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A duality principle and a related convex dual formulation suitable for global non-convex variational optimization

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Abstract

This article develops a duality principle and a related convex dual formulation suitable for the global optimization of a non-convex primal formulation for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a Ginzburg-Landau type equation.

Key words: Convex dual variational formulation, duality principle for non-convex global optimization, Ginzburg-Landau type equation

MSC 49N15

1 Introduction

In this section we establish a duality principle and a related convex dual formulation suitable for the global optimization of the primal formulation for a large class of models in non-convex optimization.

The main duality principle is applied to the Ginzburg-Landau system in superconductivity in an absence of a magnetic field.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 13, 14] and on a D.C. optimization approach developed in Toland [15].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [9, 5, 6, 7, 12]. Finally, similar models on the superconductivity physics may be found in [4, 11].

Remark 1.1. *It is worth highlighting, we may generically denote*

$$\int_\Omega [(-\gamma\nabla^2 + KI_d)^{-1} v^*] v^* \, dx$$

 doi:10.20944/preprints202210.0207.v1

simply by

$$\int_\Omega \frac{(v^*)^2}{-\gamma\nabla^2 + K}\,dx.$$

Other similar notations may be use along this text as their indicated meaning are sufficiently clear.

Finally, ∇^2 denotes the Laplace operator and for real constants $K_2 > 0$ and $K_1 > 0$, the notation $K_2 \gg K_1$ means that $K_2 > 0$ is much larger than $K_1 > 0$.

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u\,dx \\ &+\frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\,dx - \langle u, f\rangle_{L^2}. \end{aligned} \tag{1}$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functionals $F_1 : V \times Y \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $G : V \times Y \to \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_2^*) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u\,dx - \frac{K}{2}\int_\Omega u^2\,dx \\ &+\frac{K_1}{2}\int_\Omega (v_2^* - K_2 u)^2\,dx + \frac{K_2}{2}\int_\Omega u^2\,dx, \end{aligned} \tag{2}$$

$$F_2(u) = \frac{K_2}{2}\int_\Omega u^2\,dx,$$

and

$$G(u,v) = \frac{\alpha}{2}\int_\Omega (u^2 - \beta + v)^2\,dx + \frac{K}{2}\int_\Omega u^2\,dx - \langle u, f\rangle_{L^2}.$$

We define also $F_1^* : [Y^*]^2 \to \mathbb{R}$, $F_2^* : Y^* \to \mathbb{R}$, and $G^* : [Y^*]^2 \to \mathbb{R}$, by

$$\begin{aligned} &F_1^*(v_2^*, v_1^*) \\ =\ &\sup_{u \in V}\{\langle u, v_1^* + v_2^*\rangle_{L^2} - F_1(u, v_2^*)\} \\ =\ &\frac{1}{2}\int_\Omega \frac{(v_1^*)^2 + (1 + \gamma\nabla^2 K + KK_1 + K_1K_2)(v_2^*)^2 + 2v_1^*(v_2^* + K_1K_2v_2^*)}{-\gamma\nabla^2 - K + K_2 + K_1K_2^2}\,dx, \end{aligned} \tag{3}$$

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u \in V}\{\langle u, v_2^*\rangle_{L^2} - F_2(u)\} \\ &= \frac{1}{2K_2}\int_\Omega (v_2^*)^2\,dx, \end{aligned} \tag{4}$$

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and

$$\begin{aligned} G^*(v_1^*,v_0^*) &= \sup_{(u,v)\in V\times Y}\{-\langle u,v_1^*\rangle_{L^2}+\langle v,v_0^*\rangle_{L^2}-G(u,v)\} \\ &= \frac{1}{2}\int_\Omega \frac{(v_1^*-f)^2}{2v_0^*+K}\,dx+\frac{1}{2\alpha}\int_\Omega (v_0^*)^2\,dx \\ &\quad +\beta\int_\Omega v_0^*\,dx \end{aligned} \tag{5}$$

if $v_0^*\in B^*$ where

$$B^*=\{v_0^*\in Y^*\ :\ \|v_0^*\|_\infty\leq K/8\},$$

Finally, we also define

$$D^*=\{v_1^*\in Y^*\ :\ \|v_1^*\|_\infty\leq (3/2)K\}$$

and $J_1^*:Y^*\times D^*\times B^*\to\mathbb{R}$,

$$J_1^*(v_2^*,v_1^*,v_0^*)=-F_1^*(v_2^*,v_1^*)+F_2^*(v_2^*)-G^*(v_1^*,v_0^*).$$

Assuming

$$K_2\gg K_1\gg K\gg \max\{1,\gamma,\alpha,\|f\|_\infty\}$$

by directly computing $\delta^2 J_1^*(v_2^*,v_1^*,v_0^*)$ we may obtain that for such specified real constants, J_1^* in convex in v_2^* and it is concave in (v_1^*,v_0^*) on $Y^*\times D^*\times B^*$.

Considering such statements and definitions, we may prove the following theorem.

Theorem 1.2. *Let $(\hat v_2^*,\hat v_1^*,\hat v_0^*)\in Y^*\times D^*\times B^*$ be such that*

$$\delta J_1^*(\hat v_2^*,\hat v_1^*,\hat v_0^*)=\mathbf{0}$$

and $u_0\in V$ be such that

$$u_0=\frac{\partial F_2^*(\hat v_2^*)}{\partial v_2^*}.$$

Under such hypotheses, we have

$$\delta J(u_0)=\mathbf{0},$$

and

$$\begin{aligned} J(u_0) &= \inf_{u\in V} J(u) \\ &= \inf_{v_2^*\in Y^*}\left\{\sup_{(v_1^*,v_0^*)\in D^*\times B^*} J_1^*(v_2^*,v_1^*,v_0^*)\right\} \\ &= J_1^*(\hat v_2^*,\hat v_1^*,\hat v_0^*). \end{aligned} \tag{6}$$

Proof. Observe that $\delta J_1^*(\hat v_2^*,\hat v_1^*,\hat v_0^*)=\mathbf{0}$ so that, since J_1^* is convex in v_2^* and concave in (v_1^*,v_0^*) on $Y^*\times D^*\times B^*$, from the Min-Max theorem, we obtain

$$J_1^*(\hat v_2^*,\hat v_1^*,\hat v_0^*)=\inf_{v_2^*\in Y^*}\left\{\sup_{(v_1^*,v_0^*)\in D^*\times B^*} J_1^*(v_2^*,v_1^*,v_0^*)\right\}.$$

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Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*)}{\partial v_2^*} + u_0 = 0$$

and

$$\hat{v}_2^* = K_2 u_0.$$

Observe now that denoting

$$H(v_2^*, v_1^*, u) = \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, \hat{v}_2^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_1^*, \hat{v_2}^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*) = H(\hat{v}_1^*, \hat{v_2}^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_1^*, \hat{v_2}^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_1^*, \hat{v_2}^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u} + K_1(\hat{v}_2^* - K_2 \hat{u}) \\ &= \hat{u} + K_1(K_2 u_0 - K_2 \hat{u}). \end{aligned} \tag{7}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*)}{\partial v_2^*} = \hat{u} + K_1(K_2 u_0 - K_2 \hat{u}),$$

so that

$$\hat{u} = u_0.$$

From this and

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0},$$

we obtain

$$-\hat{u} - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = -u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = 0,$$

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and thus
$$\hat{v}_1^* = -\gamma\nabla^2 u_0 - Ku_0 = -2\hat{v}_0^* u_0 - Ku_0 + f.$$

Therefore
$$-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f = 0.$$

Moreover, from
$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$
we have
$$-\frac{\hat{v}_0^*}{\alpha} + u_0^2 - \beta = 0,$$
so that
$$v_0^* = \alpha(u_0^2 - \beta).$$

From such last results we have
$$-\gamma\nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = 0,$$
so that
$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, from the Legendre transform properties
$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \langle u_0, \hat{v}_2^* + \hat{v}_1^* \rangle_{L^2} - F_1(u_0, \hat{v}_2^*),$$
$$F_2^*(\hat{v}_2^*) = \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0),$$
$$G^*(\hat{v}_1^*, \hat{v}_0^*) = -\langle u_0, \hat{v}_1^* \rangle_{L^2} + \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, 0),$$
so that
$$\begin{aligned} & J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ = & -F_1^*(\hat{v}_2^*, \hat{v}_2^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{v}_1^*, \hat{v}_0^*) \\ = & F_1(u_0, \hat{v}_2^*) - F_2(u_0) + G(u_0, 0) \\ = & J(u_0). \end{aligned} \tag{8}$$

Finally, observe that
$$J_1^*(v_2^*, v_1, v_0^*) \leq -\langle u, v_1^* + v_2^* \rangle_{L^2} + F_1(u, v_2^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*),$$
$\forall u \in V,\ v_2^* \in Y^*,\ v_1^* \in D^*, v_0^* \in B^*$.

From this, we get
$$\begin{aligned} & \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \\ \leq & \sup_{(v_1^*, v_0^*) \in D^* \times B^*} \{-\langle u, v_1^* + v_2^* \rangle_{L^2} + F_1(u, v_2^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*)\} \\ \leq & \sup_{(v_1^*, v_0^*) \in Y^* \times B^*} \{-\langle u, v_1^* + v_2^* \rangle_{L^2} + F_1(u, v_2^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*)\} \\ \leq & -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_2^*) + F_2^*(v_2^*) + G(u, 0) \end{aligned} \tag{9}$$

doi:10.20944/preprints202210.0207.v1

Consequently, we may obtain

$$\begin{aligned}
& \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\
\leq & \inf_{v_2^* \in Y^*} \{ -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_2^*) + F_2^*(v_2^*) + G(u,0) \} \\
= & F_1(u, K_2 u) - F_2(u) + G(u,0) \\
= & J(u), \; \forall u \in V. \qquad (10)
\end{aligned}$$

Summarizing, we have obtained

$$\inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \leq \inf_{u \in V} J(u).$$

Joining the pieces, we have got

$$\begin{aligned}
J(u_0) &= \inf_{u \in V} J(u) \\
&= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\
&= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \qquad (11)
\end{aligned}$$

The proof is complete.

□

2 Conclusion

In this article we have developed a convex dual variational formulation suitable for the global optimization of a non-convex primal formulation.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principle here presented is applied to a Ginzburg-Landau type equation. In a future research, we intend to extend such results for some models of plates and shells and other models in the elasticity theory.

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