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Article

On Duality Principles and Related Convex Dual Formulations Suitable for Local and Global Non-Convex Variational Optimization

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Abstract: This article develops duality principles and related convex dual formulations suitable for the local and global optimization of non-convex primal formulations for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a Ginzburg-Landau type equation.

Keywords: convex dual variational formulation; duality principle for non-convex local primal optimization; Ginzburg-Landau type equation

MSC: 49N15

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1. Introduction

In this article we establish a duality principle and a related convex dual formulation suitable for the local optimization of the primal formulation for a large class of models in non-convex optimization.

The main duality principle is applied to the Ginzburg-Landau system in superconductivity in the absence of a magnetic field.

Such results are based on the works of J.J. Telega and W.R. Bielski [2,3,13,14] and on a D.C. optimization approach developed in Toland [15].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [5–7,9,12]. Finally, similar models on the superconductivity physics may be found in [4,11].

Remark 1. *It is worth highlighting, we may generically denote*

$$\int_\Omega [(-\gamma\nabla^2 + KI_d)^{-1}v^*]v^*\,dx$$

simply by

$$\int_\Omega \frac{(v^*)^2}{-\gamma\nabla^2 + K}\,dx,$$

where I_d denotes a concerning identity operator.

Other similar notations may be used along this text as their indicated meaning are sufficiently clear.

Finally, ∇^2 denotes the Laplace operator and for real constants $K_2 > 0$ and $K_1 > 0$, the notation $K_2 \gg K_1$ means that $K_2 > 0$ is much larger than $K_1 > 0$.

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

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For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx \\ &+ \frac{\alpha}{2}\int_\Omega (u^2-\beta)^2 \, dx - \langle u, f\rangle_{L^2}. \end{aligned} \tag{1}$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.
Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.
Define the functionals $F_1 : V \times Y \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $G : V \times Y \to \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx - \frac{K}{2}\int_\Omega u^2 \, dx \\ &+ \frac{K_1}{2}\int_\Omega (-\gamma \nabla^2 u + 2v_0^* u - f)^2 \, dx + \frac{K_2}{2}\int_\Omega u^2 \, dx, \end{aligned} \tag{2}$$

$$F_2(u) = \frac{K_2}{2}\int_\Omega u^2 \, dx$$

and

$$G(u, v) = \frac{\alpha}{2}\int_\Omega (u^2 - \beta + v)^2 \, dx + \frac{K}{2}\int_\Omega u^2 \, dx - \langle u, f\rangle_{L^2}.$$

We define also $F_1^* : [Y^*]^3 \to \mathbb{R}$, $F_2^* : Y^* \to \mathbb{R}$, and $G^* : [Y^*]^2 \to \mathbb{R}$, by

$$\begin{aligned} & F_1^*(v_2^*, v_1^*, v_0^*) \\ &= \sup_{u \in V}\{\langle u, v_1^* + v_2^*\rangle_{L^2} - F_1(u, v_0^*)\} \\ &= \int_\Omega \frac{(v_1^* + v_2^* + K_1(-\gamma\nabla^2 + 2v_0^*)f)^2}{2[K_2 - K - \gamma\nabla^2 + K_1(-\gamma\nabla^2 + 2v_0^*)^2]} \, dx \\ & - \frac{K_1}{2}\int_\Omega f^2 \, dx, \end{aligned} \tag{3}$$

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u \in V}\{\langle u, v_2^*\rangle_{L^2} - F_2(u)\} \\ &= \frac{1}{2K_2}\int_\Omega (v_2^*)^2 \, dx \end{aligned} \tag{4}$$

and

$$\begin{aligned} G^*(v_1^*, v_0^*) &= \sup_{(u,v) \in V \times Y}\{-\langle u, v_1^*\rangle_{L^2} + \langle v, v_0^*\rangle_{L^2} - G(u, v)\} \\ &= \frac{1}{2}\int_\Omega \frac{(v_1^* - f)^2}{2v_0^* + K} \, dx + \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx \\ & + \beta\int_\Omega v_0^* \, dx \end{aligned} \tag{5}$$

if $v_0^* \in B^*$ where

$$B^* = \left\{ v_0^* \in Y^* \; : \; \|2v_0^*\|_\infty < K/8 \text{ and } -\gamma\nabla^2 + 2v_0^* > \varepsilon I_d \right\},$$

for a small parameter $0 < \varepsilon \ll 1$.

Furthermore, we define

$$D^* = \{v_1^* \in Y^* \; : \; \|v_1^*\|_\infty \leq (3/2)K\}$$

and $J_1^* : Y^* \times D^* \times B^* \to \mathbb{R}$, by

$$J_1^*(v_2^*, v_1^*, v_0^*) = -F_1^*(v_2^*, v_1^*, v_0^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*).$$

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Assuming
$$K_2 \gg K_1 \gg K \gg \max\{\|f\|_\infty, \alpha, \beta, \gamma, 1/\varepsilon^2\}$$
by directly computing $\delta^2 J_1^*(v_2^*, v_1^*, v_0^*)$ we may obtain that for such specified real constants, J_1^* in convex in v_2^* and it is concave in (v_1^*, v_0^*) on $Y^* \times D^* \times B^*$.

2. The Main Duality Principle and a Concerning Convex Dual Formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

Theorem 1. *Let $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \in Y^* \times D^* \times B^*$ be such that*
$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$$
and $u_0 \in V$ be such that
$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*}.$$

Under such hypotheses, we have
$$\delta J(u_0) = \mathbf{0},$$
and
$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx \right\} \\ &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \end{aligned} \tag{6}$$

Proof. Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$ so that, since J_1^* is convex in v_2^* and concave in (v_1^*, v_0^*) on $Y^* \times D^* \times B^*$, from the Min-Max theorem, we obtain
$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\}.$$

Now we are going to show that
$$\delta J(u_0) = \mathbf{0}.$$

From
$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$
and
$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$
we have
$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, v_0^*)}{\partial v_2^*} + u_0 = \mathbf{0}$$
and
$$\hat{v}_2^* - K_2 u_0 = \mathbf{0}.$$

Observe now that denoting
$$H(v_2^*, v_1^*, v_0^*, u) = \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_0^*),$$

there exists $\hat{u} \in V$ such that
$$\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$
and
$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = H(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{u}),$$
so that
$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial v_2^*} \\ &+ \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u}\frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}. \end{aligned} \tag{7}$$

Summarizing, we have got
$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \hat{u}.$$

Also, denoting
$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$
from
$$\frac{\partial H(\hat{v}_1^*, \hat{v}_2{}^*, \hat{v}_0^*, u_0)}{\partial u} = \mathbf{0},$$
we have
$$-\hat{v}_1^* + K u_0 + \gamma \nabla^2 u_0 + K_1(-\gamma \nabla^2 + 2\hat{v}_0^*) A(u_0, \hat{v}_0^*) - \hat{v}_2^* + K_2 u_0 = \mathbf{0},$$
so that
$$-\hat{v}_1^* + K u_0 + \gamma \nabla^2 u_0 + K_1(-\gamma \nabla^2 + 2\hat{v}_0^*) A(u_0, \hat{v}_0^*) = \mathbf{0}. \tag{8}$$

From such results, we may infer that
$$\begin{aligned} &\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} \\ =\; &\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial v_1^*} \\ &+ \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u}\frac{\partial \hat{u}}{\partial v_1^*} \\ =\; &\hat{u} \\ =\; &u_0. \end{aligned} \tag{9}$$

Now observe that from the variation of J_1^* in v_1^*, we have
$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0}$$
so that
$$-u_0 - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0}$$
that is
$$-u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = \mathbf{0}.$$

From this and (8), we may infer that

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$$\hat{v}_1^* = -\gamma\nabla^2 u_0 - K u_0 - K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = -(2\hat{v}_0^* + K)u_0 + f,$$

so that

$$-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f - K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = 0.$$

From this and the concerning boundary conditions, since

$$A(u_0, v_0^*) = -\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

we may obtain

$$-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f = A(u_0, \hat{v}_0^*) = 0.$$

Moreover, from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$A(u_0, \hat{v}_0^*) 2u_0 - \frac{\hat{v}_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

From such last results we get

$$-\gamma\nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0},$$

and thus

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \langle u_0, \hat{v}_2^* + \hat{v}_1^* \rangle_{L^2} - F_1(u_0, \hat{v}_0^*),$$

$$F_2^*(\hat{v}_2^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0),$$

$$G^*(\hat{v}_1^*, \hat{v}_0^*) = -\langle u_0, \hat{v}_1^* \rangle_{L^2} + \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, \mathbf{0}),$$

so that

$$\begin{aligned} & J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ = \; & -F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{v}_1^*, \hat{v}_0^*) \\ = \; & F_1(u_0, \hat{v}_0^*) - F_2(u_0) + G(u_0, \mathbf{0}) \\ = \; & J(u_0). \end{aligned} \qquad (10)$$

Finally, observe that

$$J_1^*(v_2^*, v_1^*, v_0^*) \leq -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_0^*) + F_2^*(v_2^*) + G(u, \mathbf{0}),$$

$\forall u \in V, \; v_2^* \in Y^*, \; v_1^* \in D^*, v_0^* \in B^*$.

Thus, we may obtain

$$
\begin{aligned}
& \inf_{v_2^* \in Y^*} J_1^*(v_2^*, \hat{v}_1^*, \hat{v}_0^*) \\
\leq & \inf_{v_2^* \in Y^*} \{-\langle u, v_2^* \rangle_{L^2} + F_1(u, \hat{v}_0^*) + F_2^*(v_2^*) + G(u, \mathbf{0})\} \\
= & F_1(u, \hat{v}_0^*) - F_2(u) + G(u, \mathbf{0}) \\
= & J(u) + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx, \ \forall u \in V. \qquad (11)
\end{aligned}
$$

From this and (11), we obtain

$$
\begin{aligned}
& J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\
= & \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\
\leq & \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx \right\}. \qquad (12)
\end{aligned}
$$

Joining the pieces, from a concerning convexity in u, we have got

$$
\begin{aligned}
J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 \, dx \right\} \\
&= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\
&= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \qquad (13)
\end{aligned}
$$

The proof is complete. □

Remark 2. *We could have also defined*

$$
B^* = \left\{ v_0^* \in Y^* \ : \ \|2v_0^*\|_\infty < K/8 \text{ and } -\gamma \nabla^2 + 2v_0^* < -\varepsilon I_d \right\},
$$

for a small parameter $0 < \varepsilon \ll 1$. This corresponds to $-\gamma \nabla^2 + 2v_0^*$ be negative definite, whereas the previous case corresponds to $-\gamma \nabla^2 + 2v_0^*$ be positive definite. It is worth recalling the inequality

$$
-\gamma \nabla^2 + 2v_0^* < -\varepsilon I_d
$$

necessarily refers to a finite dimensional version for the model in question, in a finite elements or finite differences context.

3. One More Duality Principle Suitable for the Primal Formulation Global Optimization

In this section we establish one more duality principle and related convex dual formulation suitable for a global optimization of the primal variational formulation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, we define $V = W_0^{1,2}(\Omega)$ and consider a functional $J : V \to \mathbb{R}$ where

$$
\begin{aligned}
J(u) = & \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx \\
& -\langle u, f \rangle_{L^2}. \qquad (14)
\end{aligned}
$$

Here we assume $f \in L^2(\Omega)$, and define $Y = Y^* = L^2(\Omega)$

$$V_2 = \{u \in V \ : \ \|u\|_\infty \leq K_4\},$$

$$A^+ = \{u \in V \ : \ u\, f > 0, \text{ a.e. in } \Omega\},$$

and

$$V_1^* = A^+ \cap V_2,$$

for an appropriate constant $K_4 > 0$ to be specified.

Define also the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V \times Y \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ by

$$F_1(u) = \frac{K_2}{2}\int_\Omega (\nabla^2 u)^2\, dx - \langle u, f\rangle_{L^2},$$

$$\begin{aligned} F_2(u, v_3^*, v_0^*) &= -\frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u\, dx - \langle u^2, v_0^*\rangle_{L^2} + \frac{K_2}{2}\int_\Omega (\nabla^2 u)^2\, dx \\ & -\frac{K_1}{2}\int_\Omega (v_3^* u - K_3)^2\, dx, \end{aligned} \tag{15}$$

and

$$G(u^2) = \frac{\alpha}{2}\int_\Omega (u^2 - \beta)^2\, dx,$$

for appropriate positive constants K_1, K_2, K_3, K_4 to be specified.

Moreover, define $F_1^* : Y^* \to \mathbb{R}$, and $F_2^* : [Y^*]^2 \to \mathbb{R}$ and $G^* : Y^* \to \mathbb{R}$, by

$$\begin{aligned} F_1^*(v_2^*) &= \sup_{u \in V}\{\langle u, v_2^*\rangle_{L^2} - F_1(u)\} \\ &= \frac{1}{2K_2}\int_\Omega \frac{(v_2^* + f)^2}{\nabla^4}\, dx, \end{aligned} \tag{16}$$

and

$$\begin{aligned} F_2^*(v_2^*, v_3^*, v_0^*) &= \sup_{u \in V}\{\langle u, v_2^*\rangle_{L^2} - F_2(u, v_3^*, v_0^*)\} \\ &= \frac{1}{2}\int_\Omega \frac{(v_2^* - K_1 K_3 v_3^*)^2}{K_2\nabla^4 + \gamma\nabla^2 - 2v_0^* - K_1(v_3^*)^2} \\ & -\frac{K_1}{2}\int_\Omega K_3^2\, dx \end{aligned}$$

and

$$\begin{aligned} G^*(v_0^*) &= \sup_{v \in Y}\{\langle v, v_0^*\rangle_{L^2} - G(v)\} \\ &= \frac{1}{2\alpha}\int_\Omega (v_0^*)^2\, dx + \beta\int_\Omega v_0^*\, dx. \end{aligned} \tag{17}$$

Furthermore, we define

$$D^* = \{v_2^* \in Y^* \ : \ \|v_2^*\|_\infty \leq (3/2)K_2\},$$

$$B^* = \{v_3^* \in Y^* \ : \ u_1(v_3^*) \in V_1\},$$

where

$$u_1(v_3^*) = \frac{K_3}{v_3^*}.$$

Define also

$$C_1^* = \{v_0^* \in Y^* \ : \ \|v_0^*\|_\infty \leq K_4\}.$$

and $J_1^*: D^* \times C_1^* \to \mathbb{R}$ by

$$J_1^*(v_2^*, v_3^*, v_0^*) = -F_1^*(v_2^*) + F_2^*(v_2^*, v_3^*, v_0^*) - G^*(v_0^*).$$

Moreover, assuming $K_2 \gg K_1 \gg K_4 \gg \max\{1, K_3, \alpha, \beta, \gamma, \|f\|_\infty\}$.
By directly computing $\delta^2 J_1^*(v_2^*, v_3^*, v_0^*)$ denoting

$$A = -K_1 K_3,$$

$$B = 2K_1 v_3^*,$$

$$\varphi = -K_2 \nabla^4 - \gamma \nabla^2 + 2v_0^* + K_1(v_3^*)^2),$$

$$\varphi_1 = v_2^* - K_1 K_3 v_3^*,$$

$$u = -\frac{\varphi_1}{\varphi},$$

we may obtain, considering that $\varphi < 0$

$$\frac{\partial^2 J_1^*(v_2^*, v_3^*, v_0^*)}{\partial (v_3^*)^2} = 4K_1 u^2 - \frac{(A + 2uB)^2}{\varphi} > \mathbf{0}$$

on $D^* \times B^*$.

Moreover,

$$\begin{aligned} & \frac{\partial^2 J_1^*(v_2^*, v_3^*, v_0^*)}{\partial (v_2^*)^2} \frac{\partial^2 J_1^*(v_2^*, v_3^*, v_0^*)}{\partial (v_3^*)^2} - \left(\frac{\partial^2 J_1^*(v_2^*, v_3^*, v_0^*)}{\partial v_2^* \partial v_3^*} \right)^2 \\ = & \frac{K_1(-K_1 K_3^2(3u^2 - 4uu_1 + u_1^2) + u_1^2[(G + 2v_0^*)u]u)}{K_2(\nabla^4)(-K_1 K_3^2 + u_1(K_2(\nabla^4) + \gamma \nabla^2 - 2v_0^*)u_1)} \\ = & \frac{K_1^2 H_1 + K_1 H_2}{K_2(\nabla^4)(-K_1 K_3^2 + u_1(K_2 \nabla^4 + \gamma \nabla^2 - 2v_0^*)u_1)}, \end{aligned} \tag{18}$$

where

$$u_1 = u_1(v_3^*) = \frac{K_3}{v_3^*},$$

$$H_1 = -K_3^2(3u^2 - 4uu_1 + u_1^2),$$

and

$$H_2 = u_1^2[(-\gamma \nabla^2 + 2v_0^*)u]u.$$

At a critical point we have $H_1 = \mathbf{0}$ and

$$H_2 = u_0^2 f u_0 > 0, \text{ a.e in } \Omega.$$

With such results, we may define the restrictions

$$C_2^* = \{v_0^* \in Y^* \; : \; H_1(v_2^*, v_3^*, v_0^*) \geq 0, \text{ in } \Omega, \; \forall v_2^* \in D^*, v_3^* \in B^*\}.$$

$$C_3^* = \{v_0^* \in Y^* \; : \; H_2(v_2^*, v_3^*, v_0^*) \geq 0, \text{ in } \Omega, \; \forall v_2^* \in D^*, \; v_3^* \in B^*\}.$$

Here, we define $C^* = C_1^* \cap C_2^* \cap C_3^*$.

On the other hand, clearly we have

$$\frac{\partial^2 J_1^*(v_2^*, v_3^*, v_0^*)}{\partial (v_0^*)^2} < \mathbf{0}$$

From such results, we may obtain that J_1^* in convex in (v_2^*, v_3^*) and it is concave in v_0^* on $D^* \times B^* \times C^*$.

3.1. The Main Duality Principle and a Related Convex Dual Formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

Theorem 2. *Let* $(\hat{v}_2^*, \hat{v}_3^* \hat{v}_0^*) \in D^* \times B^* \times C^*$ *be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V_1$ *be such that*

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*)}{\partial v_2^*}.$$

Assume also

$$u_0 \neq 0, \text{ a.e. in } \Omega.$$

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

$$\hat{v}_3^* u_0 - K_3 = 0, \text{ a.e. in } \Omega,$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} J(u) \\ &= \inf_{(v_2^*, v_3^*) \in D^* \times B^*} \left\{ \sup_{v_0^* \in C^*} J_1^*(v_2^*, v_3^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*). \end{aligned} \tag{19}$$

Proof. Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}$ so that, since J_1^* is convex in $(v_2^*, v_3^*) \in D^* \times B^* \times C^*$, $\hat{v}_0^* \in C^*$ and

$$\frac{\partial^2 J_1^*(\hat{v}_2^*, \hat{v}_3^*, v_0^*)}{\partial (v_0^*)^2} < \mathbf{0}, \ \forall v_0^* \in C_1^*,$$

we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*) = \inf_{(v_2^*, v_3^*) \in D^* \times B^*} J_1^*(v_2^*, v_3^*, \hat{v}_0^*) = \sup_{v_0^* \in C^*} J_1^*(\hat{v}_2^*, \hat{v}_3^*, v_0^*).$$

Consequently, from this and the Saddle Point Theorem, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*) = \inf_{(v_2^*, v_3^*) \in D^* \times B^*} \left\{ \sup_{v_0^* \in C^*} J_1^*(v_2^*, v_3^*, v_0^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_1^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_2^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*)}{\partial v_2^*} - u_0 = \mathbf{0}$$

and

$$\hat{v}_2^* = K_2 \nabla^4 u_0 - f.$$

Observe now that

$$F_2^*(\hat{v}_2^*, \hat{v}_3^*, v_0^*) = \sup_{u \in V} \{\langle u, v_2^* \rangle_{L^2} - F_2(u, v_3^*, v_0^*)\}.$$

Denoting

$$H(v_2^*, v_3^*, v_)^*, u) = \langle u, v_2^* \rangle_{L^2} - F_2(u, v_3^*, v_0^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_2^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*) = H(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_2^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}. \end{aligned} \tag{20}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*)}{\partial v_2^*} = \hat{u}.$$

From such results and the Legendre tranform proprieties we get

$$v_2^* = \frac{\partial F_1(u_0)}{\partial u}$$

and

$$v_2^* = \frac{\partial F_2(u_0, \hat{v}_3^*, \hat{v}_0^*)}{\partial u}.$$

On the other hand, from the variation of J_1^* in v_3^*, we have

$$\begin{aligned} &\frac{\partial F_2^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*)}{\partial v_3^*} \\ &= -K_1(\hat{v}_3^* u_0 - K_3) u_0 + \frac{\partial H(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_3^*} \\ &= -K_1(\hat{v}_3^* u_0 - K_3) u_0 \\ &= \mathbf{0}. \end{aligned} \tag{21}$$

From such results, since

$$u_0 \neq 0, \text{ a.e. in } \Omega,$$

we get

$$\hat{v}_3^* u_0 - K_3 = 0, \text{ a.e. in } \Omega.$$

Finally, from the variation of J_1^* in v_0^* we obtain

$$\frac{\partial F_2^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*)}{\partial v_0^*} - \frac{\partial G^*(v_0^*)}{\partial v_0^*} = 0,$$

so that

$$u_0^2 + \frac{\partial H(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_0^*} - \frac{v_0^*}{\alpha} - \beta = \mathbf{0}.$$

Thus,

$$v_0^* = \alpha(u_0^2 - \beta).$$

Consequently, from such last results, we have

$$\begin{aligned} \mathbf{0} &= \hat{v}_2^* - \hat{v}_2^* \\ &= \frac{\partial F_1(u_0)}{\partial u} - \frac{\partial F_2(u_0, \hat{v}_3^*, \hat{v}_0^*)}{\partial u} \\ &= K_2 \nabla^4 u_0 - f - K_2 \nabla^4 u_0 - \gamma \nabla^2 u_0 + 2 v_0^* u_0 \\ &= -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta) u_0 - f \\ &= \delta J(u_0). \end{aligned} \tag{22}$$

Summarizing,

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_2^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_1(u_0),$$

$$F_2^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0, \hat{v}_3^*, \hat{v}_0^*),$$

$$G^*(\hat{v}_0^*) = \langle u_0^2, v_0^* \rangle_{L^2} - G(u_0^2),$$

so that

$$\begin{aligned} & J_1^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*) \\ = & -F_1^*(\hat{v}_2^*) + F_2^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*) - G^*(\hat{v}_0^*) \\ = & J(u_0). \end{aligned} \tag{23}$$

Finally, observe that

$$J_1^*(v_2^*, v_3^*, v_0^*) \leq F_1(u) - \langle u, v_2^* \rangle_{L^2} + F_2^*(v_2^*, v_3^*, v_0^*) - G^*(v_0^*),$$

$\forall u \in V_1,\ v_2^* \in D^*,\ v_3^* \in B^*,\ v_0^* \in C^*$.

Therefore,

$$\sup_{v_0^* \in C^*} J_1^*(v_2^*, v_3^*, v_0^*) \leq \sup_{v_0^* \in C_1^*} \{-\langle u, v_2^* \rangle_{L^2} + F_1(u) + F_2^*(v_2^*, v_3^*, v_0^*) - G^*(v_0^*)\},$$

so that

$$\begin{aligned} & \inf_{(v_2^*, v_3^*) \in D^* \times B^*} \left\{ \sup_{v_0^* \in C^*} J_1^*(v_2^*, v_3^*, v_0^*) \right\} \\ \leq & \inf_{(v_2^*, v_3^*) \in D^* \times B^*} \left\{ \sup_{v_0^* \in C_1^*} \{-\langle u, v_2^* \rangle_{L^2} + F_1(u) + F_2^*(v_2^*, v_3^*, v_0^*) - G^*(v_0^*)\} \right\} \\ = & J(u), \forall u \in V_1. \end{aligned} \tag{24}$$

Summarizing, we have got

$$\begin{aligned} J_1^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*) &= \inf_{(v_2^*, v_3^*) \in D^* \times B^*} \left\{ \sup_{v_0^* \in C^*} J_1^*(v_2^*, v_3^*, v_0^*) \right\} \\ &\leq \inf_{u \in V_1} J(u). \end{aligned} \tag{25}$$

Joining the pieces, we have got

$$\begin{aligned} J(u_0) &= \inf_{u\in V_1} J(u) \\ &= \inf_{(v_2^*,v_3^*)\in D^*\times B^*}\left\{\sup_{v_0^*\in C^*} J_1^*(v_2^*,v_3^*,v_0^*)\right\} \\ &= J_1^*(\hat{v}_2^*,\hat{v}_3^*,\hat{v}_0^*). \end{aligned} \tag{26}$$

The proof is complete. □

4. Another Duality Principle Suitable for the Primal Formulation Global Optimization

In this section we establish one more duality principle and related convex dual formulation suitable for a global optimization of the primal variational formulation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, we define $V = W_0^{1,2}(\Omega)$ and consider a functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx + \frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\,dx \\ &\quad -\langle u,f\rangle_{L^2}. \end{aligned} \tag{27}$$

Here we assume $f \in L^2(\Omega)$, and define $Y = Y^* = L^2(\Omega)$

$$V_2 = \{u \in V \;:\; \|u\|_\infty \leq K_4\},$$

$$A^+ = \{u \in V \;:\; u\,f > 0, \text{ a.e. in } \Omega\},$$

and

$$V_1^* = A^+ \cap V_2,$$

for an appropriate constant $K_4 > 0$ to be specified.

Define also the functionals $F_1 : V \times [Y]^2 \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ by

$$\begin{aligned} F_1(u,v_3^*,v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx + \langle u^2, v_0^*\rangle_{L^2} \\ &\quad -\langle u,f\rangle_{L^2} + \frac{K_1}{2}\int_\Omega (v_3^* u - K_3)^2\,dx, \end{aligned} \tag{28}$$

and

$$G(u^2) = \frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\,dx,$$

for appropriate positive constants K_1, K_2, K_3, K_4 to be specified.

Moreover, define $F_1^* : [Y^*]^2 \to \mathbb{R}$ and $G^* : Y^* \to \mathbb{R}$, by

$$\begin{aligned} F_1^*(v_3^*,v_0^*) &= \sup_{u\in V}\{-F_2(u,v_3^*,v_0^*)\} \\ &= \frac{1}{2}\int_\Omega \frac{(f+K_1K_3v_3^*)^2}{-\gamma\nabla^2 + 2v_0^* + K_1(v_3^*)^2}\,dx \\ &\quad -\frac{K_1}{2}\int_\Omega K_3^2\,dx \end{aligned}$$

and

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$$\begin{aligned} G^*(v_0^*) &= \sup_{v \in Y}\{\langle v, v_0^*\rangle_{L^2} - G(v)\} \\ &= \frac{1}{2\alpha}\int_\Omega (v_0^*)^2\,dx + \beta\int_\Omega v_0^*\,dx. \end{aligned} \tag{29}$$

Furthermore, we define

$$B^* = \{v_3^* \in Y^* \ : \ u_1(v_3^*) \in V_1\},$$

where

$$u_1(v_3^*) = \frac{K_3}{v_3^*}.$$

Define also

$$C_1^* = \{v_0^* \in Y^* \ : \ \|v_0^*\|_\infty \leq K_4\}.$$

and $J_1^* : B^* \times C_1^* \to \mathbb{R}$ by

$$J_1^*(v_3^*, v_0^*) = -F_2^*(v_3^*, v_0^*) - G^*(v_0^*).$$

Moreover, assuming $K_1 \gg K_4 \gg \max\{1, K_3, \alpha, \beta, \gamma, \|f\|_\infty\}$.

By directly computing $\delta^2 J_1^*(v_3^*, v_0^*)$ denoting

$$A = -K_1 K_3,$$

$$B = 2K_1 v_3^*,$$

$$\varphi = -\gamma\nabla^2 + 2v_0^* + K_1(v_3^*)^2,$$

$$\varphi_1 = f + K_1 K_3 v_3^*,$$

$$u = \frac{\varphi_1}{\varphi},$$

we may obtain,

$$\begin{aligned} &\frac{\partial^2 J_1^*(v_3^*, v_0^*)}{\partial (v_3^*)^2} \\ &= -\frac{(A - uB)^2}{\varphi} + K_1 u^2 \\ &= -\frac{K_1(K_1 K_3^2(3u^2 - 4uu_1 + u_1^2) - u^2 u_1(-\gamma\nabla^2 + 2v_0^*)u_1)}{K_1 K_3^2 + u_1(-\gamma\nabla^2 + 2v_0^*)u_1} \\ &= \frac{K_1^2 H_1 + K_1 H_2}{K_1 K_3^2 + u_1(-\gamma\nabla^2 - 2v_0^*)u_1)} \end{aligned} \tag{30}$$

on $D^* \times B^*$.

where

$$u_1 = u_1(v_3^*) = \frac{K_3}{v_3^*},$$

$$H_1 = K_3^2(3u^2 - 4uu_1 + u_1^2),$$

and

$$H_2 = u^2[(-\gamma\nabla^2 + 2v_0^*)u_1]u_1.$$

At a critical point we have $H_1 = \mathbf{0}$ and

$$H_2 = u_0^2\, f u_0 > 0, \text{ a.e in } \Omega.$$

With such results, we may define the restrictions

$$C_2^* = \{v_0^* \in Y^* \ : \ H_1(v_2^*, v_3^*, v_0^*) \leq 0, \text{ in } \Omega, \ \forall v_2^* \in D^*, v_3^* \in B^*\}.$$

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$$C_3^* = \{v_0^* \in Y^* \ : \ H_2(v_2^*, v_3^*, v_0^*) \geq 0, \text{ in } \Omega, \ \forall v_2^* \in D^*, \ v_3^* \in B^*\}.$$

Here, we define $C^* = C_1^* \cap C_2^* \cap C_3^*$.

On the other hand, clearly we have

$$\frac{\partial^2 J_1^*(v_3^*, v_0^*)}{\partial (v_0^*)^2} < \mathbf{0}$$

4.1. A Concerning Duality Principle and a Related Convex Dual Formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

Theorem 3. *Let* $(\hat{v}_3^* \hat{v}_0^*) B^* \times C^*$ *be such that*

$$\delta J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V_1$ *be such that*

$$u_0 = \frac{\varphi_1}{\varphi} = \frac{f + K_1 K_3 \hat{v}_3^*}{-\gamma \nabla^2 + 2\hat{v}_0^* + K_1 (\hat{v}_3^*)^2}.$$

Assume also

$$u_0 \neq 0, \ a.e. \ in \ \Omega.$$

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

$$\hat{v}_3^* u_0 - K_3 = 0, \ a.e. \ in \ \Omega,$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} J(u) \\ &= \inf_{v_3^* \in B^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_3^*, \hat{v}_0^*). \end{aligned} \tag{31}$$

Proof. Observe that $\delta J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \mathbf{0}$ so that, since $\hat{v}_0^* \in C^*$ and

$$\frac{\partial^2 J_1^*(\hat{v}_3^*, v_0^*)}{\partial (v_0^*)^2} < \mathbf{0}, \ \forall v_0^* \in C_1^*,$$

we obtain

$$J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \inf_{v_3^* \in B^*} J_1^*(v_3^*, \hat{v}_0^*) = \sup_{v_0^* \in C_1^*} J_1^*(\hat{v}_3^*, v_0^*).$$

Consequently, from this and the Saddle Point Theorem, we have

$$J_1^*(\hat{v}_3^*, \hat{v}_0^*) = \inf_{v_3^* \in B^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

Firstly, observe that

$$F_2^*(v_3^*, v_0^*) = \sup_{u \in V} \{-F_2(u, v_3^*, v_0^*)\}.$$

Denoting

$$H(v_3^*, v_0^*, u) = -F_2(u, v_3^*, v_0^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_3^*, \hat{v}_0^*) = H(\hat{v}_3^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} u_0^2 = \frac{\partial F_1^*(\hat{v}_3^*, \hat{v}_0^*)}{\partial v_0^*} &= \frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial v_0^*} \\ &+ \frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u}\frac{\partial \hat{u}}{\partial v_0^*} \\ &= \hat{u}^2. \end{aligned} \tag{32}$$

Summarizing, we this last equation is satisfied through the relation

$$u_0 = \hat{u}.$$

Hence from the variation of J_1^* in v_0^*, we obtain

$$u_0^2 - \frac{v_0^*}{\alpha} - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

On the other hand, from the variation of J_1^* in v_3^*, we have

$$\begin{aligned} &\frac{\partial F_1^*(\hat{v}_3^*, \hat{v}_0^*)}{\partial v_3^*} \\ = &-K_1(\hat{v}_3^* u_0 - K_3)u_0 + \frac{\partial H(\hat{v}_3^*, \hat{v}_0^*, \hat{u})}{\partial u}\frac{\partial \hat{u}}{\partial v_3^*} \\ = &-K_1(\hat{v}_3^* u_0 - K_3)u_0 \\ = &\ \mathbf{0}. \end{aligned} \tag{33}$$

From such results, since

$$u_0 \neq 0, \text{ a.e. in } \Omega,$$

we get

$$\hat{v}_3^* u_0 - K_3 = 0, \text{ a.e. in } \Omega.$$

Consequently, from such last results and from

$$u_0 = \frac{f + K_1 K_3 \hat{v}_3^*}{-\gamma \nabla^2 u_0 + 2\hat{v}_0^* + K_1(\hat{v}_3^*)^2},$$

we obtain

$$\begin{aligned} &-\gamma \nabla^2 u_0 + 2v_0^* u_0 + K_1(\hat{v}_3^*)^2 u_0 - f - K_1 K_3 \hat{v}_3^* \\ = &-\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f \\ = &\ \delta J(u_0) \\ = &\ \mathbf{0}. \end{aligned} \tag{34}$$

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Summarizing,

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_3^*, \hat{v}_0^*) = -F_1(u_0, \hat{v}_3^*, \hat{v}_0^*),$$

$$G^*(\hat{v}_0^*) = \langle u_0^2, v_0^* \rangle_{L^2} - G(u_0^2),$$

so that

$$\begin{aligned} & J_1^*(\hat{v}_0^*) \\ = & -F_1^*(\hat{v}_3^*, \hat{v}_0^*) - G^*(\hat{v}_0^*) \\ = & J(u_0). \end{aligned} \tag{35}$$

Finally, observe that

$$J_1^*(v_3^*, v_0^*) \leq F_1(u, v_3^*, v_0^*) - G^*(v_0^*),$$

$\forall u \in V_1,\ v_3^* \in B^*,\ v_0^* \in C_1^*$.

Therefore,

$$\begin{aligned} J_1^*(v_3^*, v_0^*) &\leq \sup_{v_0^* \in C_1^*} \{F_2(u, v_3^*, v_0^*) - G^*(v_0^*)\} \\ &= J(u) + \frac{K_1}{2} \int_\Omega (v_3^* u - K_3)^2 \, dx, \end{aligned} \tag{36}$$

$\forall u \in V_1,\ v_3^* \in B^*$. so that

$$\begin{aligned} \inf_{v_3^* \in B^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\} &\leq \inf_{v_3^* \in B^*} \left\{ \sup_{v_0^* \in C_1^*} \{F_2(u, v_3^*, v_0^*) - G^*(v_0^*)\} \right\} \\ &= \inf_{v_3^* \in B^*} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (v_3^* u - K_3)^2 \, dx \right\} \\ &\leq J(u) + \frac{K_1}{2} \int_\Omega (v_3^*(u) u - K_3)^2 \, dx \\ &= J(u) + \frac{K_1}{2} \int_\Omega (K_3 - K_3)^2 \, dx \\ &= J(u), \end{aligned} \tag{37}$$

$\forall u \in V_1$.

Summarizing, we have obtained

$$\begin{aligned} J_1^*(\hat{v}_3^*, \hat{v}_0^*) &= \inf_{v_3^* \in B^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\} \\ &\leq \inf_{u \in V_1} J(u). \end{aligned} \tag{38}$$

Joining the pieces, we have got

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} J(u) \\ &= \inf_{v_3^* \in B^*} \left\{ \sup_{v_0^* \in C_1^*} J_1^*(v_3^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_3^*, \hat{v}_0^*). \end{aligned} \tag{39}$$

The proof is complete. □

5. Conclusions

In this article we have developed convex dual variational formulations suitable for the local optimization of non-convex primal formulations.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principles here presented are applied to a Ginzburg-Landau type equation. In a future research, we intend to extend such results for some models of plates and shells and other models in the elasticity theory.

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