 doi:10.20944/preprints202210.0116.v6

A duality principle and a concerning convex dual formulation suitable for non-convex variational optimization

Fabio Silva Botelho
Department of Mathematics
Federal University of Santa Catarina
Florianópolis - SC, Brazil

Abstract

This article develops a duality principle and a related convex dual formulation suitable for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a model in non-linear elasticity.

Key words: Convex dual variational formulation, duality principle for non-convex optimization, model in non-linear elasticity

MSC 49N15

1 Introduction

In this article we establish a duality principle and a related convex dual formulation for a large class of models in non-convex optimization.

More specifically, the main duality principle is applied to a model in non-linear elasticity.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 10, 11] and on a D.C. optimization approach developed in Toland [12].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [7, 4, 5, 6, 9]. Finally, the model in non-linear elasticity here presented may be found in [8].

Remark 1.1. *In this text we adopt the standard Einstein convention of summing up repeated indices unless otherwise indicated.*

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 December 2022 doi:10.20944/preprints202210.0116.v6

For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = \frac{1}{2}\int_\Omega H_{ijkl} e_{ij}(u) e_{kl}(u)\,dx - \langle u_i, f_i\rangle_{L^2}. \quad (1)$$

Here $\{H_{ijkl}\}$ is a fourth order symmetric positive definite tensor and

$$\{e_{ij}(u)\} = \left\{\frac{1}{2}(u_{i,j}+u_{j,i}) + \frac{1}{2}(u_{m,i}u_{m,j})\right\},$$

where

$$u = (u_1, u_2, u_3) \in V = W_0^{1,4}(\Omega; \mathbb{R}^3)$$

denotes the field of displacements resulting from the action of the external forces $f = (f_1, f_2, f_2) \in L^2(\Omega; \mathbb{R}^3)$ on the elastic solid comprised by $\Omega \subset \mathbb{R}^3$.

Moreover, denoting $Y = Y^* = L^2(\Omega; \mathbb{R}^{3\times 3})$, the stress tensor $\sigma \in Y^*$ is defined by

$$\{\sigma_{ij}(u)\} = \{H_{ijkl} e_{kl}(u)\}.$$

At this point we define the functionals $F_1 : V \times Y \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $G : V \to \mathbb{R}$ by

$$\begin{aligned} F_1(u,\sigma) &= -\sum_{i=1}^3 \frac{K}{2}\int_\Omega (u_{i,i})^2\,dx \\ &+\sum_{i=1}^3 \frac{K_1}{2}\int_\Omega (\sigma_{ij,j} + (\sigma_{im}u_{m,j})_{,j} + f_i)^2\,dx \\ &+\frac{K_2}{2}\int_\Omega u_{i,j}u_{i,j}\,dx - \langle u_i, f_i\rangle_{L^2}, \end{aligned} \quad (2)$$

for appropriate positive real constants, K, K_1, K_2 to be specified,

$$F_2(u) = \frac{K_2}{2}\int_\Omega u_{i,j}u_{i,j}\,dx,$$

and

$$G(u) = \frac{1}{2}\int_\Omega H_{ijkl} e_{ij}(u) e_{kl}(u)\,dx + \sum_{i=1}^3 \frac{K}{2}\int_\Omega (u_{i,i})^2\,dx.$$

Here, it is worth highlighting that

$$F_1(u,\sigma) - F_2(u) + G(u) = J(u) + \sum_{i=1}^3 \frac{K_1}{2}\int_\Omega (\sigma_{ij,j} + (\sigma_{im}u_{m,j})_{,j} + f_i)^2\,dx, \forall u \in V,\ \sigma \in Y^*.$$

Furthermore, we define the functionals $F_1^* : [Y^*]^3 \to \mathbb{R}$, $F_2^* : Y^* \to \mathbb{R}$ and $G^* : [Y^*]^2 \to \mathbb{R}$ by

$$F_1^*(\sigma, Q, \tilde{Q}) = \sup_{u\in V}\{-\langle u_{i,j}, \sigma_{ij}\rangle_{L^2} - \langle u_{i,j}, Q_{ij}\rangle_{L^2} + \langle u_{i,j}, \tilde{Q}_{ij}\rangle_{L^2} - F_1(u,\sigma)\},$$

$$\begin{aligned} F_2^*(\tilde{Q}) &= \sup_{v_2 \in Y} \left\{ \langle (v_2)_{ij}, \tilde{Q}_{ij} \rangle_{L^2} - \frac{K_2}{2} \int_\Omega (v_2)_{ij}(v_2)_{ij}\, dx \right\} \\ &= \frac{1}{2K_2} \int_\Omega \tilde{Q}_{ij}\tilde{Q}_{ij}\, dx, \end{aligned} \tag{3}$$

and

$$G^*(\sigma, Q) = \sup_{(v_1,v_2)\in Y\times Y} \{\langle (v_1)_{ij}, \sigma_{ij}\rangle_{L^2} + \langle (v_2)_{ij}, Q_{ij}\rangle_{L^2} - \hat{G}(v_1, v_2)\},$$

where

$$\begin{aligned} \hat{G}(v_1, v_2) &= \frac{1}{2}\int_\Omega H_{ijkl}\left((v_1)_{ij} + \frac{1}{2}(v_2)_{mi}(v_2)_{mj}\right)\left((v_1)_{kl} + \frac{1}{2}(v_2)_{mk}(v_2)_{ml}\right)\, dx \\ &+ \sum_{i=1}^3 \frac{K}{2}\int_\Omega ((v_2)_{ii})^2\, dx, \end{aligned} \tag{4}$$

so that

$$G^*(\sigma, Q) = \frac{1}{2}\int_\Omega (\overline{\sigma_{ij}^K}) Q_{mi}Q_{mj}\, dx + \frac{1}{2}\int_\Omega \overline{H}_{ijkl}\sigma_{ij}\sigma_{kl}\, dx,$$

if $\sigma \in B^*$ where

$$B^* = \{\sigma \in Y^* \;:\; \|\sigma_{ij}\|_\infty \leq K/8,\ \forall i,j \in \{1,2,3\} \text{ and } \{\sigma_{ij}\} < -\varepsilon I_d\},$$

for some small parameter $\varepsilon > 0$ and where I_d denotes the 3×3 identity matrix. Observe that such a definition for B^* corresponds to the case of negative definite stress tensors, which refers to compression in a solid mechanics context.

Here

$$\{\sigma_{ij}^K\} = \begin{bmatrix} \sigma_{11}+K & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}+K & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}+K \end{bmatrix} \tag{5}$$

$$\overline{\sigma_{ij}^K} = \{\sigma_{ij}^K\}^{-1},$$

and

$$\{\overline{H}_{ijkl}\} = \{H_{ijkl}\}^{-1}$$

in an appropriate tensor sense.

At this point we define

$$J^*(\sigma, Q, \tilde{Q}) = -F_1^*(\sigma, Q, \tilde{Q}) + F_2^*(\tilde{Q}) - G^*(\sigma, Q).$$

Specifically for

$$K_2 \gg K_1 \gg K \gg \max\{1/\varepsilon^2, 1, K_3, \|H_{ijkl}\|\},$$

we define

$$D^* = \{Q \in Y^* \;:\; \|Q_{ij}\|_\infty \leq K_3,\ \forall i,j \in \{1,2,3\}\}.$$

By direct computation, we may obtain

$$\left\{\frac{\partial^2 J^*(\sigma, Q, \tilde{Q})}{\partial\sigma\partial Q}\right\} < \mathbf{0},$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 December 2022 doi:10.20944/preprints202210.0116.v6

and

$$\left\{ \frac{\partial^2 J^*(\sigma, Q, \tilde{Q})}{\partial \tilde{Q}^2} \right\} > \mathbf{0},$$

on $B^* \times D^* \times Y^*$, so that J^* is concave in (σ, Q) and convex in $\tilde{Q}$ on $B^* \times D^* \times Y^*$.

2 The main duality principle and a related convex dual variational formulation

Our main duality principle is summarized by the following theorem.

Theorem 2.1. *Considering the statements and definitions of the previous section, suppose* $(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) \in B^* \times D^* \times Y^*$ *is such that*

$$\delta J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) = \mathbf{0}.$$

Let $u_0 \in V$ *be such that*

$$(u_0)_{i,j} = \frac{\partial F_2^*(\tilde{Q})}{\partial \tilde{Q}_{ij}}.$$

Under such hypotheses, we have

$$\begin{aligned} J(u_0) &= \min_{u \in V} \left\{ J(u) + \sum_{i=1}^{3} \frac{K_1}{2} \int_\Omega (\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im} u_{m,j})_{,j} + f_i)^2 \, dx \right\} \\ &= \inf_{\tilde{Q} \in Y^*} \left\{ \sup_{(\sigma, Q) \in B^* \times D^*} J^*(\sigma, Q, \tilde{Q}) \right\} \\ &= J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}). \end{aligned} \tag{6}$$

Proof. Observe that there exists $\hat{u} \in V$ such that, defining

$$H(u, \sigma, Q, \tilde{Q}) = -\langle u_{i,j}, \sigma_{ij} \rangle_{L^2} - \langle u_{i,j}, Q_{ij} \rangle_{L^2} + \langle u_{i,j}, \tilde{Q}_{ij} \rangle_{L^2} - F_1(u, \sigma),$$

we have

$$\frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial u} = \mathbf{0}$$

and

$$F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) = H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}).$$

Moreover, from the variation in of J^* in $\tilde{Q}$, we obtain

$$-\frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} + \frac{\partial F_2^*(\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} = \mathbf{0},$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 December 2022 doi:10.20944/preprints202210.0116.v6

where

$$\begin{aligned}
&\frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} \\
=\ &\frac{\partial H(\hat{u},\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} \\
&+\frac{\partial H(\hat{u},\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial u}\frac{\partial \hat{u}}{\partial \tilde{Q}_{ij}} \\
=\ &\hat{u}_{i,j}. \qquad (7)
\end{aligned}$$

From such last two equations we get

$$(u_0)_{i,j}=\frac{\partial F_2^*(\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}}=\frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}}=\hat{u}_{i,j},$$

so that from the concerning boundary conditions,

$$u_0=\hat{u}.$$

On the other hand, from the variation of J^* in Q we have

$$-\frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial Q_{ij}}-\frac{\partial G^*(\hat{\sigma},\hat{Q})}{\partial Q_{ij}}=\mathbf{0},$$

so that

$$(u_0)_{i,j}-\overline{\hat{\sigma}_{im}^K}\hat{Q}_{mj}=0,$$

and therefore

$$\hat{Q}_{ij}=\hat{\sigma}_{im}(u_0)_{m,j}+K\delta_{ij}(u_0)_{i,j}.$$

Finally, from the variation of J^* in σ we obtain

$$-\frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \sigma_{ij}}-\frac{\partial G^*(\hat{\sigma},\hat{Q})}{\partial \sigma_{ij}}=\mathbf{0},$$

so that

$$(u_0)_{i,j}+\frac{1}{2}(u_0)_{m,i}(u_0)_{m,j}-\overline{H}_{ijkl}\hat{\sigma}_{kl}=0.$$

Thus, since $\{H_{ijkl}\}$ is symmetric, we get

$$\hat{\sigma}_{ij}=H_{ijkl}e_{kl}(u_0).$$

From these last results and from

$$\frac{\partial H(\hat{u},\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial u}=\mathbf{0}$$

we obtain

$$\hat{\sigma}_{ij,j}+(\hat{\sigma}_{im}(u_0)_{m,j})_{,j}+f_i=0,\ \forall i\in\{1,2,3\},$$

 doi:10.20944/preprints202210.0116.v6

so that

$$\delta J(u_0) = \mathbf{0}.$$

Finally, from such last results and the Legendre transform properties, we have

$$\begin{aligned} & F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) \\ = & -\langle (u_0)_{i,j}, \hat{\sigma}_{ij}\rangle_{L^2} - \langle (u_0)_{i,j}, \hat{Q}_{ij}\rangle_{L^2} + \langle (u_0)_{i,j}, \hat{\tilde{Q}}_{ij}\rangle_{L^2} - F_1(u_0, \hat{\sigma}), \end{aligned} \tag{8}$$

$$F_2^*(\hat{\tilde{Q}}) = \langle (u_0)_{i,j}, \hat{\tilde{Q}}_{ij}\rangle_{L^2} - F_2(u_0),$$

and

$$G^*(\hat{\sigma}, \hat{Q}) = \langle (u_0)_{i,j}, \hat{\sigma}_{ij}\rangle_{L^2} + \langle (u_0)_{i,j}, \hat{Q}_{ij}\rangle_{L^2} - G(u_0).$$

From these results, we obtain

$$\begin{aligned} J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) &= -F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) + F_2^*(\hat{\tilde{Q}}) - G^*(\hat{\sigma}, \hat{Q}) \\ &= F_1(u_0, \hat{\sigma}) - F_2(u_0) + G(u_0) \\ &= J(u_0). \end{aligned} \tag{9}$$

Joining the pieces, we have got

$$\begin{aligned} J(u_0) &= \min_{u \in V} \left\{ J(u) + \sum_{i=1}^{3} \frac{K_1}{2} \int_\Omega (\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im} u_{m,j})_{,j} + f_i)^2 \, dx \right\} \\ &= \inf_{\tilde{Q} \in Y^*} \left\{ \sup_{(\sigma, Q) \in B^* \times D^*} J^*(\sigma, Q, \tilde{Q}) \right\} \\ &= J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}). \end{aligned} \tag{10}$$

The proof is complete.

□

Remark 2.2. *A similar result is valid if we would define*

$$B^* = \{\sigma \in Y^* \ : \ \|\sigma_{ij}\|_\infty \leq K/8, \ \forall i, j \in \{1, 2, 3\} \ \textit{and} \ \{\sigma_{ij}\} > \varepsilon I_d\}.$$

This case refers to a positive definite tensor $\{\sigma_{ij}\}$ *and the previous case to a negative definite one.*

3 A closely related primal-dual variational formulation for a related model

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected with a regular (Lipschitzian) boundary denoted by $\partial\Omega$. Consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx - \langle u, f\rangle_{L^2},$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 December 2022 doi:10.20944/preprints202210.0116.v6

where $\alpha > 0$, $\gamma > 0$, $\beta > 0$, $f \in L^2(\Omega) = Y = Y^*$ and $V = W_0^{1,2}(\Omega)$.

Observe that

$$\begin{aligned} J(u) &= J(u) + \langle u, v_1^* \rangle_{L^2} - \langle u, v_1^* \rangle_{L^2} \\ &\geq \inf_{u \in V} \left\{ -\langle u, v_1^* \rangle_{L^2} + \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx \right\} \\ &\quad + \langle u, v_1 \rangle_{L^2} + \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2} \\ &= -\frac{1}{2} \int_\Omega \frac{(v_1^*)^2}{-\gamma \nabla^2} \, dx + \langle u, v_1 \rangle_{L^2} + \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2} \\ &= J_1^*(u, v_1^*) \end{aligned} \tag{11}$$

Having obtained $J_1^*(u, v_1^*)$, we propose the following penalized primal-dual formulation $J_2^*(u, v_1^*)$, where

$$J_2^*(u, v_1^*) = J_1^*(u, v_1^*) + \frac{K_1}{2} \int_\Omega (v_1^* + \gamma \nabla^2 u)^2 \, dx,$$

so that

$$\begin{aligned} J_2^*(u, v_1^*) &= -\frac{1}{2} \int_\Omega \frac{(v_1^*)^2}{-\gamma \nabla^2} \, dx + \langle u, v_1 \rangle_{L^2} \\ &\quad + \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2} \\ &\quad + \frac{K_1}{2} \int_\Omega (v_1^* + \gamma \nabla^2 u)^2 \, dx. \end{aligned} \tag{12}$$

Here we define

$$J_3(u) = \sup_{v_1^* \in E(u)} J_2^*(u, v_1^*),$$

where

$$E(u) = \{ v_1^* \in Y^* \; : \; 0 \leq v_1^* + \gamma \nabla^2 u \leq \varepsilon, \text{ a.e. in } \Omega \}.$$

For a large $K_1 > 0$ and a sufficiently small $\varepsilon > 0$, such last supremum in v_1^* translates into a quadractic optimization problem.

It is clear that a sufficiently large $K_1 > 0$ improves a lot the convexity conditions of J_3.

Summarizing, we have obtained an interesting approximate convex variational formulation for the original problem represented by the functional J_3.

4 Conclusion

In this article we have developed a convex dual variational formulation suitable for non-convex variational primal formulations.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principle here presented is applied to a model in non-linear elasticity. In a future research, we intend to extend such results for other related models of plates and shells.

 doi:10.20944/preprints202210.0116.v6

References

[1] R.A. Adams and J.F. Fournier, Sobolev Spaces, 2nd edn. (Elsevier, New York, 2003).

[2] W.R. Bielski, A. Galka, J.J. Telega, The Complementary Energy Principle and Duality for Geometrically Nonlinear Elastic Shells. I. Simple case of moderate rotations around a tangent to the middle surface. Bulletin of the Polish Academy of Sciences, Technical Sciences, Vol. 38, No. 7-9, 1988.

[3] W.R. Bielski and J.J. Telega, A Contribution to Contact Problems for a Class of Solids and Structures, Arch. Mech., 37, 4-5, pp. 303-320, Warszawa 1985.

[4] F.S. Botelho, Functional Analysis, Calculus of Variations and Numerical Methods in Physics and Engineering, CRC Taylor and Francis, Florida, 2020.

[5] F.S. Botelho, *Variational Convex Analysis*, Ph.D. thesis, Virginia Tech, Blacksburg, VA -USA, (2009).

[6] F. Botelho, *Topics on Functional Analysis, Calculus of Variations and Duality*, Academic Publications, Sofia, (2011).

[7] F. Botelho, Functional Analysis and Applied Optimization in Banach Spaces, Springer Switzerland, 2014.

[8] P.Ciarlet, *Mathematical Elasticity*, Vol. I – Three Dimensional Elasticity, North Holland Elsivier (1988).

[9] R.T. Rockafellar, Convex Analysis, Princeton Univ. Press, (1970).

[10] J.J. Telega, *On the complementary energy principle in non-linear elasticity. Part I: Von Karman plates and three dimensional solids*, C.R. Acad. Sci. Paris, Serie II, 308, 1193-1198; Part II: Linear elastic solid and non-convex boundary condition. Minimax approach, ibid, pp. 1313-1317 (1989)

[11] A.Galka and J.J.Telega *Duality and the complementary energy principle for a class of geometrically non-linear structures. Part I. Five parameter shell model; Part II. Anomalous dual variational priciples for compressed elastic beams*, Arch. Mech. 47 (1995) 677-698, 699-724.

[12] J.F. Toland, A duality principle for non-convex optimisation and the calculus of variations, Arch. Rat. Mech. Anal., **71**, No. 1 (1979), 41-61.