

On duality principles and related convex dual formulations suitable for local and global non-convex variational optimization

Fabio Silva Botelho

Department of Mathematics

Federal University of Santa Catarina

Florianópolis - SC, Brazil

Abstract

This article develops duality principles and related convex dual formulations suitable for the local and global optimization of non-convex primal formulations for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a Ginzburg-Landau type equation.

Key words: Convex dual variational formulation, duality principle for non-convex local primal optimization, Ginzburg-Landau type equation

MSC 49N15

1 Introduction

In this article we establish a duality principle and a related convex dual formulation suitable for the local optimization of the primal formulation for a large class of models in non-convex optimization.

The main duality principle is applied to the Ginzburg-Landau system in superconductivity in the absence of a magnetic field.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 13, 14] and on a D.C. optimization approach developed in Toland [15].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [9, 5, 6, 7, 12]. Finally, similar models on the superconductivity physics may be found in [4, 11].

Remark 1.1. *It is worth highlighting, we may generically denote*

$$\int_{\Omega} [(-\gamma \nabla^2 + K I_d)^{-1} v^*] v^* dx$$

simply by

$$\int_{\Omega} \frac{(v^*)^2}{-\gamma \nabla^2 + K} dx,$$

where I_d denotes a concerning identity operator.

Other similar notations may be used along this text as their indicated meaning are sufficiently clear.

Finally, ∇^2 denotes the Laplace operator and for real constants $K_2 > 0$ and $K_1 > 0$, the notation $K_2 \gg K_1$ means that $K_2 > 0$ is much larger than $K_1 > 0$.

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \rightarrow \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx \\ &\quad + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}. \end{aligned} \quad (1)$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

At this point we define $F_1 : V \times Y \rightarrow \mathbb{R}$, $F_2 : V \rightarrow \mathbb{R}$ and $G : V \times Y \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_0^*) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx - \frac{K}{2} \int_{\Omega} u^2 \, dx \\ &\quad + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2v_0^* u - f)^2 \, dx + \frac{K_2}{2} \int_{\Omega} u^2 \, dx, \end{aligned} \quad (2)$$

$$F_2(u) = \frac{K_2}{2} \int_{\Omega} u^2 \, dx + \langle u, f \rangle_{L^2},$$

and

$$G(u, v) = \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta + v)^2 \, dx + \frac{K}{2} \int_{\Omega} u^2 \, dx.$$

We define also

$$\begin{aligned} J_1(u, v_0^*) &= F_1(u, v_0^*) - F_2(u) + G(u, 0), \\ J(u) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}, \end{aligned}$$

and $F_1^* : [Y^*]^3 \rightarrow \mathbb{R}$, $F_2^* : Y^* \rightarrow \mathbb{R}$, and $G^* : [Y^*]^2 \rightarrow \mathbb{R}$, by

$$\begin{aligned} &F_1^*(v_2^*, v_1^*, v_0^*) \\ &= \sup_{u \in V} \{ \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_0^*) \} \\ &= \frac{1}{2} \int_{\Omega} \frac{(v_1^* + v_2^* + K_1(-\gamma \nabla^2 + 2v_0^*)f)^2}{(-\gamma \nabla^2 - K + K_2 + K_1(-\gamma \nabla^2 + 2v_0^*)^2)} \, dx \\ &\quad - \frac{K_1}{2} \int_{\Omega} f^2 \, dx, \end{aligned} \quad (3)$$

$$\begin{aligned}
F_2^*(v_2^*) &= \sup_{u \in V} \{ \langle u, v_2^* \rangle_{L^2} - F_2(u) \} \\
&= \frac{1}{2K_2} \int_{\Omega} (v_2^* - f)^2 dx,
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
G^*(v_1^*, v_0^*) &= \sup_{(u,v) \in V \times Y} \{ \langle u, v_1^* \rangle_{L^2} - \langle v, v_0^* \rangle_{L^2} - G(u, v) \} \\
&= \frac{1}{2} \int_{\Omega} \frac{(v_1^*)^2}{2v_0^* + K} dx + \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 dx \\
&\quad + \beta \int_{\Omega} v_0^* dx
\end{aligned} \tag{5}$$

if $v_0^* \in B^*$ where

$$B^* = \{v_0^* \in Y^* : \|v_0^*\|_{\infty} \leq K/2\}.$$

Define also

$$\begin{aligned}
V_2 &= \{u \in V : \|u\|_{\infty} \leq K_3\}, \\
A^+ &= \{u \in V : u f \geq 0 \text{ a.e. in } \Omega\}, \\
V_1 &= V_2 \cap A^+,
\end{aligned}$$

$$B_2^* = \{v_0^* \in Y^* : -\gamma \nabla^2 - K + K_1(-\gamma \nabla^2 + 2v_0^*)^2 > \mathbf{0}\},$$

$$D_3^* = \{(v_1^*, v_2^*) \in Y^* \times Y^* : -1/\alpha + 4K_1[u(v_1^*, v_2^*, v_0^*)^2] + 100/K_2 \leq \mathbf{0}, \forall v_0^* \in B^*\},$$

where

$$u(v_2^*, v_0^*) = \frac{\varphi_1}{\varphi},$$

$$\varphi_1 = (v_1^* + v_2^* + K_1(-\gamma \nabla^2 + 2v_0^*)f)$$

and

$$\varphi = (-\gamma \nabla^2 - K + K_1(-\gamma \nabla^2 + 2v_0^*)^2 + K_2),$$

$$D^* = \{v_2^* \in Y^* : \|v_2^*\|_{\infty} < K_4\}$$

$$E^* = \{v_1^* \in Y^* : \|v_1^*\|_{\infty} \leq K_5\},$$

for some $K_3, K_4, K_5 > 0$ to be specified,

Finally, we also define $J_1^* : [Y^*]^2 \times B^* \rightarrow \mathbb{R}$,

$$J_1^*(v_2^*, v_1^*, v_0^*) = -F_1^*(v_2^*, v_1^*, v_0^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*).$$

Assume now $K_1 = 1/[4(\alpha + \varepsilon)K_3^2]$,

$$K_2 \gg K_1 \gg \max\{K_3, K_4, K_5, 1, \gamma, \alpha, \beta\}.$$

Observe that, by direct computation, we may obtain

$$\frac{\partial^2 J_1^*(v_2^*, v_1^*, v_0^*)}{\partial (v_0^*)^2} = -\frac{1}{\alpha} + 4K_1 u(v^*)^2 + \mathcal{O}(1/K_2) < \mathbf{0},$$

for $v_0^* \in B_3^*$.

Considering such statements and definitions, we may prove the following theorem.

Theorem 1.2. *Let $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \in ((D^* \times E^*) \cap D_3^*) \times (B_2^* \cap B^*)$ be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V_1$, where

$$u_0 = \frac{\hat{v}_1^* + \hat{v}_2^* + K_1(-\gamma \nabla^2 + 2v_0^*)f}{K_2 - K - \gamma \nabla^2 + K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)^2}.$$

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

so that

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx \right\} \\ &= \inf_{v_2^* \in D^*} \left\{ \sup_{(v_1^*, v_0^*) \in E^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \end{aligned} \quad (6)$$

Proof. Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$ so that, since $(\hat{v}_2^*, \hat{v}_1^*) \in D_3^*$, $\hat{v}_0^* \in B_2^*$ and J_1^* is quadratic in v_2^* , we may infer that

$$\begin{aligned} J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) &= \inf_{v_2^* \in Y^*} J_1^*(v_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ &= \sup_{(v_1^*, v_0^*) \in E^* \times B^*} J_1^*(\hat{v}_2^*, v_1^*, v_0^*). \end{aligned} \quad (7)$$

Therefore, from a standard saddle point theorem, we have that

$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in E^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

we have

$$-u_0 + \frac{\hat{v}_2^*}{K_2} = 0,$$

and thus

$$\hat{v}_2^* = K_2 u_0.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0},$$

we obtain

$$-u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = 0,$$

and thus

$$\hat{v}_1^* = -2\hat{v}_0^*u_0 - Ku_0 + f.$$

Finally, denoting

$$D = -\gamma\nabla^2 u_0 + 2\hat{v}_0^*u_0 - f,$$

from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$-2Du_0 + u_0^2 - \frac{\hat{v}_0^*}{\alpha} - \beta = 0,$$

so that

$$\hat{v}_0^* = \alpha(u_0^2 - \beta - 2Du_0). \quad (8)$$

Observe now that

$$\hat{v}_1^* + \hat{v}_2^* + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)f = (K_2 - K - \gamma\nabla^2 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)^2)u_0$$

so that

$$\begin{aligned} & K_2u_0 - 2\hat{v}_0^*u_0 - Ku_0 + f \\ = & K_2u_0 - Ku_0 - \gamma\nabla^2 u_0 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)(-\gamma\nabla^2 u_0 + 2\hat{v}_0^*u_0 - f). \end{aligned} \quad (9)$$

The solution for this last system of equations (8) and (9) is obtained through the relations

$$\hat{v}_0^* = \alpha(u_0^2 - \beta)$$

and

$$-\gamma\nabla^2 u_0 + 2\hat{v}_0^*u_0 - f = D = 0,$$

so that

$$\delta J(u_0) = -\gamma\nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = 0$$

and

$$\delta \left\{ J(u_0) + \frac{K_1}{2} \int_{\Omega} (-\gamma\nabla^2 u_0 + 2\hat{v}_0^*u_0 - f)^2 dx \right\} = 0.$$

Moreover, from the Legendre transform properties

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \langle u_0, \hat{v}_2^* + \hat{v}_1^* \rangle_{L^2} - F_1(u_0, \hat{v}_0^*),$$

$$F_2^*(\hat{v}_2^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0),$$

$$G^*(\hat{v}_1^*, \hat{v}_0^*) = -\langle u_0, \hat{v}_1^* \rangle_{L^2} - \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, 0),$$

so that

$$\begin{aligned}
J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) &= -F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{v}_1^*, \hat{v}_0^*) \\
&= F_1(u_0, \hat{v}_0^*) - F_2(u_0) + G(u_0, 0) \\
&= J(u_0).
\end{aligned} \tag{10}$$

Observe now that

$$\begin{aligned}
J(u_0) &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\
&\leq \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx - \frac{K}{2} \int_{\Omega} u^2 \, dx \\
&\quad + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + \hat{v}_0^* u - f)^2 \, dx + \langle u, \hat{v}_1^* \rangle_{L^2} - \langle u, f \rangle_{L^2} \\
&\quad - \frac{1}{2} \int_{\Omega} \frac{(v_1^*)^2}{2v_0^* + K} \, dx - \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx - \beta \int_{\Omega} v_0^* \, dx \\
&\leq \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx - \langle u, f \rangle_{L^2} \\
&\quad + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + \hat{v}_0^* u - f)^2 \, dx \\
&\quad + \sup_{(v_1^*, v_0^*) \in D^* \times B^*} \left\{ \langle u, \hat{v}_1^* \rangle_{L^2} - \frac{1}{2} \int_{\Omega} \frac{(v_1^*)^2}{2v_0^* + K} \, dx \right. \\
&\quad \left. - \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx - \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx - \beta \int_{\Omega} v_0^* \, dx \right\} \\
&= J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx,
\end{aligned} \tag{11}$$

$\forall u \in V_1$.

Hence, we have got

$$J(u_0) = \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx \right\}.$$

Joining the pieces, we have got

$$\begin{aligned}
J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx \right\} \\
&= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in E^* \times (B^* \cap B_r(\hat{v}_0^*))} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\
&= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*).
\end{aligned} \tag{12}$$

The proof is complete. $\square$

2 Another duality principle suitable for a local optimization of the primal formulation

In this section we develop a second duality principle which the dual formulation is concave.

We start by describing the primal formulation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \rightarrow \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx \\ &\quad + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}. \end{aligned} \quad (13)$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functionals $F_1 : V \times Y \rightarrow \mathbb{R}$, $F_2 : V \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_0^*) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^* \rangle_{L^2} \\ &\quad - \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2v_0^* u - f)^2 \, dx + \frac{K_2}{2} \int_{\Omega} (\nabla^2 u)^2 \, dx \\ &\quad - \langle u, f \rangle_{L^2} - \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx \\ &\quad - \beta \int_{\Omega} v_0^* \, dx, \end{aligned} \quad (14)$$

and

$$F_2(u) = \frac{K_2}{2} \int_{\Omega} (\nabla^2 u)^2 \, dx.$$

We define also $F_1^* : [Y^*]^2 \rightarrow \mathbb{R}$ and $F_2^* : Y^* \rightarrow \mathbb{R}$, by

$$\begin{aligned} &F_1^*(v_2^*, v_0^*) \\ &= \sup_{u \in V} \{ \langle u, v_2^* \rangle_{L^2} - F_1(u, v_0^*) \} \\ &= \frac{1}{2} \int_{\Omega} \frac{(v_2^* + f - K_1(-\gamma \nabla^2 + 2v_0^*)f)^2}{K_2 \nabla^4 - \gamma \nabla^2 + 2v_0^* - K_1(-\gamma \nabla^2 + 2v_0^*)^2} \, dx \\ &\quad + \frac{K_1}{2} \int_{\Omega} f^2 \, dx \end{aligned} \quad (15)$$

and,

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u \in V} \{ \langle u, v_2^* \rangle_{L^2} - F_2(u) \} \\ &= \frac{1}{2K_2} \int_{\Omega} \frac{(v_2^*)^2}{\nabla^4} \, dx. \end{aligned} \quad (16)$$

Here we denote

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_\infty < K/2\},$$

for an appropriate real constant $K > 0$.

Furthermore, we define

$$D^* = \{v_2^* \in Y^* : \|v_2^*\|_\infty \leq 5K_2/4\}$$

and $J_1^* : D^* \times B^* \rightarrow \mathbb{R}$, by

$$J_1^*(v_2^*, v_0^*) = -F_1^*(v_2^*, v_0^*) + F_2^*(v_2^*).$$

Assuming $0 < \alpha \ll 1$ (through a re-scaling, if necessary) and

$$K_2 \gg K_1 \gg K \gg \max\{\|f\|_\infty, \alpha, \beta, \gamma, 1\}$$

by directly computing $\delta^2 J_1^*(v_2^*, v_0^*)$ we may easily obtain that for such specified real constants, J_1^* is concave in (v_2^*, v_0^*) on $D^* \times B^*$.

2.1 The main duality principle and a concerning convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

Theorem 2.1. *Let $(\hat{v}_2^*, \hat{v}_0^*) \in D^* \times B^*$ be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V$ be such that

$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*}.$$

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{1}{2K_2} \int_{\Omega} \left(\frac{\hat{v}_2^*}{-\nabla^2} - K_2(-\nabla^2 u) \right)^2 dx \right\} \\ &= \sup_{(v_2^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_0^*) \\ &= J_1^*(\hat{v}_2^*, \hat{v}_0^*). \end{aligned} \tag{17}$$

Proof. Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_0^*) = \mathbf{0}$ so that, since J_1^* is concave in (v_2^*, v_0^*) on $D^* \times B^*$, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_0^*) = \sup_{(v_2^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_0^*).$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, v_0^*)}{\partial v_2^*} + u_0 = \mathbf{0}$$

and

$$\hat{v}_2^* - K_2 \nabla^4 u_0 = \mathbf{0}.$$

Observe now that denoting

$$H(v_2^*, v_0^*, u) = \langle u, v_2^* \rangle_{L^2} - F_1(u, v_0^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_2^*, \hat{v}_0^*) = H(\hat{v}_2^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}. \end{aligned} \tag{18}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_2^*} = \hat{u}.$$

Also, denoting

$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

from

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, u_0)}{\partial u} = \mathbf{0},$$

we have

$$-(-\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f - K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) - \hat{v}_2^* + K_2 \nabla^4 u_0) = \mathbf{0},$$

so that

$$A(u_0, \hat{v}_0^*) - K_1(-\gamma \nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = \mathbf{0}. \tag{19}$$

From such results, we may infer that

$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f = 0, \text{ in } \Omega.$$

Moreover, from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$-K_1 A(u_0, \hat{v}_0^*) 2u_0 - \frac{\hat{v}_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

From such last results we get

$$-\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0},$$

and thus

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_2^*, \hat{v}_0^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_1(u_0, \hat{v}_0^*),$$

$$F_2^*(\hat{v}_2^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0),$$

so that

$$\begin{aligned} & J_1^*(\hat{v}_2^*, \hat{v}_0^*) \\ &= -F_1^*(\hat{v}_2^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) \\ &= F_1(u_0, \hat{v}_0^*) - F_2(u_0) \\ &= J(u_0). \end{aligned} \tag{20}$$

Finally, observe that

$$J_1^*(v_2^*, v_0^*) \leq -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_0^*) + F_2^*(v_2^*),$$

$\forall u \in V, v_2^* \in D^*, v_0^* \in B^*.$

Thus, we may obtain

$$\begin{aligned} & J_1^*(\hat{v}_2^*, \hat{v}_0^*) \\ &\leq -\langle u, \hat{v}_2^* \rangle_{L^2} + \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \\ &\quad - \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx + F_2(u) + F_2^*(\hat{v}_2^*) - \langle u, f \rangle_{L^2} \\ &\quad - \frac{1}{2\alpha} \int_{\Omega} (\hat{v}_0^*)^2 \, dx - \beta \int_{\Omega} \hat{v}_0^* \, dx \\ &\leq -\langle u, \hat{v}_2^* \rangle_{L^2} + \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \\ &\quad + F_2(u) + F_2^*(\hat{v}_2^*) - \langle u, f \rangle_{L^2} \\ &\quad - \frac{1}{2\alpha} \int_{\Omega} (\hat{v}_0^*)^2 \, dx - \beta \int_{\Omega} \hat{v}_0^* \, dx \\ &\leq \sup_{v_0^* \in Y^*} \left\{ -\langle u, \hat{v}_2^* \rangle_{L^2} + \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \right. \\ &\quad \left. + F_2(u) + F_2^*(\hat{v}_2^*) - \langle u, f \rangle_{L^2} \right. \\ &\quad \left. - \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx - \beta \int_{\Omega} v_0^* \, dx \right\} \\ &= J(u) + F_2(u) - \langle u, \hat{v}_2^* \rangle_{L^2} + F_2^*(\hat{v}_2^*), \quad \forall u \in V. \end{aligned} \tag{21}$$

Summarizing, we have got

$$J_1^*(\hat{v}_2^*, \hat{v}_0^*) \leq J(u) + F_2(u) - \langle u, \hat{v}_2^* \rangle_{L^2} + F_2^*(\hat{v}_2^*), \quad \forall u \in V. \quad (22)$$

Joining the pieces, from a concerning convexity in u , we have got

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{1}{2K_2} \int_{\Omega} \left(\frac{\hat{v}_2^*}{-\nabla^2} - K_2(-\nabla^2 u) \right)^2 dx \right\} \\ &= \sup_{(v_2^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_0^*) \\ &= J_1^*(\hat{v}_2^*, \hat{v}_0^*). \end{aligned} \quad (23)$$

The proof is complete. $\square$

3 A third duality principle also suitable for the primal formulation local optimization

In this section we establish one more duality principle and related convex dual formulation suitable for a local optimization of the primal variational formulation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, we define $V = W_0^{1,2}(\Omega)$ and consider a functional $J : V \rightarrow \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx \\ &\quad - \langle u, f \rangle_{L^2}. \end{aligned} \quad (24)$$

Here we assume $f \in L^2(\Omega)$, and define $Y = Y^* = L^2(\Omega)$,

$$V_2 = \{u \in V : \|u\|_{\infty} \leq K_4\},$$

$$A^+ = \{u \in V : u f > 0, \text{ a.e. in } \Omega\},$$

and

$$V_1 = A^+ \cap V_2,$$

for an appropriate constant $K_4 > 0$ to be specified.

Define also the functionals $F_1 : V \times [Y]^2 \rightarrow \mathbb{R}$ and $G : Y \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_3^*, v_0^*) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^* \rangle_{L^2} \\ &\quad - \langle u, f \rangle_{L^2} + \frac{K_1}{2} \int_{\Omega} (v_3^* u - K_3)^2 \, dx, \end{aligned} \quad (25)$$

and

$$G(u^2) = \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx,$$

for appropriate positive constants K_1, K_3, K_4 to be specified.

Moreover, define $F_1^* : [Y^*]^2 \rightarrow \mathbb{R}$ and $G^* : Y^* \rightarrow \mathbb{R}$, by

$$\begin{aligned} F_1^*(v_3^*, v_0^*) &= \sup_{u \in V} \{-F_2(u, v_3^*, v_0^*)\} \\ &= \frac{1}{2} \int_{\Omega} \frac{(f + K_1 K_3 v_3^*)^2}{-\gamma \nabla^2 + 2v_0^* + K_1(v_3^*)^2} dx \\ &\quad - \frac{K_1}{2} \int_{\Omega} K_3^2 dx \end{aligned}$$

and

$$\begin{aligned} G^*(v_0^*) &= \sup_{v \in Y} \{\langle v, v_0^* \rangle_{L^2} - G(v)\} \\ &= \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 dx + \beta \int_{\Omega} v_0^* dx. \end{aligned} \quad (26)$$

Furthermore, we define

$$B^* = \{v_3^* \in Y^* : u_1(v_3^*) \in V_1\},$$

where

$$u_1(v_3^*) = \frac{K_3}{v_3^*}.$$

Define also

$$C_1^* = \{v_0^* \in Y^* : \|v_0^*\|_{\infty} \leq K_2\}$$

for an appropriate real constant $K_2 > 0$ to be specified, and $J_1^* : B^* \times C_1^* \rightarrow \mathbb{R}$ by

$$J_1^*(v_3^*, v_0^*) = -F_2^*(v_3^*, v_0^*) - G^*(v_0^*).$$

Moreover, we assume $K_1 \gg K_2 \gg \max\{1, K_3, K_4, \alpha, \beta, \gamma, \|f\|_{\infty}\}$.

By directly computing $\delta^2 J_1^*(v_3^*, v_0^*)$ denoting

$$A = -K_1 K_3,$$

$$B = 2K_1 v_3^*,$$

$$\varphi = -\gamma \nabla^2 + 2v_0^* + K_1(v_3^*)^2,$$

$$\varphi_1 = f + K_1 K_3 v_3^*,$$

$$u = \frac{\varphi_1}{\varphi},$$

we may also obtain,

$$\begin{aligned} &\frac{\partial^2 J_1^*(v_3^*, v_0^*)}{\partial (v_3^*)^2} \\ &= -\frac{(A - uB)^2}{\varphi} + K_1 u^2 \\ &= -\frac{K_1(K_1 K_3^2(3u^2 - 4uu_1 + u_1^2) - u^2 u_1(-\gamma \nabla^2 + 2v_0^*)u_1)}{K_1 K_3^2 + u_1(-\gamma \nabla^2 + 2v_0^*)u_1} \\ &= \frac{K_1^2 H_1 + K_1 H_2}{K_1 K_3^2 + u_1(-\gamma \nabla^2 + 2v_0^*)u_1} \end{aligned} \quad (27)$$

on $B^* \times C_1^*$.
where

$$u_1 = u_1(v_3^*) = \frac{K_3}{v_3^*},$$

$$H_1 = K_3^2(3u^2 - 4uu_1 + u_1^2),$$

and

$$H_2 = u^2[(-\gamma \nabla^2 + 2v_0^*)u_1]u_1.$$

At a critical point we have $H_1 = \mathbf{0}$ and

$$H_2 = u_0^2 f u_0 > 0, \text{ a.e in } \Omega.$$

With such results, for a sufficiently small $\varepsilon_1 > 0$, we may define the restrictions

$$C_{v_0^*} = \{v_3^* \in B^* : |H_1(v_3^*, v_0^*)| \leq \varepsilon_1(v_3^*)^2 \text{ in } \Omega\}$$

and

$$(C_3)_{v_0^*} = \{v_3^* \in B^* : H_2(v_3^*, v_0^*) \geq 0 \text{ in } \Omega\}.$$

At this point, we prove that $(C)_{v_0^*}$ is a convex set.

Firstly, fixing $v_0^* \in C_1^*$ observe that for K_7 sufficiently large

$$|H_1(v_3^*, v_0^*)| - \varepsilon_1(v_3^*)^2 + K_7(v_3^*)^2$$

is convex in v_3^* on B^* .

Observe also that

$$|H_1(v_3^*, v_0^*)| \leq \varepsilon_1(v_3^*)^2$$

is equivalent to

$$|H_1(v_3^*, v_0^*)| - \varepsilon_1(v_3^*)^2 + K_7(v_3^*)^2 \leq K_7(v_3^*)^2,$$

which is equivalent to

$$\sqrt{|H_1(v_3^*, v_0^*)| - \varepsilon_1(v_3^*)^2 + K_7(v_3^*)^2} \leq \sqrt{K_7} |v_3^*|.$$

Moreover, this last inequality is equivalent to

$$H_5(v_3^*, v_0^*) = \sqrt{|H_1(v_3^*, v_0^*)| - \varepsilon_1(v_3^*)^2 + K_7(v_3^*)^2} - \sqrt{K_7} |v_3^*| \leq \mathbf{0}.$$

Since for $v_3^* \in B^*$ we have $v_3^* f \geq 0$, a.e. in Ω , we may obtain that $-\sqrt{K_7} |v_3^*|$ is a convex function on B^* , so that H_5 is convex in v_3^* on B^*

From such results, we may easily infer that $C_{v_0^*}$ is a convex set, $\forall v_0^* \in C_1^*$.

Similarly, we may prove that $(C_3)_{v_0^*}$ is a convex set, $\forall v_0^* \in C_1^*$.

On the other hand, clearly we have

$$\frac{\partial^2 J_1^*(v_3^*, v_0^*)}{\partial (v_0^*)^2} < \mathbf{0} \text{ in } B^* \times C_1^*.$$

3.1 A concerning duality principle and a related convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

Theorem 3.1. *Let $(\hat{v}_3^*, \hat{v}_0^*) \in (C_3)_{\hat{v}_0^*} \times C_1^*$ be such that*

$$\delta J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V_1$ be such that

$$u_0 = \frac{\varphi_1}{\varphi} = \frac{f + K_1 K_3 \hat{v}_3^*}{-\gamma \nabla^2 + 2\hat{v}_0^* + K_1(\hat{v}_3^*)^2}.$$

Assume also

$$u_0 \neq 0, \text{ a.e. in } \Omega.$$

Under such hypotheses, denoting $B_1^ = C_{\hat{v}_0^*} \cap (C_3)_{\hat{v}_0^*}$, we have*

$$\delta J(u_0) = \mathbf{0},$$

$$\hat{v}_3^* u_0 - K_3 = 0, \text{ a.e. in } \Omega,$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (\hat{v}_3^* u - K_3)^2 dx \right\} \\ &= \inf_{v_3^* \in B_1^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_3^*, \hat{v}_0^*). \end{aligned} \tag{28}$$

Proof. Observe that $\delta J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}$ so that, since $\hat{v}_0^* \in C^*$ and $\hat{v}_3^* \in C_{\hat{v}_0^*}$, from the results in the previous lines, for a sufficiently small $\varepsilon_1 > 0$, we have that

$$J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \inf_{v_3^* \in B_1^*} J_1^*(v_3^*, \hat{v}_0^*) = \sup_{v_0^* \in C_1^*} J_1^*(\hat{v}_3^*, v_0^*).$$

Consequently, from this and the Saddle Point Theorem, we have

$$J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \inf_{v_3^* \in B_1^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

Firstly, observe that

$$F_2^*(v_3^*, v_0^*) = \sup_{u \in V} \{-F_2(u, v_3^*, v_0^*)\}.$$

Denoting

$$H(v_3^*, v_0^*, u) = -F_2(u, v_3^*, v_0^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_3^*, \hat{v}_0^*) = H(\hat{v}_3^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} u_0^2 = \frac{\partial F_1^*(\hat{v}_3^*, \hat{v}_0^*)}{\partial v_0^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial v_0^*} \\ &\quad + \frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_0^*} \\ &= \hat{u}^2. \end{aligned} \tag{29}$$

Summarizing, we this last equation is satisfied through the relation

$$u_0 = \hat{u}.$$

Hence from the variation of J_1^* in v_0^* , we obtain

$$u_0^2 - \frac{v_0^*}{\alpha} - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

On the other hand, from the variation of J_1^* in v_3^* , we have

$$\begin{aligned} &\frac{\partial F_1^*(\hat{v}_3^*, \hat{v}_0^*)}{\partial v_3^*} \\ &= -K_1(\hat{v}_3^* u_0 - K_3)u_0 + \frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_3^*} \\ &= -K_1(\hat{v}_3^* u_0 - K_3)u_0 \\ &= \mathbf{0}. \end{aligned} \tag{30}$$

From such results, since

$$u_0 \neq 0, \text{ a.e. in } \Omega,$$

we get

$$\hat{v}_3^* u_0 - K_3 = 0, \text{ a.e. in } \Omega.$$

Consequently, from such last results and from

$$u_0 = \frac{f + K_1 K_3 \hat{v}_3^*}{-\gamma \nabla^2 + 2\hat{v}_0^* + K_1(\hat{v}_3^*)^2},$$

we obtain

$$\begin{aligned} &-\gamma \nabla^2 u_0 + 2v_0^* u_0 + K_1(\hat{v}_3^*)^2 u_0 - f - K_1 K_3 \hat{v}_3^* \\ &= -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f \\ &= \delta J(u_0) \\ &= \mathbf{0}. \end{aligned} \tag{31}$$

Summarizing,

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_3^*, \hat{v}_0^*) = -F_1(u_0, \hat{v}_3^*, \hat{v}_0^*),$$

$$G^*(\hat{v}_0^*) = \langle u_0^2, v_0^* \rangle_{L^2} - G(u_0^2),$$

so that

$$\begin{aligned} J_1^*(\hat{v}_0^*) &= -F_1^*(\hat{v}_3^*, \hat{v}_0^*) - G^*(\hat{v}_0^*) \\ &= J(u_0). \end{aligned} \tag{32}$$

Finally, observe that

$$J_1^*(v_3^*, v_0^*) \leq F_1(u, v_3^*, v_0^*) - G^*(v_0^*),$$

$$\forall u \in V_1, \quad v_3^* \in B^*, \quad v_0^* \in C_1^*.$$

Therefore,

$$\begin{aligned} J_1^*(\hat{v}_3^*, \hat{v}_0^*) &\leq \sup_{v_0^* \in C_1^*} \{F_2(u, \hat{v}_3^*, v_0^*) - G^*(v_0^*)\} \\ &= J(u) + \frac{K_1}{2} \int_{\Omega} (\hat{v}_3^* u - K_3)^2 dx, \end{aligned} \tag{33}$$

$$\forall u \in V_1.$$

Summarizing, we have obtained

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (\hat{v}_3^* u - K_3)^2 dx \right\} \\ &= \inf_{v_3^* \in B_1^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_3^*, \hat{v}_0^*). \end{aligned} \tag{34}$$

The proof is complete. $\square$

4 A convex primal dual for a local optimization of the primal formulation

In this section we develop a convex primal dual formulation corresponding to a non-convex primal formulation.

We start by describing the primal formulation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \rightarrow \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx \\ &\quad + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}. \end{aligned} \quad (35)$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functional $J_1^* : V \times [Y^*]^2 \rightarrow \mathbb{R}$, by

$$\begin{aligned} J_1^*(u, v_3^*, v_0^*) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^* \rangle_{L^2} \\ &\quad + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2v_3^* u - f)^2 \, dx + \frac{K_1}{2} \int_{\Omega} (v_3^* - \alpha(u^2 - \beta))^2 \, dx \\ &\quad - \langle u, f \rangle_{L^2} - \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx \\ &\quad - \beta \int_{\Omega} v_0^* \, dx. \end{aligned} \quad (36)$$

We define also

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_\infty < K/2\},$$

for an appropriate real constant $K > 0$.

Furthermore, we define

$$\begin{aligned} D^* &= \{v_3^* \in Y^* : \|v_3^*\|_\infty \leq K_2\} \\ A^+ &= \{u \in V : u f \geq 0, \text{ a.e. in } \Omega\}, \\ V_2 &= \{u \in V : \|u\|_\infty \leq K_3\} \end{aligned}$$

for an appropriate real constant $K_3 > 0$ and

$$V_1 = A^+ \cap V_2.$$

Now observe that

$$\begin{aligned} \frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial u^2} &= K_1(-\gamma \nabla^2 + 2v_3^*)^2 + 4K_1\alpha^2 u^2 \\ &\quad - \gamma \nabla^2 + 2v_0^* \end{aligned} \quad (37)$$

and

$$\frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial (v_3^*)^2} = K_1 + 4K_1 u^2. \quad (38)$$

Denoting $\varphi = -\gamma \nabla^2 u + 2v_0^* u - f$ we have also that

$$\frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial u \partial v_3^*} = K_1(-2\gamma \nabla^2 u + 2v_3^* u) + 2K_1\varphi - 2K_1\alpha u. \quad (39)$$

In such a case, we obtain

$$\begin{aligned} \det \left\{ \frac{\partial^2 J_1^*(u, v_3^*, v_0^*)}{\partial u \partial v_3^*} \right\} &= K_1^2 (-\gamma \nabla^2 + 2v_3^* + 4\alpha u^2)^2 + 4K_1^2 \alpha^2 u^2 \\ &\quad + (-\gamma \nabla^2 + 2v_0^*) \mathcal{O}(K_1) \\ &\quad - 4K_1^2 \varphi^2 - 4K_1^2 \varphi [(-\gamma \nabla^2 u + 2v_0^* u) - 2\alpha u]. \end{aligned} \quad (40)$$

Observe that at a critical point

$$\varphi = \mathbf{0}$$

so that we may set the non-active restriction

$$C^* = \{(u, v_3^*) \in V_1 \times D^* : (\varphi)^2 = (\gamma \nabla^2 u + 2v_3^* u - f)^2 \leq \varepsilon u^2, \text{ in } \Omega\}$$

for a small parameter $0 < \varepsilon \ll 1$.

Now we are going to prove that C^* is a convex subset of $V_1 \times D^*$.

For a $K_7 > 0$ observe that

$$\varphi^2 \leq \varepsilon u^2$$

is equivalent to

$$\varphi^2 + K_7 u^2 \leq (K_7 + \varepsilon) u^2,$$

which is equivalent to

$$\sqrt{\varphi^2 + K_7 u^2} - \sqrt{K_7 + \varepsilon} |u| \leq 0, \text{ in } \Omega.$$

Define

$$H(u, v_3^*) = \sqrt{\varphi^2 + K_7 u^2} - \sqrt{K_7 + \varepsilon} |u|.$$

Observe that since for $(u, v_3^*) \in V_1 \times D^*$ we have $u f \geq 0$ in Ω , we have also that

$$-\sqrt{K_7 + \varepsilon} |u|$$

is convex on $V_1 \times D^*$.

Moreover, for $K_7 > 0$ sufficiently large, the function

$$\sqrt{\varphi^2 + K_7 u^2}$$

is also convex on $V_1 \times D^*$.

Summarizing, $H(u, v_3^*)$ is convex on $V_1 \times D^*$ so that from such results, we may infer that C^* is a convex set.

Finally, observe that for $\varepsilon > 0$ sufficiently small,

$$\det\{\delta_{u v_3^*}^2 J_1^*(u, v_3^*, v_0^*)\}$$

is positive definite on $C^* \times B^*$

From such results we may infer that J_1^* is convex in (u, v_3^*) and concave in v_0^* on $C^* \times B^*$.

With such results in mind, we may prove the following theorem.

Theorem 4.1. Let $(u_0, \hat{v}_3^*, \hat{v}_0^*) \in V_1 \times D^* \times B^*$ be such that

$$\delta J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}.$$

Under such hypotheses, we have

$$\delta J(u_0) = -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0}$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\alpha(u^2 - \beta)u - f)^2 dx \right\} \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{(u, v_3^*) \in V_1 \times D^*} J_1^*(u, v_3^*, v_0^*) \right\} \\ &= J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*). \end{aligned} \quad (41)$$

Proof. The proof that

$$\delta J(u_0) = \mathbf{0}$$

and

$$J(u_0) = J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*)$$

may be done similarly as in the previous sections.

Observe that J_1^* is convex in (u, v_3^*) and concave in v_0^* on $C^* \times B^*$, where C^* and B^* are convex sets.

From such results and Min-Max Theorem, we may infer that

$$J(u_0) = J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*) = \sup_{v_0^* \in B^*} \left\{ \inf_{(u, v_3^*) \in V_1 \times D^*} J_1^*(u, v_3^*, v_0^*) \right\}.$$

Finally, observe that

$$J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*) \leq J_1^*(u, v_3^*, \hat{v}_0^*), \forall u \in V_1, v_3^* \in D^*. \quad (42)$$

In particular for $v_3^* = \alpha(u^2 - \beta)$ we obtain

$$\begin{aligned} J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*) &\leq \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \\ &\quad - \frac{1}{2\alpha} \int_{\Omega} (\hat{v}_0^*)^2 dx - \beta \int_{\Omega} \hat{v}_0^* dx \\ &\quad + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\alpha(u^2 - \beta)u - f)^2 dx - \langle u, f \rangle_{L^2} \\ &\leq \sup_{v_0^* \in Y^*} \left\{ \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u dx + \langle u^2, v_0^* \rangle_{L^2} \right. \\ &\quad - \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 dx - \beta \int_{\Omega} v_0^* dx \\ &\quad \left. + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\alpha(u^2 - \beta)u - f)^2 dx - \langle u, f \rangle_{L^2} \right\} \\ &= J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\alpha(u^2 - \beta)u - f)^2 dx, \forall u \in V_1. \end{aligned} \quad (43)$$

Joining the pieces, we have got

$$\begin{aligned}
 J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\alpha(u^2 - \beta)u - f)^2 dx \right\} \\
 &= \sup_{v_0^* \in B^*} \left\{ \inf_{(u, v_3^*) \in V_1 \times D^*} J_1^*(u, v_3^*, v_0^*) \right\} \\
 &= J_1^*(u_0, \hat{v}_3^*, \hat{v}_0^*).
 \end{aligned} \tag{44}$$

The proof is complete. $\square$

5 Conclusion

In this article we have developed convex dual variational formulations suitable for the local optimization of non-convex primal formulations.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principles here presented are applied to a Ginzburg-Landau type equation. In a future research, we intend to extend such results for some models of plates and shells and other models in the elasticity theory.

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