} \\
= & J(u) + F_2(u) - \langle u, \hat{v}_2^* \rangle_{L^2} + F_2^*(\hat{v}_2^*), \ \forall u \in V.
\end{aligned} \tag{21}$$

Summarizing, we have got

$$J_1^*(\hat{v}_2^*, \hat{v}_0^*) \leq J(u) + F_2(u) - \langle u, \hat{v}_2^* \rangle_{L^2} + F_2^*(\hat{v}_2^*), \; \forall u \in V. \tag{22}$$

Joining the pieces, from a concerning convexity in u, we have got

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{1}{2K_2} \int_\Omega \left(\frac{\hat{v}_2^*}{-\nabla^2} - K_2(-\nabla^2 u) \right)^2 \, dx \right\} \\ &= \sup_{(v_2^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_0^*) \\ &= J_1^*(\hat{v}_2^*, \hat{v}_0^*). \end{aligned} \tag{23}$$

The proof is complete.

□

3 A third duality principle also suitable for the primal formulation local optimization

In this section we establish one more duality principle and related convex dual formulation suitable for a local optimization of the primal variational formulation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, we define $V = W_0^{1,2}(\Omega)$ and consider a functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx \\ &\quad -\langle u, f \rangle_{L^2}. \end{aligned} \tag{24}$$

Here we assume $f \in L^2(\Omega)$, and define $Y = Y^* = L^2(\Omega)$,

$$V_2 = \{u \in V \; : \; \|u\|_\infty \leq K_4\},$$

$$A^+ = \{u \in V \; : \; u\, f > 0, \text{ a.e. in } \Omega\},$$

and

$$V_1 = A^+ \cap V_2,$$

for an appropriate constant $K_4 > 0$ to be specified.

Define also the functionals $F_1 : V \times [Y]^2 \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_3^*, v_0^*) &= \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^* \rangle_{L^2} \\ &\quad -\langle u, f \rangle_{L^2} + \frac{K_1}{2} \int_\Omega (v_3^* u - K_3)^2 \, dx, \end{aligned} \tag{25}$$

and

$$G(u^2) = \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx,$$

for appropriate positive constants K_1, K_3, K_4 to be specified.

Moreover, define $F_1^* : [Y^*]^2 \to \mathbb{R}$ and $G^* : Y^* \to \mathbb{R}$, by

$$\begin{aligned} F_1^*(v_3^*, v_0^*) &= \sup_{u \in V}\{-F_2(u, v_3^*, v_0^*)\} \\ &= \frac{1}{2}\int_\Omega \frac{(f + K_1K_3v_3^*)^2}{-\gamma\nabla^2 + 2v_0^* + K_1(v_3^*)^2}\,dx \\ &\quad -\frac{K_1}{2}\int_\Omega K_3^2\,dx \end{aligned}$$

and

$$\begin{aligned} G^*(v_0^*) &= \sup_{v \in Y}\{\langle v, v_0^*\rangle_{L^2} - G(v)\} \\ &= \frac{1}{2\alpha}\int_\Omega (v_0^*)^2\,dx + \beta\int_\Omega v_0^*\,dx. \end{aligned} \tag{26}$$

Furthermore, we define

$$B^* = \{v_3^* \in Y^* \ : \ u_1(v_3^*) \in V_1\},$$

where

$$u_1(v_3^*) = \frac{K_3}{v_3^*}.$$

Define also

$$C_1^* = \{v_0^* \in Y^* \ : \ \|v_0^*\|_\infty \leq K_2\}$$

for an appropriate real constant $K_2 > 0$ to be specified, and $J_1^* : B^* \times C_1^* \to \mathbb{R}$ by

$$J_1^*(v_3^*, v_0^*) = -F_2^*(v_3^*, v_0^*) - G^*(v_0^*).$$

Moreover, we assume $K_1 \gg K_2 \gg \max\{1, K_3, K_4, \alpha, \beta, \gamma, \|f\|_\infty\}$.

By directly computing $\delta^2 J_1^*(v_3^*, v_0^*)$ denoting

$$A = -K_1K_3,$$

$$B = 2K_1v_3^*,$$

$$\varphi = -\gamma\nabla^2 + 2v_0^* + K_1(v_3^*)^2,$$

$$\varphi_1 = f + K_1K_3v_3^*,$$

$$u = \frac{\varphi_1}{\varphi},$$

we may also obtain,

$$\begin{aligned} &\frac{\partial^2 J_1^*(v_3^*, v_0^*)}{\partial(v_3^*)^2} \\ =\ & -\frac{(A - uB)^2}{\varphi} + K_1u^2 \\ =\ & -\frac{K_1(K_1K_3^2(3u^2 - 4uu_1 + u_1^2) - u^2u_1(-\gamma\nabla^2 + 2v_0^*)u_1)}{K_1K_3^2 + u_1(-\gamma\nabla^2 + 2v_0^*)u_1} \\ =\ & \frac{K_1^2H_1 + K_1H_2}{K_1K_3^2 + u_1(-\gamma\nabla^2 - 2v_0^*)u_1} \end{aligned} \tag{27}$$

on $B^* \times C_1^*$.

where

$$u_1 = u_1(v_3^*) = \frac{K_3}{v_3^*},$$

$$H_1 = K_3^2(3u^2 - 4uu_1 + u_1^2),$$

and

$$H_2 = u^2[(-\gamma\nabla^2 + 2v_0^*)u_1]u_1.$$

At a critical point we have $H_1 = \mathbf{0}$ and

$$H_2 = u_0^2\, f u_0 > 0, \text{ a.e in } \Omega.$$

With such results, for a sufficiently small $\varepsilon_1 > 0$, we may define the restrictions

$$C_{v_0^*} = \{v_3^* \in B^* \ : \ |H_1(v_3^*, v_0^*)| \leq \varepsilon_1 (v_3^*)^2 \ \text{ in } \Omega\}$$

and

$$(C_3)_{v_0^*} = \{v_3^* \in B^* \ : \ H_2(v_3^*, v_0^*) \geq 0 \text{ in } \Omega\}.$$

At this point, we prove that $(C)_{v_0^*}$ is a convex set.

Firstly, fixing $v_0^* \in C_1^*$ observe that for K_7 sufficiently large

$$|H_1(v_3^*, v_0^*)| - \varepsilon_1(v_3^*)^2 + K_7(v_3^*)^2$$

is convex in v_3^* on B^*.

Observe also that

$$|H_1(v_3^*, v_0^*)| \leq \varepsilon_1(v_3^*)^2$$

is equivalent to

$$|H_1(v_3^*, v_0^*)| - \varepsilon_1(v_3^*)^2 + K_7(v_3^*)^2 \leq K_7(v_3^*)^2,$$

which is equivalent to

$$\sqrt{|H_1(v_3^*, v_0^*)| - \varepsilon_1(v_3^*)^2 + K_7(v_3^*)^2} \leq \sqrt{K_7}|v_3^*|.$$

Moreover, this last inequality is equivalent to

$$H_5(v_3^*, v_0^*) = \sqrt{|H_1(v_3^*, v_0^*)| - \varepsilon_1(v_3^*)^2 + K_7(v_3^*)^2} - \sqrt{K_7}|v_3^*| \leq \mathbf{0}.$$

Since for $v_3^* \in B^*$ we have $v_3^* f \geq 0$, a.e. in Ω, we may obtain that $-\sqrt{K_7}|v_3^*|$ is a convex function on B^*, so that H_5 is convex in v_3^* on B^*

From such results, we may easily infer that $C_{v_0^*}$ is a convex set, $\forall v_0^* \in C_1^*$.

Similarly, we may prove that $(C_3)_{v_0^*}$ is a convex set, $\forall v_0^* \in C_1^*$.

On the other hand, clearly we have

$$\frac{\partial^2 J_1^*(v_3^*, v_0^*)}{\partial (v_0^*)^2} < \mathbf{0} \text{ in } B^* \times C_1^*.$$

3.1 A concerning duality principle and a related convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

Theorem 3.1. *Let* $(\hat{v}_3^*, \hat{v}_0^*) \in (C_3)_{\hat{v}_0^*} \times C_1^*$ *be such that*

$$\delta J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V_1$ *be such that*

$$u_0 = \frac{\varphi_1}{\varphi} = \frac{f + K_1 K_3 \hat{v}_3^*}{-\gamma \nabla^2 + 2\hat{v}_0^* + K_1 (\hat{v}_3^*)^2}.$$

Assume also

$$u_0 \neq 0, \text{ a.e. in } \Omega.$$

Under such hypotheses, denoting $B_1^* = C_{\hat{v}_0^*} \cap (C_3)_{\hat{v}_0^*}$, *we have*

$$\delta J(u_0) = \mathbf{0},$$

$$\hat{v}_3^* u_0 - K_3 = 0, \text{ a.e. in } \Omega,$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (\hat{v}_3^* u - K_3)^2 \, dx \right\} \\ &= \inf_{v_3^* \in B_1^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_3^*, \hat{v}_0^*). \end{aligned} \tag{28}$$

Proof. Observe that $\delta J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}$ so that, since $\hat{v}_0^* \in C^*$ and $\hat{v}_3^* \in C_{\hat{v}_0^*}$, from the results in the previous lines, for a sufficiently small $\varepsilon_1 > 0$, we have that

$$J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \inf_{v_3^* \in B_1^*} J_1^*(v_3^*, \hat{v}_0^*) = \sup_{v_0^* \in C_1^*} J_1^*(\hat{v}_3^*, v_0^*).$$

Consequently, from this and the Saddle Point Theorem, we have

$$J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \inf_{v_3^* \in B_1^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

Firstly, observe that

$$F_2^*(v_3^*, v_0^*) = \sup_{u \in V} \{-F_2(u, v_3^*, v_0^*)\}.$$

Denoting

$$H(v_3^*, v_0^*, u) = -F_2(u, v_3^*, v_0^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_3^*, \hat{v}_0^*) = H(\hat{v}_3^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} u_0^2 = \frac{\partial F_1^*(\hat{v}_3^*, \hat{v}_0^*)}{\partial v_0^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial v_0^*} \\ &+ \frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u}\frac{\partial \hat{u}}{\partial v_0^*} \\ &= \hat{u}^2. \end{aligned} \tag{29}$$

Summarizing, we this last equation is satisfied through the relation

$$u_0 = \hat{u}.$$

Hence from the variation of J_1^* in v_0^*, we obtain

$$u_0^2 - \frac{v_0^*}{\alpha} - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

On the other hand, from the variation of J_1^* in v_3^*, we have

$$\begin{aligned} &\frac{\partial F_1^*(\hat{v}_3^*, \hat{v}_0^*)}{\partial v_3^*} \\ &= -K_1(\hat{v}_3^* u_0 - K_3)u_0 + \frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u}\frac{\partial \hat{u}}{\partial v_3^*} \\ &= -K_1(\hat{v}_3^* u_0 - K_3)u_0 \\ &= \mathbf{0}. \end{aligned} \tag{30}$$

From such results, since

$$u_0 \neq 0, \text{ a.e. in } \Omega,$$

we get

$$\hat{v}_3^* u_0 - K_3 = 0, \text{ a.e. in } \Omega.$$

Consequently, from such last results and from

$$u_0 = \frac{f + K_1 K_3 \hat{v}_3^*}{-\gamma\nabla^2 + 2\hat{v}_0^* + K_1(\hat{v}_3^*)^2},$$

we obtain

$$\begin{aligned} &-\gamma\nabla^2 u_0 + 2v_0^* u_0 + K_1(\hat{v}_3^*)^2 u_0 - f - K_1 K_3 \hat{v}_3^* \\ &= -\gamma\nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f \\ &= \delta J(u_0) \\ &= \mathbf{0}. \end{aligned} \tag{31}$$

Summarizing,

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_3^*, \hat{v}_0^*) = -F_1(u_0, \hat{v}_3^*, \hat{v}_0^*),$$

$$G^*(\hat{v}_0^*) = \langle u_0^2, v_0^* \rangle_{L^2} - G(u_0^2),$$

so that

$$\begin{aligned} & J_1^*(\hat{v}_0^*) \\ = & -F_1^*(\hat{v}_3^*, \hat{v}_0^*) - G^*(\hat{v}_0^*) \\ = & J(u_0). \end{aligned} \tag{32}$$

Finally, observe that

$$J_1^*(v_3^*, v_0^*) \leq F_1(u, v_3^*, v_0^*) - G^*(v_0^*),$$

$\forall u \in V_1, \;\; v_3^* \in B^*, \; v_0^* \in C_1^*$.

Therefore,

$$\begin{aligned} J_1^*(\hat{v}_3^*, \hat{v}_0^*) & \leq \sup_{v_0^* \in C_1^*} \{F_2(u, \hat{v}_3^*, v_0^*) - G^*(v_0^*)\} \\ & = J(u) + \frac{K_1}{2} \int_\Omega (\hat{v}_3^* u - K_3)^2 \, dx, \end{aligned} \tag{33}$$

$\forall u \in V_1$.

Summarizing, we have obtained

$$\begin{aligned} J(u_0) & = \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (\hat{v}_3^* u - K_3)^2 \, dx \right\} \\ & = \inf_{v_3^* \in B_1^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\} \\ & = J_1^*(\hat{v}_3^*, \hat{v}_0^*). \end{aligned} \tag{34}$$

The proof is complete.

□

4 A convex primal dual for a local optimization of the primal formulation

In this section we develop a convex primal dual formulation corresponding to a non-convex primal formulation.

We start by describing the primal formulation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx \\ &+\frac{\alpha}{2}\int_\Omega (u^2-\beta)^2 \, dx - \langle u, f\rangle_{L^2}. \end{aligned} \tag{35}$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.
Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.
Define the functional $J_1^* : V \times [Y^*]^2 \to \mathbb{R}$, by

$$\begin{aligned} J_1^*(u, v_3^*, v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^*\rangle_{L^2} \\ &+\frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2v_3^* u - f)^2 \, dx + \frac{K_1}{2}\int_\Omega (v_3^* - \alpha(u^2-\beta))^2 \, dx \\ &-\langle u, f\rangle_{L^2} - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx \\ &-\beta\int_\Omega v_0^* \, dx. \end{aligned} \tag{36}$$

We define also

$$B^* = \{v_0^* \in Y^* \ : \ \|2v_0^*\|_\infty < K/2\},$$

for an appropriate real constant $K > 0$.
Furthermore, we define

$$D^* = \{v_3^* \in Y^* \ : \ \|v_3^*\|_\infty \leq K_2\}$$

$$A^+ = \{u \in V \ : \ u\, f \geq 0, \text{ a.e. in } \Omega\},$$

$$V_2 = \{u \in V \ : \ \|u\|_\infty \leq K_3\}$$

for an appropriate real constant $K_3 > 0$ and

$$V_1 = A^+ \cap V_2.$$

Now observe that denoting $\varphi_1 = v_3^* - \alpha(u^2 - \beta)$, we have

$$\begin{aligned} \frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial u^2} &= K_1(-\gamma\nabla^2 + 2v_3^*)^2 + 4K_1\alpha^2 u^2 \\ &-2K_1\alpha\varphi_1 - \gamma\nabla^2 + 2v_0^* \end{aligned} \tag{37}$$

and

$$\frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial (v_3^*)^2} = K_1 + 4K_1 u^2. \tag{38}$$

Denoting $\varphi = -\gamma\nabla^2 u + 2v_0^* u - f$ we have also that

$$\frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial u \partial v_3^*} = K_1(-2\gamma\nabla^2 u + 2v_3^* u) + 2K_1\varphi - 2K_1\alpha u. \tag{39}$$

In such a case, we obtain

$$\begin{aligned}\det\left\{\frac{\partial^2 J_1^*(u,v_3^*,v_0^*)}{\partial u \partial v_3^*}\right\} &= \frac{\partial^2 J_1^*(u,v_3^*,v_0^*)}{\partial (v_3^*)^2}\frac{\partial^2 J_1^*(u,v_3^*,v_0^*)}{\partial u^2} - \left(\frac{\partial^2 J_1^*(u,v_3^*,v_0^*)}{\partial v_3^* \partial u}\right)^2 \\ &= K_1^2(-\gamma\nabla^2+2v_3^*+4\alpha u^2)^2 \\ &+(-\gamma\nabla^2+2v_0^*)\mathcal{O}(K_1) \\ &-4K_1^2\varphi^2-4K_1^2\varphi\left[(-\gamma\nabla^2 u+2v_0^* u)-2\alpha u\right] \\ &-2K_1^2\alpha\,\varphi_1(1+4u^2). \qquad (40)\end{aligned}$$

Observe that at a critical point

$$\varphi = \mathbf{0}$$

so that we may set the non-active restriction

$$C_1^* = \{(u,v_3^*) \in V_1 \times D^* \;:\; (\varphi)^2 = (\gamma\nabla^2 u + 2v_3^* u - f)^2 \leq \varepsilon u^2, \text{ in } \Omega\}$$

for a small parameter $0<\varepsilon \ll 1$.

Now we are going to prove that C_1^* is a convex subset of $V_1 \times D^*$.

For a $K_7>0$ observe that

$$\varphi^2 \leq \varepsilon u^2$$

is equivalent to

$$\varphi^2 + K_7 u^2 \leq (K_7+\varepsilon)u^2,$$

which is equivalent to

$$\sqrt{\varphi^2+K_7u^2}-\sqrt{K_7+\varepsilon}\,|u| \leq 0, \text{ in } \Omega.$$

Define

$$H(u,v_3^*) = \sqrt{\varphi^2+K_7u^2}-\sqrt{K_7+\varepsilon}\,|u|.$$

Observe that since for $(u,v_3^*) \in V_1 \times D^*$ we have $u\,f \geq 0$ in Ω, we have also that

$$-\sqrt{K_7+\varepsilon}\,|u|$$

is convex on $V_1 \times D^*$.

Moreover, for $K_7>0$ sufficiently large, the function

$$\sqrt{\varphi^2+K_7u^2}$$

is also convex on $V_1 \times D^*$.

Summarizing, $H(u,v_3^*)$ is convex on $V_1 \times D^*$ so that from such results, we may infer that C_1^* is a convex set.

On the other hand, at a critical point we have also $\varphi_1 = \mathbf{0}$. Now define the non-active constraint

$$C_2^* = \{(u,v_3^*) \in V_1 \times D^* \;:\; \varphi_1^2 = (v_3^* - \alpha(u^2-\beta))^2 \leq \varepsilon, \text{ in } \Omega\}.$$

Similarly as it was made for C_1^* we may prove that C_2^* is convex in $V_1 \times D^*$.

Furthermore, define

$$C_3^* = \{(u,v_3^*) \in V_1 \times D^* \;:\; D_2(u,v_3^*) \equiv (-\gamma\nabla^2+2v_3^*+4\alpha u^2) \geq \varepsilon_1 (v_3^*)^2\}.$$

We are going to prove that C_3^* is convex as well.

Observe that

$$D_2(u,v_3^*) \equiv (-\gamma\nabla^2 + 2v_3^* + 4\alpha u^2) \geq \varepsilon_1 (v_3^*)^2$$

is equivalent to

$$K_7 u^2 \geq K_7 u^2 - D_2(u,v_3^*) + \varepsilon_1 (v_3^*)^2,$$

which is equivalent to

$$\sqrt{K_7 u^2 - D_2(u,v_3^*) + \varepsilon_1 (v_3^*)^2} - \sqrt{K_7}|u| \leq 0.$$

At this point we highlight that, for $K_7 > 0$ sufficiently large, the function

$$\sqrt{K_7 u^2 - D_2(u,v_3^*) + \varepsilon_1 (v_3^*)^2}$$

is convex on $V_1 \times D^*$.

Moreover, since for $(u,v_3^*) \in V_1 \times D^*$, we have $uf \geq 0$, in Ω, the function

$$-\sqrt{K_7}|u|$$

is also convex on $V_1 \times D^*$. Therefore, we have got the function

$$\sqrt{K_7 u^2 - D_2(u,v_3^*) + \varepsilon_1 (v_3^*)^2} - \sqrt{K_7}|u|$$

is convex on $V_1 \times D^*$.

From such results we may infer that C_3^* is a convex set.

At this point, we define the convex set $C^* = C_1^* \cap C_2 \cap C_3^*$

Finally, observe that for $0 < \varepsilon \ll \varepsilon_1 \ll 1$, we have that

$$\det\{\delta^2_{u\ v_3^*} J_1^*(u,v_3^*,v_0^*)\}$$

is positive definite on $C^* \times B^*$

From such results we may infer that J_1^* is convex in (u,v_3^*) and concave in v_0^* on $C^* \times B^*$.

With such results in mind, we may prove the following theorem.

Theorem 4.1. *Let* $(u_0, \hat{v}_3^*, \hat{v}_0^*) \in C^* \times B^*$ *be such that*

$$\delta J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}.$$

Under such hypotheses, we have

$$\delta J(u_0) = -\gamma\nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0}$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2\alpha(u^2-\beta)u - f)^2\,dx \right\} \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{(u,v_3^*) \in V_1 \times D^*} J_1^*(u,v_3^*,v_0^*) \right\} \\ &= J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*). \end{aligned} \quad (41)$$

Proof. The proof that

$$\delta J(u_0) = \mathbf{0}$$

and

$$J(u_0) = J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*)$$

may be done similarly as in the previous sections.

Observe that J_1^* is convex in (u, v_3^*) and concave in v_0^* on $C^* \times B^*$, where C^* and B^* are convex sets.

From such results and Min-Max Theorem, we may infer that

$$J(u_0) = J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*) = \sup_{v_0^* \in B^*} \left\{ \inf_{(u, v_3^*) \in V_1 \times D^*} J_1^*(u, v_3^*, v_0^*) \right\}.$$

Finally, observe that

$$J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*) \leq J_1^*(u, v_3^*, \hat{v}_0^*), \forall u \in V_1, v_3^* \in D^*. \tag{42}$$

In particular for $v_3^* = \alpha(u^2 - \beta)$ we obtain

$$\begin{aligned}
J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*) &\leq \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \\
&\quad - \frac{1}{2\alpha} \int_\Omega (\hat{v}_0^*)^2 \, dx - \beta \int_\Omega \hat{v}_0^* \, dx \\
&\quad + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\alpha(u^2 - \beta)u - f)^2 \, dx - \langle u, f \rangle_{L^2} \\
&\leq \sup_{v_0^* \in Y^*} \left\{ \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^* \rangle_{L^2} \right. \\
&\quad - \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx - \beta \int_\Omega v_0^* \, dx \\
&\quad \left. + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\alpha(u^2 - \beta)u - f)^2 \, dx - \langle u, f \rangle_{L^2} \right\} \\
&= J(u) + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\alpha(u^2 - \beta)u - f)^2 \, dx, \forall u \in V_1.
\end{aligned} \tag{43}$$

Joining the pieces, we have got

$$\begin{aligned}
J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\alpha(u^2 - \beta)u - f)^2 \, dx \right\} \\
&= \sup_{v_0^* \in B^*} \left\{ \inf_{(u, v_3^*) \in V_1 \times D^*} J_1^*(u, v_3^*, v_0^*) \right\} \\
&= J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*).
\end{aligned} \tag{44}$$

The proof is complete. □

5 Conclusion

In this article we have developed convex dual variational formulations suitable for the local optimization of non-convex primal formulations.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principles here presented are applied to a Ginzburg-Landau type equation. In a future research, we intend to extend such results for some models of plates and shells and other models in the elasticity theory.

References

[1] R.A. Adams and J.F. Fournier, Sobolev Spaces, 2nd edn. (Elsevier, New York, 2003).

[2] W.R. Bielski, A. Galka, J.J. Telega, The Complementary Energy Principle and Duality for Geometrically Nonlinear Elastic Shells. I. Simple case of moderate rotations around a tangent to the middle surface. Bulletin of the Polish Academy of Sciences, Technical Sciences, Vol. 38, No. 7-9, 1988.

[3] W.R. Bielski and J.J. Telega, A Contribution to Contact Problems for a Class of Solids and Structures, Arch. Mech., 37, 4-5, pp. 303-320, Warszawa 1985.

[4] J.F. Annet, Superconductivity, Superfluids and Condensates, 2nd edn. (Oxford Master Series in Condensed Matter Physics, Oxford University Press, Reprint, 2010)

[5] F.S. Botelho, Functional Analysis, Calculus of Variations and Numerical Methods in Physics and Engineering, CRC Taylor and Francis, Florida, 2020.

[6] F.S. Botelho, *Variational Convex Analysis*, Ph.D. thesis, Virginia Tech, Blacksburg, VA -USA, (2009).

[7] F. Botelho, *Topics on Functional Analysis, Calculus of Variations and Duality*, Academic Publications, Sofia, (2011).

[8] F. Botelho, *Existence of solution for the Ginzburg-Landau system, a related optimal control problem and its computation by the generalized method of lines*, Applied Mathematics and Computation, 218, 11976-11989, (2012).

[9] F. Botelho, Functional Analysis and Applied Optimization in Banach Spaces, Springer Switzerland, 2014.

[10] J.C. Strikwerda, *Finite Difference Schemes and Partial Differential Equations*, SIAM, second edition (Philadelphia, 2004).

[11] L.D. Landau and E.M. Lifschits, Course of Theoretical Physics, Vol. 5- Statistical Physics, part 1. (Butterworth-Heinemann, Elsevier, reprint 2008).

[12] R.T. Rockafellar, Convex Analysis, Princeton Univ. Press, (1970).

[13] J.J. Telega, *On the complementary energy principle in non-linear elasticity. Part I: Von Karman plates and three dimensional solids*, C.R. Acad. Sci. Paris, Serie II, 308, 1193-1198; Part II: Linear elastic solid and non-convex boundary condition. Minimax approach, ibid, pp. 1313-1317 (1989)

[14] A.Galka and J.J.Telega *Duality and the complementary energy principle for a class of geometrically non-linear structures. Part I. Five parameter shell model; Part II. Anomalous dual variational priciples for compressed elastic beams*, Arch. Mech. 47 (1995) 677-698, 699-724.

[15] J.F. Toland, A duality principle for non-convex optimisation and the calculus of variations, Arch. Rat. Mech. Anal., **71**, No. 1 (1979), 41-61.