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A duality principle and a concerning convex dual formulation suitable for non-convex variational optimization

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Abstract

This article develops a duality principle and a related convex dual formulation suitable for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a model in non-linear elasticity.

Key words: Convex dual variational formulation, duality principle for non-convex optimization, model in non-linear elasticity

MSC 49N15

1 Introduction

In this article we establish a duality principle and a related convex dual formulation for a large class of models in non-convex optimization.

More specifically, the main duality principle is applied to a model in non-linear elasticity.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 10, 11] and on a D.C. optimization approach developed in Toland [12].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [7, 4, 5, 6, 9]. Finally, the model in non-linear elasticity here presented may be found in [8].

Remark 1.1. *In this text we adopt the standard Einstein convention of summing up repeated indices unless otherwise indicated.*

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

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For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = \frac{1}{2}\int_\Omega H_{ijkl}e_{ij}(u)e_{kl}(u)\,dx - \langle u_i, f_i\rangle_{L^2}. \tag{1}$$

Here $\{H_{ijkl}\}$ is a fourth order symmetric positive definite tensor and

$$\{e_{ij}(u)\} = \left\{\frac{1}{2}(u_{i,j}+u_{j,i}) + \frac{1}{2}(u_{m,i}u_{m,j})\right\},$$

where

$$u = (u_1, u_2, u_3) \in V = W_0^{1,4}(\Omega;\mathbb{R}^3)$$

denotes the field of displacements resulting from the action of the external forces $f = (f_1, f_2, f_2) \in L^2(\Omega;\mathbb{R}^3)$ on the elastic solid comprised by $\Omega \subset \mathbb{R}^3$.

Moreover, denoting $Y = Y^* = L^2(\Omega;\mathbb{R}^{3\times 3})$, the stress tensor $\sigma \in Y^*$ is defined by

$$\{\sigma_{ij}(u)\} = \{H_{ijkl}e_{kl}(u)\}.$$

At this point we define the functionals $F_1 : V \times Y \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $G : V \to \mathbb{R}$ by

$$\begin{aligned} F_1(u,\sigma) &= -\sum_{i=1}^3 \frac{K}{2}\int_\Omega (u_{i,i})^2\,dx \\ &+ \sum_{i=1}^3 \frac{K_1}{2}\int_\Omega (\sigma_{ij,j} + (\sigma_{im}u_{m,j})_{,j} + f_i)^2\,dx \\ &+ \frac{K_2}{2}\int_\Omega u_{i,j}u_{i,j}\,dx - \langle u_i, f_i\rangle_{L^2}, \end{aligned} \tag{2}$$

for appropriate positive real constants, K, K_1, K_2 to be specified,

$$F_2(u) = \frac{K_2}{2}\int_\Omega u_{i,j}u_{i,j}\,dx,$$

and

$$G(u) = \frac{1}{2}\int_\Omega H_{ijkl}e_{ij}(u)e_{kl}(u)\,dx + \sum_{i=1}^3 \frac{K}{2}\int_\Omega (u_{i,i})^2\,dx.$$

Here, it is worth highlighting that

$$F_1(u,\sigma) - F_2(u) + G(u) = J(u) + \sum_{i=1}^3 \frac{K_1}{2}\int_\Omega (\sigma_{ij,j} + (\sigma_{im}u_{m,j})_{,j} + f_i)^2\,dx, \forall u \in V,\ \sigma \in Y^*.$$

Furthermore, we define the functionals $F_1^* : [Y^*]^3 \to \mathbb{R}$, $F_2^* : Y^* \to \mathbb{R}$ and $G^* : [Y^*]^2 \to \mathbb{R}$ by

$$F_1^*(\sigma, Q, \tilde{Q}) = \sup_{u\in V}\{-\langle u_{i,j}, \sigma_{ij}\rangle_{L^2} - \langle u_{i,j}, Q_{ij}\rangle_{L^2} + \langle u_{i,j}, \tilde{Q}_{ij}\rangle_{L^2} - F_1(u,\sigma)\},$$

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$$\begin{aligned} F_2^*(\tilde{Q}) &= \sup_{v_2 \in Y} \left\{ \langle (v_2)_{ij}, \tilde{Q}_{ij} \rangle_{L^2} - \frac{K_2}{2} \int_\Omega (v_2)_{ij}(v_2)_{ij}\, dx \right\} \\ &= \frac{1}{2K_2} \int_\Omega \tilde{Q}_{ij}\tilde{Q}_{ij}\, dx, \end{aligned} \tag{3}$$

and

$$G^*(\sigma, Q) = \sup_{(v_1,v_2)\in Y\times Y} \{\langle (v_1)_{ij}, \sigma_{ij}\rangle_{L^2} + \langle (v_2)_{ij}, Q_{ij}\rangle_{L^2} - \hat{G}(v_1, v_2)\},$$

where

$$\begin{aligned} \hat{G}(v_1, v_2) &= \frac{1}{2}\int_\Omega H_{ijkl}\left((v_1)_{ij} + \frac{1}{2}(v_2)_{mi}(v_2)_{mj}\right)\left((v_1)_{kl} + \frac{1}{2}(v_2)_{mk}(v_2)_{ml}\right)\, dx \\ &+ \sum_{i=1}^{3} \frac{K}{2}\int_\Omega ((v_2)_{ii})^2\, dx, \end{aligned} \tag{4}$$

so that

$$G^*(\sigma, Q) = \frac{1}{2}\int_\Omega (\overline{\sigma_{ij}^K}) Q_{mi}Q_{mj}\, dx + \frac{1}{2}\int_\Omega \overline{H}_{ijkl}\sigma_{ij}\sigma_{kl}\, dx,$$

if $\sigma \in B^*$ where

$$B^* = \{\sigma \in Y^* \ : \ \|\sigma_{ij}\|_\infty \le K/8,\ \forall i,j \in \{1,2,3\} \text{ and } \{\sigma_{ij}\} < -\varepsilon I_d\},$$

for some small parameter $\varepsilon > 0$ and where I_d denotes the 3×3 identity matrix. Observe that such a definition for B^* corresponds to the case of negative definite stress tensors, which refers to compression in a solid mechanics context.

Here

$$\{\sigma_{ij}^K\} = \begin{bmatrix} \sigma_{11} + K & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} + K & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} + K \end{bmatrix} \tag{5}$$

$$\overline{\sigma_{ij}^K} = \{\sigma_{ij}^K\}^{-1},$$

and

$$\{\overline{H}_{ijkl}\} = \{H_{ijkl}\}^{-1}$$

in an appropriate tensor sense.

At this point we define

$$J^*(\sigma, Q, \tilde{Q}) = -F_1^*(\sigma, Q, \tilde{Q}) + F_2^*(\tilde{Q}) - G^*(\sigma, Q).$$

Specifically for

$$K_2 \gg K_1 \gg K \gg \max\{1/\varepsilon^2, 1, K_3, \|H_{ijkl}\|\},$$

we define

$$D^* = \{Q \in Y^* \ : \ \|Q_{ij}\|_\infty \le K_3,\ \forall i,j \in \{1,2,3\}\}.$$

By direct computation, we may obtain

$$\left\{\frac{\partial^2 J^*(\sigma, Q, \tilde{Q})}{\partial\sigma\partial Q}\right\} < \mathbf{0},$$

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and

$$\left\{\frac{\partial^2 J^*(\sigma,Q,\tilde{Q})}{\partial \tilde{Q}^2}\right\} > \mathbf{0},$$

on $B^* \times D^* \times Y^*$, so that J^* is concave in (σ, Q) and convex in $\tilde{Q}$ on $B^* \times D^* \times Y^*$.

2 The main duality principle and a related convex dual variational formulation

Our main duality principle is summarized by the following theorem.

Theorem 2.1. *Considering the statements and definitions of the previous section, suppose* $(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) \in B^* \times D^* \times Y^*$ *is such that*

$$\delta J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) = \mathbf{0}.$$

Let $u_0 \in V$ *be such that*

$$(u_0)_{i,j} = \frac{\partial F_2^*(\tilde{Q})}{\partial \tilde{Q}_{ij}}.$$

Under such hypotheses, we have

$$\begin{aligned} J(u_0) &= \min_{u \in V}\left\{J(u) + \sum_{i=1}^{3}\frac{K_1}{2}\int_\Omega (\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im}u_{m,j})_{,j} + f_i)^2\,dx\right\} \\ &= \inf_{\tilde{Q}\in Y^*}\left\{\sup_{(\sigma,Q)\in B^*\times D^*} J^*(\sigma,Q,\tilde{Q})\right\} \\ &= J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}). \end{aligned} \tag{6}$$

Proof. Observe that there exists $\hat{u} \in V$ such that, defining

$$H(u,\sigma,Q,\tilde{Q}) = -\langle u_{i,j}, \sigma_{ij}\rangle_{L^2} - \langle u_{i,j}, Q_{ij}\rangle_{L^2} + \langle u_{i,j}, \tilde{Q}_{ij}\rangle_{L^2} - F_1(u,\sigma),$$

we have

$$\frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial u} = \mathbf{0}$$

and

$$F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) = H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}).$$

Moreover, from the variation in of J^* in $\tilde{Q}$, we obtain

$$-\frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} + \frac{\partial F_2^*(\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} = \mathbf{0},$$

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where

$$\begin{aligned}
& \frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} \\
= & \frac{\partial H(\hat{u},\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} \\
& + \frac{\partial H(\hat{u},\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial u}\frac{\partial \hat{u}}{\partial \tilde{Q}_{ij}} \\
= & \hat{u}_{i,j}.
\end{aligned} \tag{7}$$

From such last two equations we get

$$(u_0)_{i,j} = \frac{\partial F_2^*(\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} = \frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} = \hat{u}_{i,j},$$

so that from the concerning boundary conditions,

$$u_0 = \hat{u}.$$

On the other hand, from the variation of J^* in Q we have

$$-\frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial Q_{ij}} - \frac{\partial G^*(\hat{\sigma},\hat{Q})}{\partial Q_{ij}} = \mathbf{0},$$

so that

$$(u_0)_{i,j} - \overline{\hat{\sigma}^K_{im}}\hat{Q}_{mj} = 0,$$

and therefore

$$\hat{Q}_{ij} = \hat{\sigma}_{im}(u_0)_{m,j} + K\delta_{ij}(u_0)_{i,j}.$$

Finally, from the variation of J^* in σ we obtain

$$-\frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \sigma_{ij}} - \frac{\partial G^*(\hat{\sigma},\hat{Q})}{\partial \sigma_{ij}} = \mathbf{0},$$

so that

$$(u_0)_{i,j} + \frac{1}{2}(u_0)_{m,i}(u_0)_{m,j} - \overline{H}_{ijkl}\hat{\sigma}_{kl} = 0.$$

Thus, since $\{H_{ijkl}\}$ is symmetric, we get

$$\hat{\sigma}_{ij} = H_{ijkl}e_{kl}(u_0).$$

From these last results and from

$$\frac{\partial H(\hat{u},\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial u} = \mathbf{0}$$

we obtain

$$\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im}(u_0)_{m,j})_{,j} + f_i = 0, \ \forall i \in \{1,2,3\},$$

so that

$$\delta J(u_0) = \mathbf{0}.$$

Finally, from such last results and the Legendre transform properties, we have

$$\begin{aligned} & F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) \\ = & -\langle (u_0)_{i,j}, \hat{\sigma}_{ij} \rangle_{L^2} - \langle (u_0)_{i,j}, \hat{Q}_{ij} \rangle_{L^2} + \langle (u_0)_{i,j}, \hat{\tilde{Q}}_{ij} \rangle_{L^2} - F_1(u_0, \hat{\sigma}), \end{aligned} \tag{8}$$

$$F_2^*(\hat{\tilde{Q}}) = \langle (u_0)_{i,j}, \hat{\tilde{Q}}_{ij} \rangle_{L^2} - F_2(u_0),$$

and

$$G^*(\hat{\sigma}, \hat{Q}) = \langle (u_0)_{i,j}, \hat{\sigma}_{ij} \rangle_{L^2} + \langle (u_0)_{i,j}, \hat{Q}_{ij} \rangle_{L^2} - G(u_0).$$

From these results, we obtain

$$\begin{aligned} J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) &= -F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) + F_2^*(\hat{\tilde{Q}}) - G^*(\hat{\sigma}, \hat{Q}) \\ &= F_1(u_0, \hat{\sigma}) - F_2(u_0) + G(u_0) \\ &= J(u_0). \end{aligned} \tag{9}$$

Joining the pieces, we have got

$$\begin{aligned} J(u_0) &= \min_{u \in V} \left\{ J(u) + \sum_{i=1}^3 \frac{K_1}{2} \int_\Omega (\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im} u_{m,j})_{,j} + f_i)^2 \, dx \right\} \\ &= \inf_{\tilde{Q} \in Y^*} \left\{ \sup_{(\sigma, Q) \in B^* \times D^*} J^*(\sigma, Q, \tilde{Q}) \right\} \\ &= J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}). \end{aligned} \tag{10}$$

The proof is complete.

□

Remark 2.2. *A similar result is valid if we would define*

$$B^* = \{\sigma \in Y^* \ : \ \|\sigma_{ij}\|_\infty \leq K/8, \ \forall i, j \in \{1, 2, 3\} \text{ and } \{\sigma_{ij}\} > \varepsilon I_d\}.$$

This case refers to a positive definite tensor $\{\sigma_{ij}\}$ *and the previous case to a negative definite one.*

3 One more duality principle and concerning convex dual formulation also suitable for a local optimization of the primal formulation

In this section we develop a duality principle suitable for a global optimization of the primal variational formulation.

We start with the following remark.

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Remark 3.1. *Denoting $Y_1 = Y_1^* = L^2(\Omega;\mathbb{R}^3)$, consider the functionals $F_1 : V \times Y_1 \to \mathbb{R}$ and $F_2 : V \to \mathbb{R}$, where*

$$\begin{aligned} F_1(u, v_3^*) &= -\sum_{i=1}^3 \frac{K}{2}\int_\Omega u_{i,i}^2\,dx + \sum_{i=1}^3 \frac{K_1}{2}\int_\Omega (K_2(v_3^*)_i u_i - K_2)^2\,dx \\ &\quad + \sum_{i=1}^3 \frac{K_2}{2}\int_\Omega (u_i)^2\,dx, \end{aligned} \tag{11}$$

$$F_2(u) = \sum_{i=1}^3 \frac{K_2}{2}\int_\Omega (u_i)^2\,dx.$$

Define $F_1^* : [Y_1^*]^2 \to \mathbb{R}$ and $F_2^* : Y_1^* \to \mathbb{R}$ where denoting

$$u_{i,i} = L_i(u_i),$$

$$\begin{aligned} & F_1^*(v_2^*, v_3^*) \\ =\; & \sup_{u\in V}\{\langle u_i, (v_2^*)_i\rangle_{L^2} - F_1(u, v_3^*)\} \\ =\; & \sum_{i=1}^3 \left(-\frac{1}{2}\int_\Omega \frac{(v_2^*)_i^2 + K_1K_2^2(-K_2 + KL_i^2 + 2(v_2^*)_i(v_3^*)_i)}{K_2 - KL_i^2 + K_1K_2^2(v_3^*)_i^2}\,dx\right), \end{aligned} \tag{12}$$

and

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u\in V}\{\langle u_i, (v_2^*)_i\rangle_{L^2} - F_2(u)\} \\ &= \sum_{i=1}^3 \frac{1}{2K_2}\int_\Omega (v_2^*)_i^2\,dx. \end{aligned} \tag{13}$$

Define also

$$C^* = \{v_3^* \in Y_1^* \;:\; \|(v_3^*)_i\|_\infty \le K/8,\ \text{and } (v_3^*)_i < -\varepsilon \text{ in } \Omega,\ \forall i \in \{1,2,3\}\},$$

for some small parameter $\varepsilon > 0$ and

$$E^* = \{v_2^* \in Y_1^* \;:\; \|(v_2^*)_i\|_\infty \le K_2,\ \forall i \in \{1,2,3\}\}.$$

Assume $K = 2K_2$ and $K_1 \gg K_2 \gg \max\{1, 1/\varepsilon^2\}$.
Under such hypotheses, defining

$$F_3^*(v_2^*, v_3^*) = -F_1^*(v_2^*, v_3^*) + F_2^*(v_2^*),$$

we have that F_3^* is convex in v_2^* and concave in v_3^* on $E^* \times C^*$.

At this point we start again to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = \frac{1}{2}\int_\Omega H_{ijkl}e_{ij}(u)e_{kl}(u)\,dx - \langle u_i, f_i\rangle_{L^2}. \tag{14}$$

Here $\{H_{ijkl}\}$ is a fourth order symmetric positive definite tensor and

$$\{e_{ij}(u)\} = \left\{\frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{m,i}u_{m,j})\right\},$$

where

$$u = (u_1, u_2, u_3) \in V = W_0^{1,4}(\Omega; \mathbb{R}^3)$$

denotes the field of displacements resulting from the action of the external forces $f = (f_1, f_2, f_2) \in L^2(\Omega; \mathbb{R}^3)$ on the elastic solid comprised by $\Omega \subset \mathbb{R}^3$.

Moreover, denoting $Y = Y^* = L^2(\Omega; \mathbb{R}^{3\times 3})$ and $Y_1 = Y_1^* = L^2(\Omega; \mathbb{R}^3)$, the stress tensor $\sigma \in Y^*$ is defined by

$$\{\sigma_{ij}(u)\} = \{H_{ijkl}e_{kl}(u)\}.$$

At this point we define the functionals $F_1 : V \times Y_1 \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $G : V \to \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_3^*) &= -\sum_{i=1}^3 \frac{K}{2}\int_\Omega (u_{i,i})^2\,dx \\ &+\sum_{i=1}^3 \frac{K_1}{2}\int_\Omega (K_2(v_3^*)_i u_i - K_2)^2\,dx \\ &+\sum_{i=1}^3 \frac{K_2}{2}\int_\Omega u_i^2\,dx - \langle u_i, f_i\rangle_{L^2}, \end{aligned} \tag{15}$$

for appropriate positive real constants, K, K_1, K_2 to be specified,

$$F_2(u) = \sum_{i=1}^3 \frac{K_2}{2}\int_\Omega u_i^2\,dx,$$

and

$$G(u) = \frac{1}{2}\int_\Omega H_{ijkl}e_{ij}(u)e_{kl}(u)\,dx + \sum_{i=1}^3 \frac{K}{2}\int_\Omega (u_{i,i})^2\,dx.$$

Here, it is worth highlighting that

$$F_1(u, v_3^*) - F_2(u) + G(u) = J(u) + \sum_{i=1}^3 \frac{K_1}{2}\int_\Omega (K_2(v_3^*)_i u_i - K_2)^2\,dx, \forall u \in V,\ \sigma \in Y^*.$$

Furthermore, we define the functionals $F_1^* : [Y^*]^2 \times Y_1^* \to \mathbb{R}$, $F_2^* : Y_1^* \to \mathbb{R}$ and $G^* : [Y^*]^2 \to \mathbb{R}$ by

$$F_1^*(\sigma, Q, \tilde{Q}) = \sup_{u \in V}\{-\langle u_{i,j}, \sigma_{ij}\rangle_{L^2} - \langle u_{i,j}, Q_{ij}\rangle_{L^2} + \langle u_i, (v_2^*)_i\rangle_{L^2} - F_1(u, v_3^*)\},$$

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{v_2 \in Y}\{\langle u_i, (v_2^*)_i\rangle_{L^2} - F_2(u)\} \\ &= \sum_{i=1}^3 \frac{1}{2K_2}\int_\Omega (v_2^*)_i\, dx, \end{aligned} \quad (16)$$

and

$$G^*(\sigma, Q) = \sup_{(v_1,v_2)\in Y\times Y}\{\langle (v_1)_{ij}, \sigma_{ij}\rangle_{L^2} + \langle (v_2)_{ij}, Q_{ij}\rangle_{L^2} - \hat{G}(v_1, v_2)\},$$

where

$$\begin{aligned} \hat{G}(v_1, v_2) &= \frac{1}{2}\int_\Omega H_{ijkl}\left((v_1)_{ij} + \frac{1}{2}(v_2)_{mi}(v_2)_{mj}\right)\left((v_1)_{kl} + \frac{1}{2}(v_2)_{mk}(v_2)_{ml}\right)\, dx \\ &\quad + \sum_{i=1}^3 \frac{K}{2}\int_\Omega ((v_2)_{ii})^2\, dx, \end{aligned} \quad (17)$$

so that

$$G^*(\sigma, Q) = \frac{1}{2}\int_\Omega (\overline{\sigma_{ij}^K})Q_{mi}Q_{mj}\, dx + \frac{1}{2}\int_\Omega \overline{H}_{ijkl}\sigma_{ij}\sigma_{kl}\, dx,$$

if $\sigma \in B^*$ where

$$B^* = \{\sigma \in Y^* \ :\ \|\sigma_{ij}\|_\infty \leq K/8,\ \forall i, j \in \{1, 2, 3\}\}.$$

Here

$$\{\sigma_{ij}^K\} = \begin{bmatrix} \sigma_{11} + K & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} + K & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} + K \end{bmatrix} \quad (18)$$

$$\overline{\sigma_{ij}^K} = \{\sigma_{ij}^K\}^{-1},$$

and

$$\{\overline{H}_{ijkl}\} = \{H_{ijkl}\}^{-1}$$

in an appropriate tensor sense.

At this point we define

$$J_1^*(\sigma, Q, v_2^*, v_3^*) = -F_1^*(\sigma, Q, v_2^*, v_3^*) + F_2^*(v_2^*) - G^*(\sigma, Q).$$

Specifically for

$$K_2 \gg K_1 \gg K \gg \max\{1/\varepsilon^2, 1, \|H_{ijkl}\|\},$$

we define

$$D^* = \{Q \in Y^* \ :\ \|Q_{ij}\|_\infty \leq (3/2)K,\ \forall i, j \in \{1, 2, 3\}\}.$$

From the last theorem, we may obtain J_1^* is concave in (σ, Q) and convex in (v_2^*, v_3^*) on $B^* \times D^* \times E^* \times C^*$.

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Theorem 3.2. *Considering the statements and definitions of the previous sections, suppose* $(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*) \in B^* \times D^* \times E^* \times C^*$ *is such that*

$$\delta J_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*) = \mathbf{0}.$$

Assume $u_0 \in V$ *such that*

$$(u_0)_i = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial (v_2^*)_i}$$

is also such that

$$(u_0)_i \neq 0, \ \ a.e. \ in \ \Omega, \ \forall i \in \{1,2,3\}.$$

Under such hypotheses, we have

$$K_2(\hat{v}_3^*)_i (u_0)_i - K_2 = \mathbf{0}, \ \forall i \in \{1,2,3\},$$

and

$$\begin{aligned} J(u_0) &= \min_{u \in V} \left\{ J(u) + \sum_{i=1}^3 \frac{K_1}{2} \int_\Omega (K_2(\hat{v}_3)_i u_i - K_2)^2 \, dx \right\} \\ &= \inf_{v_2^* \in E^*} \left\{ \sup_{(\sigma, Q, v_3^*) \in B^* \times D^* \times C^*} J_1^*(\sigma, Q, v_2^*, v_3^*) \right\} \\ &= J_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*). \end{aligned} \tag{19}$$

Proof. Observe that there exists $\hat{u} \in V$ such that, defining

$$H(u, \sigma, Q, v_2^*, v_3^*) = -\langle u_{i,j}, \sigma_{ij} \rangle_{L^2} - \langle u_{i,j}, Q_{ij} \rangle_{L^2} + \langle u_i, (v_2^*)_i \rangle_{L^2} - F_1(u, v_3^*),$$

we have

$$\frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*)}{\partial u} = \mathbf{0}$$

and

$$F_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*) = H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*).$$

Moreover, from the variation in of J_1^* in $\tilde{Q}$, we obtain

$$-\frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*)}{\partial (v_2^*)_i} + \frac{\partial F_2^*(\hat{v}_2^*)}{\partial (v_2^*)_i} = \mathbf{0},$$

where

$$\begin{aligned} & \frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*)}{\partial (v_2^*)_i} \\ &= \frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*)}{\partial (v_2^*)_i} \\ & \quad + \frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*)}{\partial u_j} \frac{\partial \hat{u}_j}{\partial (\hat{v}_2^*)_i} \\ &= \hat{u}_i. \end{aligned} \tag{20}$$

From such last two equations we get

$$(u_0)_i = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial (v_2^*)_i} = \frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*)}{\partial (v_2^*)_i} = \hat{u}_i,$$

so that from the concerning boundary conditions,

$$u_0 = \hat{u}.$$

On the other hand, similarly from the variation of J_1^* in Q we have

$$-\frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*)}{\partial Q_{ij}} - \frac{\partial G^*(\hat{\sigma}, \hat{Q})}{\partial Q_{ij}} = \mathbf{0},$$

so that

$$(u_0)_{i,j} = -\frac{\partial G^*(\hat{\sigma}, \hat{Q})}{\partial Q_{ij}} = \mathbf{0},$$

and thus

$$(u_0)_{i,j} - \overline{\hat{\sigma}_{im}^K} \hat{Q}_{mj} = 0,$$

and therefore

$$\hat{Q}_{ij} = \hat{\sigma}_{im}(u_0)_{m,j} + K\delta_{ij}(u_0)_{i,j}.$$

Finally, from the variation of J_1^* in σ we obtain

$$-\frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*)}{\partial \sigma_{ij}} - \frac{\partial G^*(\hat{\sigma}, \hat{Q})}{\partial \sigma_{ij}} = \mathbf{0},$$

so that

$$(u_0)_{i,j} + \frac{1}{2}(u_0)_{m,i}(u_0)_{m,j} - \overline{H}_{ijkl}\hat{\sigma}_{kl} = 0.$$

Thus, since $\{H_{ijkl}\}$ is symmetric, we get

$$\hat{\sigma}_{ij} = H_{ijkl} e_{kl}(u_0).$$

Also from the variation of J_1^* in v_3^*, we have

$$K_1(K_2(\hat{v}_3^*)_i(u_0)_i - K_2)K_2(u_0)_i + \frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*)}{\partial u_j} \frac{\partial \hat{u}_j}{\partial (v_3^*)_i} = \mathbf{0},$$

so that, since

$$(u_0)_i \neq 0, \text{ a.e. in } \Omega,$$

we get

$$K_2(\hat{v}_3^*)_i(u_0)_i - K_2 = \mathbf{0}.$$

From these last results and from

$$\frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*)}{\partial u_i} = \mathbf{0}$$

we obtain

$$\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im}(u_0)_{m,j})_{,j} + f_i = 0, \ \forall i \in \{1, 2, 3\},$$

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so that

$$\delta J(u_0) = \mathbf{0}.$$

Finally, from such last results and the Legendre transform properties, we have

$$\begin{aligned} & F_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*) \\ = \ & -\langle (u_0)_{i,j}, \hat{\sigma}_{ij} \rangle_{L^2} - \langle (u_0)_{i,j}, \hat{Q}_{ij} \rangle_{L^2} + \langle (u_0)_i, (\hat{v}_2^*)_i \rangle_{L^2} - F_1(u_0, \hat{v}_3^*), \end{aligned} \tag{21}$$

$$F_2^*(\hat{v}_2^*) = \langle (u_0)_i, (\hat{v}_2^*)_i \rangle_{L^2} - F_2(u_0),$$

and

$$G^*(\hat{\sigma}, \hat{Q}) = \langle (u_0)_{i,j}, \hat{\sigma}_{ij} \rangle_{L^2} + \langle (u_0)_{i,j}, \hat{Q}_{ij} \rangle_{L^2} - G(u_0).$$

From these results, we obtain

$$\begin{aligned} J^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*) &= -F_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{\sigma}, \hat{Q}) \\ &= F_1(u_0, \hat{v}_3^*) - F_2(u_0) + G(u_0) \\ &= J(u_0). \end{aligned} \tag{22}$$

Finally, observe that

$$\begin{aligned} J_1^*(\hat{\sigma}, \hat{Q}, v_2^*, \hat{v}_3^*) \leq \ & \langle u_i, (v_2^*)_i \rangle_{L^2} + F_1(u, \hat{v}_3^*) \\ & + F_2^*(v_2^*) + G(u), \ \forall u \in V, v_2^* E^*. \end{aligned} \tag{23}$$

In particular

$$\begin{aligned} J_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*) &= \inf_{v_2^* \in E^*} \left\{ \sup_{(\sigma, Q, v_3^*) \in B^* \times D^* \times C^*} J_1^*(\sigma, Q, v_2^*, v_3^*) \right\} \\ &\leq \inf_{v_2^* \in E^*} \{ \langle u_i, (v_2^*)_i \rangle_{L^2} + F_1(u, \hat{v}_3^*) \\ & \quad + F_2^*(v_2^*) + G(u) \} \\ &= J(u) + \sum_{i=1}^3 \frac{K_1}{2} \int_\Omega (K_2 (\hat{v}_3)_i u_i - K_2)^2 \, dx, \ \forall u \in V. \end{aligned} \tag{24}$$

Summarizing, we have got

$$\begin{aligned} J_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*) &= \inf_{v_2^* \in E^*} \left\{ \sup_{(\sigma, Q, v_3^*) \in B^* \times D^* \times C^*} J_1^*(\sigma, Q, v_2^*, v_3^*) \right\} \\ &\leq J(u) + \sum_{i=1}^3 \frac{K_1}{2} \int_\Omega (K_2 (\hat{v}_3)_i u_i - K_2)^2 \, dx, \ \forall u \in V. \end{aligned} \tag{25}$$

Joining the pieces, we have got

$$\begin{aligned} J(u_0) &= \min_{u \in V} \left\{ J(u) + \sum_{i=1}^3 \frac{K_1}{2} \int_\Omega (K_2 (\hat{v}_3)_i u_i - K_2)^2 \, dx \right\} \\ &= \inf_{v_2^* \in E^*} \left\{ \sup_{(\sigma, Q, v_3^*) \in B^* \times D^* \times C^*} J_1^*(\sigma, Q, v_2^*, v_3^*) \right\} \\ &= J_1^*(\hat{\sigma}, \hat{Q}, \hat{v}_2^*, \hat{v}_3^*). \end{aligned} \tag{26}$$

The proof is complete.

□

Remark 3.3. *Indeed in such a infinite dimensional space context perhaps it is necessary to replace*

$$F_2(u) = \sum_{i=1}^{3} \frac{K_2}{2} \int_\Omega (u_i)^2 \, dx$$

by

$$\tilde{F}_2(u) = \sum_{i=1}^{3} \frac{K_2}{2} \int_\Omega (u_{i,i})^2 \, dx.$$

In fact this is very simple to be done with a few changes in the results obtained.

4 Conclusion

In this article we have developed a convex dual variational formulation suitable for non-convex variational primal formulations.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principle here presented is applied to a model in non-linear elasticity. In a future research, we intend to extend such results for other related models of plates and shells.

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