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Dual Variational Formulations for a Large Class of Non-Convex Models in the Calculus of Variations

Fabio Silva Botelho
Department of Mathematics
Federal University of Santa Catarina
Florianópolis - SC, Brazil

Abstract

This article develops dual variational formulations for a large class of models in variational optimization. The results are established through basic tools of functional analysis, convex analysis and duality theory. The main duality principle is developed as an application to a Ginzburg-Landau type system in superconductivity in the absence of a magnetic field. In the first part final sections, we develop new general dual convex variational formulations, more specifically, dual formulations with a large region of convexity around the critical points which are suitable for the non-convex optimization for a large class of models in physics and engineering. Finally, in the last section we present some numerical results concerning the generalized method of lines applied to a Ginzburg-Landau type equation.

Key words: Duality principles; Generalized method of lines; Ginzburg-Landau type equations

1 Introduction

In this section we establish a dual formulation for a large class of models in non-convex optimization.

The main duality principle is applied to the Ginzburg-Landau system in superconductivity in an absence of a magnetic field.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 13, 14] and on a D.C. optimization approach developed in Toland [15].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [9, 5, 6, 7, 12]. Finally, similar models on the superconductivity physics may be found in [4, 11].

Remark 1.1. *It is worth highlighting, we may generically denote*

$$\int_\Omega [(-\gamma\nabla^2 + KI_d)^{-1}v^*]v^* \, dx$$

simply by

$$\int_\Omega \frac{(v^*)^2}{-\gamma\nabla^2 + K} \, dx,$$

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where I_d denotes a concerning identity operator.

Other similar notations may be used along this text as their indicated meaning are sufficiently clear.

Also, ∇^2 denotes the Laplace operator and for real constants $K_2 > 0$ and $K_1 > 0$, the notation $K_2 \gg K_1$ means that $K_2 > 0$ is much larger than $K_1 > 0$.

Finally, we adopt the standard Einstein convention of summing up repeated indices, unless otherwise indicated.

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation we consider the functional $J : U \rightarrow \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx \\ &+\frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\, dx - \langle u, f\rangle_{L^2}. \end{aligned} \tag{1}$$

Here we assume $\alpha > 0, \beta > 0, \gamma > 0$, $U = W_0^{1,2}(\Omega)$, $f \in L^2(\Omega)$. Moreover we denote

$$Y = Y^* = L^2(\Omega).$$

Define also $G_1 : U \rightarrow \mathbb{R}$ by

$$G_1(u) = \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx,$$

$G_2 : U \times Y \rightarrow \mathbb{R}$ by

$$G_2(u, v) = \frac{\alpha}{2}\int_\Omega (u^2 - \beta + v)^2 \, dx + \frac{K}{2}\int_\Omega u^2 \, dx,$$

and $F : U \rightarrow \mathbb{R}$ by

$$F(u) = \frac{K}{2}\int_\Omega u^2 \, dx,$$

where $K \gg \gamma$.

It is worth highlighting that in such a case

$$J(u) = G_1(u) + G_2(u, 0) - F(u) - \langle u, f\rangle_{L^2}, \ \forall u \in U.$$

Furthermore, define the following specific polar functionals specified, namely, $G_1^* : [Y^*]^2 \rightarrow \mathbb{R}$ by

$$\begin{aligned} G_1^*(v_1^* + z^*) &= \sup_{u \in U}\{\langle u, v_1^* + z^*\rangle_{L^2} - G_1(u)\} \\ &= \frac{1}{2}\int_\Omega [(-\gamma\nabla^2)^{-1}(v_1^* + z^*)](v_1^* + z^*)\, dx, \end{aligned} \tag{2}$$

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$G_2^*: [Y^*]^2 \to \mathbb{R}$ by

$$\begin{aligned} G_1^*(v_2^*, v_0^*) &= \sup_{(u,v)\in U\times Y} \{\langle u, v_2^*\rangle_{L^2} + \langle v, v_0^*\rangle_{L^2} - G_2(u,v)\} \\ &= \frac{1}{2}\int_\Omega \frac{(v_2^*)^2}{2v_0^*+K}\,dx \\ &\quad +\frac{1}{2\alpha}\int_\Omega (v_0^*)^2\,dx + \beta\int_\Omega v_0^*\,dx, \end{aligned} \tag{3}$$

if $v_0^* \in B^*$ where

$$B^* = \{v_0^* \in Y^* \;:\; 2v_0^* + K > K/2 \text{ in } \Omega\},$$

and finally, $F^*: Y^* \to \mathbb{R}$ by

$$\begin{aligned} F^*(z^*) &= \sup_{u\in U}\{\langle u, z^*\rangle_{L^2} - F(u)\} \\ &= \frac{1}{2K}\int_\Omega (z^*)^2\,dx. \end{aligned} \tag{4}$$

Define also

$$A^* = \{v^* = (v_1^*, v_2^*, v_0^*) \in [Y^*]^2 \times B^* \;:\; v_1^* + v_2^* - f = 0, \text{ in } \Omega\},$$

$J^*: [Y^*]^4 \to \mathbb{R}$ by

$$J^*(v^*, z^*) = -G_1^*(v_1^* + z^*) - G_2^*(v_2^*, v_0^*) + F^*(z^*)$$

and $J_1^*: [Y^*]^4 \times U \to \mathbb{R}$ by

$$J_1^*(v^*, z^*, u) = J^*(v^*, z^*) + \langle u, v_1^* + v_2^* - f\rangle_{L^2}.$$

2 The main duality principle, a convex dual formulation and the concerning proximal primal functional

Our main result is summarized by the following theorem.

Theorem 2.1. *Considering the definitions and statements in the last section, suppose also* $(\hat{v}^*, \hat{z}^*, u_0) \in [Y^*]^2 \times B^* \times Y^* \times U$ *is such that*

$$\delta J_1^*(\hat{v}^*, \hat{z}^*, u_0) = \mathbf{0}.$$

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

$$\hat{v}^* \in A^*$$

and

$$\begin{aligned} J(u_0) &= \inf_{u\in U}\left\{J(u) + \frac{K}{2}\int_\Omega |u-u_0|^2\,dx\right\} \\ &= J^*(\hat{v}^*, \hat{z}^*) \\ &= \sup_{v^*\in A^*}\{J^*(v^*, \hat{z}^*)\}. \end{aligned} \tag{5}$$

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Proof. Since

$$\delta J_1^*(\hat{v}^*, \hat{z}^*, u_0) = \mathbf{0}$$

from the variation in v_1^* we obtain

$$-\frac{(\hat{v}_1^* + \hat{z}^*)}{-\gamma \nabla^2} + u_0 = 0 \text{ in } \Omega,$$

so that

$$\hat{v}_1^* + \hat{z}^* = -\gamma \nabla^2 u_0.$$

From the variation in v_2^* we obtain

$$-\frac{\hat{v}_2^*}{2\hat{v}_0^* + K} + u_0 = 0, \text{ in } \Omega.$$

From the variation in v_0^* we also obtain

$$\frac{(\hat{v}_2^*)^2}{(2\hat{v}_0^* + K)^2} - \frac{\hat{v}_0^*}{\alpha} - \beta = 0$$

and therefore,

$$\hat{v}_0^* = \alpha(u_0^2 - \beta).$$

From the variation in u we get

$$\hat{v}_1^* + \hat{v}_2^* - f = 0, \text{ in } \Omega$$

and thus

$$\hat{v}^* \in A^*.$$

Finally, from the variation in z^*, we obtain

$$-\frac{(\hat{v}_1^* + \hat{z}^*)}{-\gamma \nabla^2} + \frac{\hat{z}^*}{K} = 0, \text{ in } \Omega.$$

so that

$$-u_0 + \frac{\hat{z}^*}{K} = 0,$$

that is,

$$\hat{z}^* = K u_0 \text{ in } \Omega.$$

From such results and $\hat{v}^* \in A^*$ we get

$$\begin{aligned} 0 &= \hat{v}_1^* + \hat{v}_2^* - f \\ &= -\gamma \nabla^2 u_0 - \hat{z}^* + 2(v_0^*) u_0 + K u_0 - f \\ &= -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta) u_0 - f, \end{aligned} \tag{6}$$

so that

$$\delta J(u_0) = \mathbf{0}.$$

Also from this and from the Legendre transform proprieties we have

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$$G_1^*(\hat{v}_1^*+\hat{z}^*)=\langle u_0,\hat{v}_1^*+\hat{z}^*\rangle_{L^2}-G_1(u_0),$$
$$G_2^*(\hat{v}_2^*,\hat{v}_0^*)=\langle u_0,\hat{v}_2^*\rangle_{L^2}+\langle 0,v_0^*\rangle_{L^2}-G_2(u_0,0),$$
$$F^*(\hat{z}^*)=\langle u_0,\hat{z}^*\rangle_{L^2}-F(u_0)$$

and thus we obtain

$$\begin{aligned}
J^*(\hat{v}^*,\hat{z}^*) &= -G_1^*(\hat{v}_1^*+\hat{z}^*)-G_2^*(\hat{v}_2^*,\hat{v}_0^*)+F^*(\hat{z}^*)\\
&= -\langle u_0,\hat{v}_1^*+\hat{v}_2^*\rangle+G_1(u_0)+G_2(u_0,0)-F(u_0)\\
&= -\langle u_0,f\rangle_{L^2}+G_1(u_0)+G_2(u_0,0)-F(u_0)\\
&= J(u_0).
\end{aligned} \tag{7}$$

Summarizing, we have got

$$J^*(\hat{v}^*,\hat{z}^*)=J(u_0). \tag{8}$$

On the other hand

$$\begin{aligned}
J^*(\hat{v}^*,\hat{z}^*) &= -G_1^*(\hat{v}_1^*+\hat{z}^*)-G_2^*(\hat{v}_2^*,\hat{v}_0^*)+F^*(\hat{z}^*)\\
&\leq -\langle u,\hat{v}_1^*+\hat{z}^*\rangle_{L^2}-\langle u,\hat{v}_2^*\rangle_{L^2}-\langle 0,v_0^*\rangle_{L^2}+G_1(u)+G_2(u,0)+F^*(\hat{z}^*)\\
&= -\langle u,f\rangle_{L^2}+G_1(u)+G_2(u,0)-\langle u,\hat{z}^*\rangle_{L^2}+F^*(\hat{z}^*)\\
&= -\langle u,f\rangle_{L^2}+G_1(u)+G_2(u,0)-F(u)+F(u)-\langle u,\hat{z}^*\rangle_{L^2}+F^*(\hat{z}^*)\\
&= J(u)+\frac{K}{2}\int_\Omega u^2\,dx-\langle u,\hat{z}^*\rangle_{L^2}+F^*(\hat{z}^*)\\
&= J(u)+\frac{K}{2}\int_\Omega u^2\,dx-K\langle u,u_0\rangle_{L^2}+\frac{K}{2}\int_\Omega u_0^2\,dx\\
&= J(u)+\frac{K}{2}\int_\Omega |u-u_0|^2\,dx,\ \forall u\in U.
\end{aligned} \tag{9}$$

Finally by a simple computation we may obtain the Hessian

$$\left\{\frac{\partial^2 J^*(v^*,z^*)}{\partial (v^*)^2}\right\}<\mathbf{0}$$

in $[Y^*]^2\times B^*\times Y^*$, so that we may infer that J^* is concave in v^* in $[Y^*]^2\times B^*\times Y^*$.

Therefore, from this, (8) and (9), we have

$$\begin{aligned}
J(u_0) &= \inf_{u\in U}\left\{J(u)+\frac{K}{2}\int_\Omega |u-u_0|^2\,dx\right\}\\
&= J^*(\hat{v}^*,\hat{z}^*)\\
&= \sup_{v^*\in A^*}\{J^*(v^*,\hat{z}^*)\}.
\end{aligned} \tag{10}$$

The proof is complete. □

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3 A primal dual variational formulation

In this section we develop a more general primal dual variational formulation suitable for a large class of models in non-convex optimization.

Consider again $U = W_0^{1,2}(\Omega)$ and let $G : U \to \mathbb{R}$ and $F : U \to \mathbb{R}$ be three times Fréchet differentiable functionals. Let $J : U \to \mathbb{R}$ be defined by

$$J(u) = G(u) - F(u),\ \forall u \in U.$$

Assume $u_0 \in U$ is such that

$$\delta J(u_0) = \mathbf{0}$$

and

$$\delta^2 J(u_0) > \mathbf{0}.$$

Denoting $v^* = (v_1^*, v_2^*)$, define $J^* : U \times Y^* \times Y^* \to \mathbb{R}$ by

$$J^*(u, v^*) = \frac{1}{2}\|v_1^* - G'(u)\|_2^2 + \frac{1}{2}\|v_2^* - F'(u)\|_2^2 + \frac{1}{2}\|v_1^* - v_2^*\|_2^2 \tag{11}$$

Denoting $L_1^*(u, v^*) = v_1^* - G'(u)$ and $L_2^*(u, v^*) = v_2^* - F'(u)$, define also

$$C^* = \left\{(u, v^*) \in U \times Y^* \times Y^*\ :\ \|L_1^*(u, v_1^*)\|_\infty \leq \frac{1}{K} \text{ and } \|L_2^*(u, v_1^*)\|_\infty \leq \frac{1}{K}\right\},$$

for an appropriate $K > 0$ to be specified.

Observe that in C^* the Hessian of J^* is given by

$$\{\delta^2 J^*(u, v^*)\} = \left\{\begin{array}{lrr} G''(u)^2 + F''(u)^2 + \mathcal{O}(1/K) & -G''(u) & -F''(u) \\ -G''(u) & 2 & -1 \\ -F''(u) & -1 & 2 \end{array}\right\}, \tag{12}$$

Observe also that

$$\det\left\{\frac{\partial^2 J^*(u, v^*)}{\partial v_1^* \partial v_2^*}\right\} = 3,$$

and

$$\det\{\delta^2 J^*(u, v^*)\} = (G''(u) - F''(u))^2 + \mathcal{O}(1/K) = (\delta^2 J(u))^2 + \mathcal{O}(1/K).$$

Define now

$$\hat{v}_1^* = G'(u_0),$$

$$\hat{v}_2^* = F'(u_0),$$

so that

$$\hat{v}_1^* - \hat{v}_2^* = \mathbf{0}.$$

From this we may infer that $(u_0, \hat{v}_1^*, \hat{v}_2^*) \in C^*$ and

$$J^*(u_0, \hat{v}^*) = 0 = \min_{(u,v^*) \in C^*} J^*(u, v^*).$$

Moreover, for $K > 0$ sufficiently big, J^* is convex in a neighborhood of $(u_0, \hat{v}^*)$.

Therefore, in the last lines, we have proven the following theorem.

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Theorem 3.1. *Under the statements and definitions of the last lines, there exist $r_0 > 0$ and $r_1 > 0$ such that*

$$J(u_0) = \min_{u \in B_{r_0}(u_0)} J(u)$$

and $(u_0, \hat{v}_1^*, \hat{v}_2^*) \in C^*$ is such that

$$J^*(u_0, \hat{v}^*) = 0 = \min_{(u,v^*) \in U \times [Y^*]^2} J^*(u, v^*).$$

Moreover, J^* is convex in

$$B_{r_1}(u_0, \hat{v}^*).$$

4 One more duality principle and a concerning primal dual variational formulation

In this section we establish a new duality principle and a related primal dual formulation. The results are based on the approach of Toland, [15].

4.1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Let $J : V \to \mathbb{R}$ be a functional such that

$$J(u) = G(u) - F(u), \forall u \in V,$$

where $V = W_0^{1,2}(\Omega)$.

Suppose G, F are both three times Fréchet differentiable convex functionals such that

$$\frac{\partial^2 G(u)}{\partial u^2} > 0$$

and

$$\frac{\partial^2 F(u)}{\partial u^2} > 0$$

$\forall u \in V$.

Assume also there exists $\alpha_1 \in \mathbb{R}$ such that

$$\alpha_1 = \inf_{u \in V} J(u).$$

Moreover, suppose that if $\{u_n\} \subset V$ is such that

$$\|u_n\|_V \to \infty$$

then

$$J(u_n) \to +\infty, \text{ as } n \to \infty.$$

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At this point we define $J^{**} : V \to \mathbb{R}$ by

$$J^{**}(u) = \sup_{(v^*,\alpha)\in H^*} \{\langle u, v^*\rangle + \alpha\},$$

where

$$H^* = \{(v^*, \alpha) \in V^* \times \mathbb{R} \, : \, \langle v, v^*\rangle_V + \alpha \leq F(v), \; \forall v \in V\}.$$

Observe that $(0, \alpha_1) \in H^*$, so that

$$J^{**}(u) \geq \alpha_1 = \inf_{u\in V} J(u).$$

On the other hand, clearly we have

$$J^{**}(u) \leq J(u), \; \forall u \in V,$$

so that we have got

$$\alpha_1 = \inf_{u\in V} J(u) = \inf_{u\in V} J^{**}(u).$$

Let $u \in V$.

Since J is strongly continuous, there exist $\delta > 0$ and $A > 0$ such that,

$$\alpha_1 \leq J^{**}(v) \leq J(v) \leq A, \forall v \in B_\delta(u).$$

From this, considering that J^{**} is convex on V, we may infer that J^{**} is continuous at u, $\forall u \in V$.

Hence J^{**} is strongly lower semi-continuous on V, and since J^{**} is convex we may infer that J^{**} is weakly lower semi-continuous on V.

Let $\{u_n\} \subset V$ be a sequence such that

$$\alpha_1 \leq J(u_n) < \alpha_1 + \frac{1}{n}, \; \forall n \in \mathbb{N}.$$

Hence

$$\alpha_1 = \lim_{n\to\infty} J(u_n) = \inf_{u\in V} J(u) = \inf_{u\in V} J^{**}(u).$$

Suppose there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\|u_{n_k}\|_V \to \infty, \text{ as } k \to \infty.$$

From the hypothesis we have

$$J(u_{n_k}) \to +\infty, \text{ as } k \to \infty,$$

which contradicts

$$\alpha_1 \in \mathbb{R}.$$

Therefore there exists $K > 0$ such that

$$\|u_n\|_V \leq K, \; \forall u \in V.$$

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Since V is reflexive, from this and the Katutani Theorem, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u_0 \in V$ such that

$$u_{n_k} \rightharpoonup u_0, \text{ weakly in } V.$$

Consequently, from this and considering that J^{**} is weakly lower semi-continuous, we have got

$$\alpha_1 = \liminf_{k\to\infty} J^{**}(u_{n_k}) \geq J^{**}(u_0),$$

so that

$$J^{**}(u_0) = \min_{u\in V} J^{**}(u).$$

Define $G^*, F^* : V^* \to \mathbb{R}$ by

$$G^*(v^*) = \sup_{u\in V}\{\langle u, v^*\rangle_V - G(u)\},$$

and

$$F^*(v^*) = \sup_{u\in V}\{\langle u, v^*\rangle_V - F(u)\}.$$

Defining also $J^* : V \to \mathbb{R}$ by

$$J^*(v^*) = F^*(v^*) - G^*(v^*),$$

from the results in [15], we may obtain

$$\inf_{u\in V} J(u) = \inf_{v^*\in V^*} J^*(v^*),$$

so that

$$\begin{aligned} J^{**}(u_0) &= \inf_{u\in V} J^{**}(u) \\ &= \inf_{u\in V} J(u) = \inf_{v^*\in V^*} J^*(v^*). \end{aligned} \tag{13}$$

Suppose now there exists $\hat{u} \in V$ such that

$$J(\hat{u}) = \inf_{u\in V} J(u).$$

From the standard necessary conditions, we have

$$\delta J(\hat{u}) = \mathbf{0},$$

so that

$$\frac{\partial G(\hat{u})}{\partial u} - \frac{\partial F(\hat{u})}{\partial u} = \mathbf{0}.$$

Define now

$$v_0^* = \frac{\partial F(\hat{u})}{\partial u}.$$

From these last two equations we obtain

$$v_0^* = \frac{\partial G(\hat{u})}{\partial u}.$$

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From such results and the Legendre transform properties, we have

$$\hat{u} = \frac{\partial F^*(v_0^*)}{\partial v^*},$$

$$\hat{u} = \frac{\partial G^*(v_0^*)}{\partial v^*},$$

so that

$$\delta J^*(v_0^*) = \frac{\partial F^*(v_0^*)}{\partial v^*} - \frac{\partial G^*(v_0^*)}{\partial v^*} = \hat{u} - \hat{u} = \mathbf{0},$$

$$G^*(v_0^*) = \langle \hat{u}, v_0^* \rangle_V - G(\hat{u})$$

and

$$F^*(v_0^*) = \langle \hat{u}, v_0^* \rangle_V - F(\hat{u})$$

so that

$$\begin{aligned} \inf_{u \in V} J(u) &= J(\hat{u}) \\ &= G(\hat{u}) - F(\hat{u}) \\ &= \inf_{v^* \in V^*} J^*(v^*) \\ &= F^*(v_0^*) - G^*(v_0^*) \\ &= J^*(v_0^*). \end{aligned} \tag{14}$$

4.2 The main duality principle and a related primal dual variational formulation

Considering these last statements and results, we may prove the following theorem.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.*

Let $J : V \to \mathbb{R}$ be a functional such that

$$J(u) = G(u) - F(u), \forall u \in V,$$

where $V = W_0^{1,2}(\Omega)$.

Suppose G, F are both three times Fréchet differentiable functionals such that there exists $K > 0$ such that

$$\frac{\partial^2 G(u)}{\partial u^2} + K > 0$$

and

$$\frac{\partial^2 F(u)}{\partial u^2} + K > 0$$

$\forall u \in V$.

Assume also there exists $u_0 \in V$ and $\alpha_1 \in \mathbb{R}$ such that

$$\alpha_1 = \inf_{u \in V} J(u) = J(u_0).$$

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Assume $K_3 > 0$ is such that
$$\|u_0\|_\infty < K_3.$$

Define
$$\tilde{V} = \{u \in V \;:\; \|u\|_\infty \leq K_3\}.$$

Assume $K_1 > 0$ is such that if $u \in \tilde{V}$ then

$$\max\left\{\|F'(u)\|_\infty, \|G'(u)\|_\infty, \|F''(u)\|_\infty,\; \|F'''(u)\|_\infty, \|G''(u)\|_\infty, \|G'''(u)\|_\infty\right\} \leq K_1.$$

Suppose also
$$K \gg \max\{K_1, K_3\}.$$

Define $F_K, G_K : V \to \mathbb{R}$ by

$$F_K(u) = F(u) + \frac{K}{2}\int_\Omega u^2\,dx,$$

and

$$G_K(u) = G(u) + \frac{K}{2}\int_\Omega u^2\,dx,$$

$\forall u \in V$.

Define also $G_K^*, F_K^* : V^* \to \mathbb{R}$ by

$$G_K^*(v^*) = \sup_{u \in V}\{\langle u, v^*\rangle_V - G_K(u)\},$$

and
$$F_K^*(v^*) = \sup_{u \in V}\{\langle u, v^*\rangle_V - F_K(u)\}.$$

Observe that since $u_0 \in V$ is such that

$$J(u_0) = \inf_{u \in V} J(u),$$

we have
$$\delta J(u_0) = \mathbf{0}.$$

Let $\varepsilon > 0$ be a small constant.

Define
$$v_0^* = \frac{\partial F_K(u_0)}{\partial u} \in V^*.$$

Under such hypotheses, defining $J_1^* : V \times V^* \to \mathbb{R}$ by

$$\begin{aligned} J_1^*(u, v^*) &= F_K^*(v^*) - G_K^*(v^*) \\ &+ \frac{1}{2\varepsilon}\left\|\frac{\partial G_K^*(v^*)}{\partial v^*} - u\right\|_2^2 + \frac{1}{2\varepsilon}\left\|\frac{\partial F_K^*(v^*)}{\partial v^*} - u\right\|_2^2 \\ &+ \frac{1}{2\varepsilon}\left\|\frac{\partial G_K^*(v^*)}{\partial v^*} - \frac{\partial F_K^*(v^*)}{\partial v^*}\right\|_2^2, \end{aligned} \tag{15}$$

we have

$$\begin{aligned} J(u_0) &= \inf_{u \in V} J(u) \\ &= \inf_{(u,v^*) \in V \times V^*} J_1^*(u,v^*) \\ &= J_1^*(u_0,v_0^*). \end{aligned} \tag{16}$$

Proof. Observe that from the hypotheses and the results and statements of the last subsection

$$J(u_0) = \inf_{u \in V} J(u) = \inf_{v^* \in Y^*} J_K^*(v^*) = J_K^*(v_0^*),$$

where

$$J_K^*(v^*) = F_K^*(v^*) - G_K^*(v^*), \forall v^* \in V^*.$$

Moreover we have

$$J_1^*(u,v^*) \geq J_K^*(v^*), \forall u \in V,\ v^* \in V^*.$$

Also from hypotheses and the last subsection results,

$$u_0 = \frac{\partial F_K^*(v_0^*)}{\partial v^*} = \frac{\partial G_K^*(v_0^*)}{\partial v^*},$$

so that clearly we have

$$J_1^*(u_0,v_0^*) = J_K^*(v_0^*).$$

From these last results, we may infer that

$$\begin{aligned} J(u_0) &= \inf_{u \in V} J(u) \\ &= \inf_{v^* \in V^*} J_K^*(v^*) \\ &= J_K^*(v_0^*) \\ &= \inf_{(u,v^*) \in V \times V^*} J_1^*(u,v^*) \\ &= J_1^*(u_0,v_0^*). \end{aligned} \tag{17}$$

The proof is complete.

□

Remark 4.2. *At this point we highlight that J_1^* has a large region of convexity around the optimal point (u_0, v_0^*), for $K > 0$ sufficiently large and corresponding $\varepsilon > 0$ sufficiently small.*

Indeed, observe that for $v^* \in V^*$,

$$G_K^*(v^*) = \sup_{u \in V}\{\langle u, v^* \rangle_V - G_K(u)\} = \langle \hat{u}, v^* \rangle_V - G_K(\hat{u})$$

where $\hat{u} \in V$ is such that

$$v^* = \frac{\partial G_K(\hat{u})}{\partial u} = G'(\hat{u}) + K\hat{u}.$$

Taking the variation in v^* in this last equation, we obtain

$$1 = G''(u)\frac{\partial \hat{u}}{\partial v^*} + K\frac{\partial \hat{u}}{\partial v^*},$$

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so that

$$\frac{\partial \hat{u}}{\partial v^*} = \frac{1}{G''(u) + K} = \mathcal{O}\left(\frac{1}{K}\right).$$

From this we get

$$\begin{aligned} \frac{\partial^2 \hat{u}}{\partial (v^*)^2} &= -\frac{1}{(G''(u)+K)^2} G'''(u) \frac{\partial \hat{u}}{\partial v^*} \\ &= -\frac{1}{(G''(u)+K)^3} G'''(u) \\ &= \mathcal{O}\left(\frac{1}{K^3}\right). \end{aligned} \tag{18}$$

On the other hand, from the implicit function theorem

$$\frac{\partial G_K^*(v^*)}{\partial v^*} = u + [v^* - G_K'(\hat{u})] \frac{\partial \hat{u}}{\partial v^*} = u,$$

so that

$$\frac{\partial^2 G_K^*(v^*)}{\partial (v^*)^2} = \frac{\partial \hat{u}}{\partial v^*} = \mathcal{O}\left(\frac{1}{K}\right)$$

and

$$\frac{\partial^3 G_K^*(v^*)}{\partial (v^*)^3} = \frac{\partial^2 \hat{u}}{\partial (v^*)^2} = \mathcal{O}\left(\frac{1}{K^3}\right).$$

Similarly, we may obtain

$$\frac{\partial^2 F_K^*(v^*)}{\partial (v^*)^2} = \mathcal{O}\left(\frac{1}{K}\right)$$

and

$$\frac{\partial^3 F_K^*(v^*)}{\partial (v^*)^3} = \mathcal{O}\left(\frac{1}{K^3}\right).$$

Denoting

$$A = \frac{\partial^2 F_K^*(v_0^*)}{\partial (v^*)^2}$$

and

$$B = \frac{\partial^2 G_K^*(v_0^*)}{\partial (v^*)^2},$$

we have

$$\frac{\partial^2 J_1^*(u_0, v_0^*)}{\partial (v^*)^2} = A - B + \frac{1}{\varepsilon}\left(2A^2 + 2B^2 - 2AB\right),$$

$$\frac{\partial^2 J_1^*(u_0, v_0^*)}{\partial u^2} = \frac{2}{\varepsilon},$$

and

$$\frac{\partial^2 J_1^*(u_0, v_0^*)}{\partial (v^*) \partial u} = -\frac{1}{\varepsilon}(A + B).$$

From this we get

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$$\begin{aligned}
\det(\delta^2 J^*(v_0^*,u_0)) &= \frac{\partial^2 J_1^*(u_0,v_0^*)}{\partial (v^*)^2}\frac{\partial^2 J_1^*(u_0,v_0^*)}{\partial u^2} - \left[\frac{\partial^2 J_1^*(u_0,v_0^*)}{\partial (v^*)\partial u}\right]^2 \\
&= 2\frac{A-B}{\varepsilon} + 2\frac{(A-B)^2}{\varepsilon^2} \\
&= \mathcal{O}\left(\frac{1}{\varepsilon^2}\right) \\
&\gg \mathbf{0} \qquad (19)
\end{aligned}$$

about the optimal point (u_0, v_0^*).

5 A convex dual variational formulation

In this section, again for $\Omega \subset \mathbb{R}^3$ an open, bounded, connected set with a regular (Lipschitzian) boundary $\partial\Omega$, $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega)$, we denote $F_1 : V \times Y \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $G : V \times Y \to \mathbb{R}$ by

$$\begin{aligned}
F_1(u,v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx - \frac{K}{2}\int_\Omega u^2\,dx \\
&+\frac{K_1}{2}\int_\Omega(-\gamma\nabla^2 u + 2v_0^* u - f)^2\,dx + \frac{K_2}{2}\int_\Omega u^2\,dx, \qquad (20)
\end{aligned}$$

$$F_2(u) = \frac{K_2}{2}\int_\Omega u^2\,dx + \langle u,f\rangle_{L^2},$$

and

$$G(u,v) = \frac{\alpha}{2}\int_\Omega (u^2-\beta+v)^2\,dx + \frac{K}{2}\int_\Omega u^2\,dx.$$

We define also

$$J_1(u,v_0^*) = F_1(u,v_0^*) - F_2(u) + G(u,0),$$

$$J(u) = \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx + \frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\,dx - \langle u,f\rangle_{L^2},$$

and $F_1^* : [Y^*]^3 \to \mathbb{R}$, $F_2^* : Y^* \to \mathbb{R}$, and $G^* : [Y^*]^2 \to \mathbb{R}$, by

$$\begin{aligned}
&F_1^*(v_2^*,v_1^*,v_0^*) \\
&= \sup_{u\in V}\{\langle u, v_1^*+v_2^*\rangle_{L^2} - F_1(u,v_0^*)\} \\
&= \frac{1}{2}\int_\Omega \frac{\left(v_1^*+v_2^*+K_1(-\gamma\nabla^2+2v_0^*)f\right)^2}{(-\gamma\nabla^2 - K + K_2 + K_1(-\gamma\nabla^2+2v_0^*)^2)}\,dx \\
&\quad -\frac{K_1}{2}\int_\Omega f^2\,dx, \qquad (21)
\end{aligned}$$

$$\begin{aligned}
F_2^*(v_2^*) &= \sup_{u\in V}\{\langle u,v_2^*\rangle_{L^2} - F_2(u)\} \\
&= \frac{1}{2K_2}\int_\Omega (v_2^*)^2\,dx, \qquad (22)
\end{aligned}$$

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and

$$\begin{aligned} G^*(v_1^*, v_0^*) &= \sup_{(u,v)\in V\times Y} \{\langle u, v_1^*\rangle_{L^2} - \langle v, v_0^*\rangle_{L^2} - G(u,v)\} \\ &= \frac{1}{2}\int_\Omega \frac{(v_1^*)^2}{2v_0^*+K}\,dx + \frac{1}{2\alpha}\int_\Omega (v_0^*)^2\,dx \\ &+\beta\int_\Omega v_0^*\,dx \end{aligned} \tag{23}$$

if $v_0^* \in B^*$ where

$$B^* = \{v_0^* \in Y^* \;:\; \|v_0^*\|_\infty \leq K/2 \text{ and } -\gamma\nabla^2 + 2v_0^* < -\varepsilon I_d\},$$

for some small real parameter $\varepsilon > 0$ and where I_d denotes a concerning identity operator.

Finally, we also define $J_1^* : [Y^*]^2 \times B^* \to \mathbb{R}$,

$$J_1^*(v_2^*, v_1^*, v_0^*) = -F_1^*(v_2^*, v_1^*, v_0^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*).$$

Assuming

$$K_2 \gg K_1 \gg K \gg \max\{1/(\varepsilon^2), 1, \gamma, \alpha\}$$

by directly computing $\delta^2 J_1^*(v_2^*, v_1^*, v_0^*)$ we may obtain that for such specified real constants, J_1^* in convex in v_2^* and it is concave in (v_1^*, v_0^*) on $Y^* \times Y^* \times B^*$.

Considering such statements and definitions, we may prove the following theorem.

Theorem 5.1. *Let $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \in Y^* \times Y^* \times B^*$ be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V$ be such that

$$u_0 = \frac{\hat{v}_1^* + \hat{v}_2^* + K_1(-\gamma\nabla^2 + 2v_0^*)f}{K_2 - K - \gamma\nabla^2 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)^2}.$$

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

so that

$$\begin{aligned} J(u_0) &= \inf_{u\in V}\left\{J(u) + \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2\hat{v}_0^* u - f)^2\,dx\right\} \\ &= \inf_{v_2^*\in Y^*}\left\{\sup_{(v_1^*,v_0^*)\in Y^*\times B^*} J_1^*(v_2^*, v_1^*, v_0^*)\right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \end{aligned} \tag{24}$$

Proof. Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$ so that, since J_1^* is convex in v_2^* and concave in (v_1^*, v_0^*) on $Y^* \times Y^* \times B^*$, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \inf_{v_2^*\in Y^*}\left\{\sup_{(v_1^*,v_0^*)\in Y^*\times B^*} J_1^*(v_2^*, v_1^*, v_0^*)\right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

we have

$$-u_0 + \frac{\hat{v}_2^*}{K_2} = 0,$$

and thus

$$\hat{v}_2^* = K_2 u_0.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0},$$

we obtain

$$-u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = 0,$$

and thus

$$\hat{v}_1^* = -2\hat{v}_0^* u_0 - K u_0 + f.$$

Finally, denoting

$$D = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$-2Du_0 + u_0^2 - \frac{\hat{v}_0^*}{\alpha} - \beta = 0,$$

so that

$$\hat{v}_0^* = \alpha(u_0^2 - \beta - 2Du_0). \tag{25}$$

Observe now that

$$\hat{v}_1^* + \hat{v}_2^* + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)f = (K_2 - K - \gamma\nabla^2 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)^2)u_0$$

so that

$$\begin{aligned} & K_2 u_0 - 2\hat{v}_0 u_0 - K u_0 + f \\ = \quad & K_2 u_0 - K u_0 - \gamma \nabla^2 u_0 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)(-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f). \end{aligned} \tag{26}$$

The solution for this last system of equations (25) and (26) is obtained through the relations

$$\hat{v}_0^* = \alpha(u_0^2 - \beta)$$

and

$$-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f = D = 0,$$

so that

$$\delta J(u_0) = -\gamma\nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = 0$$

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and

$$\delta\left\{J(u_0)+\frac{K_1}{2}\int_\Omega(-\gamma\nabla^2 u_0+2\hat{v}_0^* u_0-f)^2\,dx\right\}=0,$$

and hence, from the concerning convexity in u on V,

$$J(u_0)=\min_{u\in V}\left\{J(u)+\frac{K_1}{2}\int_\Omega(-\gamma\nabla^2 u+2\hat{v}_0^* u-f)^2\,dx\right\}.$$

Moreover, from the Legendre transform properties

$$F_1^*(\hat{v}_2^*,\hat{v}_1^*,\hat{v}_0^*)=\langle u_0,\hat{v}_2^*+\hat{v}_1^*\rangle_{L^2}-F_1(u_0,\hat{v}_0^*),$$

$$F_2^*(\hat{v}_2^*)=\langle u_0,\hat{v}_2^*\rangle_{L^2}-F_2(u_0),$$

$$G^*(\hat{v}_1^*,\hat{v}_0^*)=-\langle u_0,\hat{v}_1^*\rangle_{L^2}-\langle 0,\hat{v}_0^*\rangle_{L^2}-G(u_0,0),$$

so that

$$\begin{aligned}J_1^*(\hat{v}_2^*,\hat{v}_1^*,\hat{v}_0^*) &= -F_1^*(\hat{v}_2^*,\hat{v}_1^*,\hat{v}_0^*)+F_2^*(\hat{v}_2^*)-G^*(\hat{v}_1^*,\hat{v}_0^*)\\ &= F_1(u_0,\hat{v}_0^*)-F_2(u_0)+G(u_0,0)\\ &= J(u_0). \end{aligned} \tag{27}$$

Joining the pieces, we have got

$$\begin{aligned}J(u_0) &= \inf_{u\in V}\left\{J(u)+\frac{K_1}{2}\int_\Omega(-\gamma\nabla^2 u+2\hat{v}_0^* u-f)^2\,dx\right\}\\ &= \inf_{v_2^*\in Y^*}\left\{\sup_{(v_1^*,v_0^*)\in Y^*\times B^*} J_1^*(v_2^*,v_1^*,v_0^*)\right\}\\ &= J_1^*(\hat{v}_2^*,\hat{v}_1^*,\hat{v}_0^*). \end{aligned} \tag{28}$$

The proof is complete.

□

Remark 5.2. *We could have also defined*

$$B^*=\{v_0^*\in Y^*\ :\ \|v_0^*\|_\infty\leq K/2 \text{ and } -\gamma\nabla^2+2v_0^*>\varepsilon I_d\},$$

for some small real parameter $\varepsilon>0$. In this case, $-\gamma\nabla^2+2v_0^*$ is positive definite, whereas in the previous case, $-\gamma\nabla^2+2v_0^*$ is negative definite.

6 A final convex dual variational formulation applicable to a related model in phase transition

In this section, again let $\Omega\subset\mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \to \mathbb{R}$ (which is related to some models in phase transition of solids) where

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega H_{ijkl}(\nabla u_i \cdot \nabla u_j)(\nabla u_k \cdot \nabla u_l)\,dx \\ &-\frac{1}{2}\int_\Omega C_{ij}\nabla u_i \cdot \nabla u_j\,dx - \langle u_i, f_i\rangle_{L^2}. \end{aligned} \tag{29}$$

Here $\{H_{ijkl}\}$ is a fourth order positive definite constant tensor and $\{C_{ij}\}$ is a second order positive definite constant tensor.

Moreover, $V = W_0^{1,4}(\Omega;\mathbb{R}^3)$, $f \in Ł^2(\Omega;\mathbb{R}^3)$ and we denote $Y = Y^* = L^2(\Omega;\mathbb{R}^3)$.

For fixed $h = (h_1, h_2, h_3) \in L^2(\Omega;\mathbb{R}^3)$ and $\gamma > 0$, define the functionals $F_1 : V \times Y \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$, $G_1 : V \times Y \to \mathbb{R}$ and $G_2 : V \to \mathbb{R}$, by

$$F_1(u, v_0^*) = J(u) + \frac{K_1}{2}\sum_{i=1}^3(-\gamma\nabla^2 u_i + 2(v_0^*)_i u_i - h_i)^2\,dx + \frac{K_2}{2}\sum_{i=1}^3\int_\Omega (u_i)^2\,dx, \tag{30}$$

$$F_2(u) = \frac{K_2}{2}\sum_{i=1}^3\int_\Omega (u_i)^2\,dx$$

$$G_1(u, v_0^*) = \frac{K_1}{2}\sum_{i=1}^3(-\gamma\nabla^2 u_i + 2(v_0^*)_i u_i - h_i)^2\,dx + \frac{K_2}{2}\sum_{i=1}^3\int_\Omega (u_i)^2\,dx,$$

and

$$G_2(u) = \frac{K_2}{2}\sum_{i=1}^3\int_\Omega (u_i)^2\,dx,$$

We define also $F_1^* : [Y^*]^2 \to \mathbb{R}$, $F_2^* : Y^* \to \mathbb{R}$, and $G_1^* : [Y^*]^2 \to \mathbb{R}$, $G_2^* : Y^* \to \mathbb{R}$, by

$$F_1^*(v_1^*, v_0^*, v_3^*) = \sup_{u\in V}\{\langle u_i, (v_1^*)_i + (v_3^*)_i\rangle_{L^2} - F_1(u, v_0^*)\}$$

$$F_2^*(v_1^*) = \sup_{u\in V}\{\langle u_i, (v_1^*)_i\rangle_{L^2} - F_2(u)\}$$

and

$$G_1^*(v_2^*, v_0^*, v_3^*) = \sup_{u\in V}\{\langle u_i, (v_2^*)_i - (v_3^*)_i\rangle_{L^2} - G_1(u, v_0^*)\}$$

$$G_2^*(v_2^*) = \sup_{u\in V}\{\langle u_i, (v_2^*)_i\rangle_{L^2} - G_2(u)\}.$$

Define also

$$D_1^* = \{v_1^* \in Y^* \ : \ \|v_1^*\|_\infty \leq (3/2)K_2,$$

$$D_2^* = \{v_2^* \in Y^* \ : \ \|v_2^*\|_\infty \leq (3/2)K_2\},$$

$$B^* = \{v_0^* \in Y_1^* \ : \ \|v_0^*\|_\infty \leq K_1/8 \text{ and } -\gamma\nabla^2 + 2(v_0^*)_i \leq -\varepsilon I_d, \ \forall i \in \{1,2,3\}\}.$$

and

$$B_1^* = \{v_3^* \in Y^* \ : \ \|v_3^*\|_\infty \leq K_1/8\}.$$

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Furthermore, we define $J_1^*: D_1^* \times D_2^* \times B^* \times B_1^* \to \mathbb{R}$, by

$$J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) = -F_1^*(v_1^*, v_0^*, v_3^*) + F_2^*(v_1^*) - G_1^*(v_2^*, v_0^*, v_3^*) + G_2^*(v_2^*).$$

Assuming

$$K_2 \gg K_1 \gg \max\{\|f\|_\infty, \|h\|_\infty \gamma, 1/\varepsilon^2\}$$

by directly computing $\delta^2 J_1^*(v_2^*, v_1^*, v_0^*, v_3^*)$ we may obtain that for such specified real constants, J_1^* in convex in (v_1, v_2^*) and it is concave in (v_0^*, v_3^*) on $D_1^* \times D_2^* \times B^* \times B_1^*$.

6.1 The main duality principle and a concerning convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

Theorem 6.1. *Let $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) \in D_1^* \times D_2^* \times B^* \times B_1^*$ be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) = \mathbf{0}$$

and $u_0 = \{(u_0)_i\} \in V$ be such that

$$(u_0)_i = \frac{\partial F_2^*(\hat{v}_1^*)}{\partial (v_1^*)_i}$$

where we also assume $(u_0)_i \neq 0$, a.e. in Ω.

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

$$-\gamma \nabla^2 (u_0)_i + 2(\hat{v}_0^*)_i (u_0)_i - h_i = 0, \ \forall i \in \{1, 2, 3\}$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \sum_{i=1}^3 \int_\Omega (-\gamma \nabla^2 u_i + 2(\hat{v}_0^*)_i u_i - h_i)^2 \, dx \right\} \\ &= \inf_{(v_1^*, v_2^*) \in D_1^* \times D_2^*} \left\{ \sup_{(v_0^*, v_3^*) \in B^* \times B_1^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*). \end{aligned} \tag{31}$$

Proof. Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, v_3^*) = \mathbf{0}$ so that, since J_1^* is convex in (v_1^*, v_2^*) and concave in (v_0^*, v_3^*) on $D_1^* \times D_2^* \times B^* \times B_1^*$, from the Min-Max theorem, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \inf_{(v_2^*, v_1^*) \in D_1^* \times D_2} \left\{ \sup_{(v_0^*, v_3^*) \in B^* \times B_1^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_1)_i^*} = \mathbf{0},$$

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and

$$\frac{\partial F_2^*(\hat{v}_1^*)}{\partial (v_1^*)_i} = (u_0)_i$$

we have

$$-\frac{\partial F_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_1^*)_i} + (u_0)_i = \mathbf{0}$$

and

$$(\hat{v}_1)_i^* - K_2(u_0)_i = \mathbf{0}.$$

Observe now that denoting

$$H(v_1^*, v_0^*, v_3^*, , u) = \langle u_i, (v_1)_i^* + (v_3^*)_i \rangle_{L^2} - F_1(u, v_0^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u_i} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) = H(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_1^*)_i} &= \frac{\partial H(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*, \hat{u})}{\partial (v_1^*)_i} \\ &\quad + \frac{\partial H(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*, \hat{u})}{\partial u_j} \frac{\partial \hat{u}_j}{\partial (v_1^*)_i} \\ &= \hat{u}_i. \end{aligned} \tag{32}$$

Summarizing, we have got

$$(u_0)_i = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial (v_1^*)_i} = \hat{u}_i.$$

Similarly denoting

$$H_1(v_2^*, v_0^*, v_3^*, u) = \langle u_i, (v_2^*)_i - (v_3^*)_i \rangle_{L^2} - F_2(u, v_0^*),$$

there exists $\hat{u}_1 \in V$ such that

$$\frac{\partial H_1(\hat{v}_2^*, \hat{v}_0^*, \hat{v}_3^*, \hat{u}_1)}{\partial u_i} = \mathbf{0},$$

and

$$G_1^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*) = H_1(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*, \hat{u}_1),$$

so that

$$\begin{aligned} \frac{\partial G_1^*(\hat{v}_2^*, \hat{v}_3^*, \hat{v}_0^*)}{\partial (v_2^*)_i} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{v}_3^*, \hat{u})}{\partial (v_2^*)_i} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_0^*, \hat{v}_3^*, \hat{u})}{\partial u_j} \frac{\partial \hat{u}_j}{\partial (v_2^*)_i} \\ &= (\hat{u}_1)_i. \end{aligned} \tag{33}$$

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Summarizing, we have got

$$\frac{\partial G_1^*(\hat{v}_2^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_2^*)_i} = (\hat{u}_1)_i.$$

Similarly, we may obtain

$$\frac{\partial F_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_3^*)_i} = (\hat{u})_i = (u_0)_i.$$

and

$$\frac{\partial G_1^*(\hat{v}_2^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_3^*)_i} = -(\hat{u}_1)_i.$$

From this and the variation of J_1^* in v_3^*, we get

$$\frac{\partial F_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_3^*)_i} + \frac{\partial G_1^*(\hat{v}_2^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_3^*)_i} = \mathbf{0}.$$

so that

$$\hat{u}_i - (\hat{u}_1)_i = \mathbf{0}.$$

Summarizing,

$$\hat{u}_i = (\hat{u}_1)_i = (u_0)_i.$$

From such results, we may obtain

$$(v_2)_i^* = K_2 u_0$$

and from this, from $\hat{v}_1^* = K_2 u_0$, and

$$\frac{\partial F_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_3)_i^*} + \frac{\partial G_1^*(\hat{v}_2^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_3^*)_i} = \mathbf{0}$$

we get

$$(\hat{v}_3^*)_i = \mathbf{0}$$

Also, denoting

$$A_i(u_j, \hat{v}_0^*) = -\gamma \nabla^2 u_j + 2(\hat{v}_0^*)_i u_j - h_i,$$

from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*)}{\partial (v_0^*)_i} = \mathbf{0},$$

we get

$$A_i((u_0)_i, (\hat{v}_0^*)_i) 2(u_0)_i + A_i((u_0)_i, (\hat{v}_0^*)_i) 2(u_0)_i = \mathbf{0},$$

so that, since $(u_0)_i \neq 0$, a.e. in Ω, we get

$$A_i((u_0)_i, (\hat{v}_0^*)_i) = \mathbf{0}.$$

Summarizing,

$$-\gamma \nabla^2 (u_0)_i + 2(\hat{v}_0^*)_i (u_0)_i - h_i = 0, \ \forall i \in \{1, 2, 3\}.$$

Finally, from

$$\frac{\partial H(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*, \hat{u})}{\partial u_i} = 0,$$

we get

$$(\hat{v}_1^*)_i + (\hat{v}_3^*)_i - \frac{\partial J(u_0)}{\partial u_i} - K_1(-\gamma\nabla^2 + (\hat{v}_0^*)_i)(-\gamma\nabla^2(u_0)_i + 2(\hat{v}_0^*)_i(u_0)_i - h_i) - K_2(u_0)_i = \mathbf{0}$$

so that

$$\frac{\partial J(u_0)}{\partial u_i} = \mathbf{0}, \ \forall i \in \{1,2,3\},$$

that is,

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$F_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) = \langle u_0, \hat{v}_1^* + \hat{v}_3^* \rangle_{L^2} - F_1(u_0, \hat{v}_0^*),$$

$$F_2^*(\hat{v}_1^*) = \langle u_0, \hat{v}_1^* \rangle_{L^2} - F_2(u_0),$$

$$G_1^*(\hat{v}_2^*, \hat{v}_0^*, \hat{v}_3^*) = -\langle u_0, \hat{v}_2^* - \hat{v}_3^* \rangle_{L^2} + \langle 0, \hat{v}_0^* \rangle_{L^2} - G_1(u_0, \hat{v}_0^*),$$

$$G_2^*(\hat{v}_2^*) = -\langle u_0, \hat{v}_2 \rangle_{L^2} - G_1(u_0),$$

so that

$$\begin{aligned}
& J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) \\
= \ & -F_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{v}_3^*) + F_2^*(\hat{v}_1^*) - G_1^*(\hat{v}_2^*, \hat{v}_0^*) + G_2^*(\hat{v}_2^*) \\
= \ & F_1(u_0, \hat{v}_0^*) - F_2(u_0) + G_1(u_0, \hat{v}_0^*) - G_2(u_0) \\
= \ & J(u_0). \qquad (34)
\end{aligned}$$

Joining the pieces, from the concerning convexity in u_i, we have got

$$\begin{aligned}
J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \sum_{i=1}^3 \int_\Omega (-\gamma\nabla^2 u_i + 2(\hat{v}_0^*)_i u_i - h_i)^2 \, dx \right\} \\
&= \inf_{(v_1^*, v_2^*) \in D_1^* \times D_2^*} \left\{ \sup_{(v_0^*, v_3^*) \in B^* \times B_1^*} J_1^*(v_2^*, v_1^*, v_0^*, v_3^*) \right\} \\
&= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \qquad (35)
\end{aligned}$$

The proof is complete.

□

Remark 6.2. *We could have also defined*

$$B^* = \left\{ v_0^* \in Y^* \ : \ \|2v_0^*\|_\infty < K_1/8 \text{ and } -\gamma\nabla^2 + 2(v_0^*)_i > \varepsilon I_d, \ \forall i \in \{1,2,3\} \right\},$$

for a small parameter $0 < \varepsilon \ll 1$. This corresponds to $-\gamma\nabla^2 + 2(v_0^*)_i$ be positive definite, whereas the previous case corresponds to $-\gamma\nabla^2 + 2(v_0^*)_i$ be negative definite.

Finally, a word of caution. Indeed the global optimal minimum point for the primal formulation may not be attained. Even so, in such a case, considering the Ekeland variational principle, the equations defining the critical points for both the primal and dual formulations may be still approximately satisfied.

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7 A related numerical computation through the generalized method of lines

We start by recalling that the generalized method of lines was originally introduced in the book entitled "Topics on Functional Analysis, Calculus of Variations and Duality" [7], published in 2011.

Indeed, the present results are extensions and applications of previous ones which have been published since 2011, in books and articles such as [7, 8, 9, 5]. About the Sobolev spaces involved we would mention [1]. Concerning the applications, related models in physics are addressed in [4, 11].

We also emphasize that, in such a method, the domain of the partial differential equation in question is discretized in lines (or more generally, in curves) and the concerning solution is written on these lines as functions of boundary conditions and the domain boundary shape.

In fact, in its previous format, this method consists of an application of a kind of a partial finite differences procedure combined with the Banach fixed point theorem to obtain the relation between two adjacent lines (or curves).

In the present article, we propose an improvement concerning the way we truncate the series solution obtained through an application of the Banach fixed point theorem to find the relation between two adjacent lines. The results obtained are very good even as a typical parameter $\varepsilon > 0$ is very small.

In the next lines and sections we develop in details such a numerical procedure.

7.1 About a concerning improvement for the generalized method of lines

Let $\Omega \subset \mathbb{R}^2$ where

$$\Omega = \{(r,\theta) \in \mathbb{R}^2 \;:\; 1 \leq r \leq 2,\; 0 \leq \theta \leq 2\pi\}.$$

Consider the problem of solving the partial differential equation

$$\begin{cases} -\varepsilon\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) + \alpha u^3 - \beta u = f, & \text{in } \Omega, \\ u = u_0(\theta), & \text{on } \partial\Omega_1, \\ u = u_f(\theta), & \text{on } \partial\Omega_2. \end{cases} \tag{36}$$

Here

$$\Omega = \{(r,\theta) \in \mathbb{R}^2 \;:\; 1 \leq r \leq 2,\; 0 \leq \theta \leq 2\pi\},$$

$$\partial\Omega_1 = \{(1,\theta) \in \mathbb{R}^2 \;:\; 0 \leq \theta \leq 2\pi\},$$

$$\partial\Omega_2 = \{(2,\theta) \in \mathbb{R}^2 \;:\; 0 \leq \theta \leq 2\pi\},$$

$\varepsilon > 0,\; \alpha > 0, \beta > 0$, and $f \equiv 1,$ on Ω.

In a partial finite differences scheme, such a system stands for

$$-\varepsilon\left(\frac{u_{n+1} - 2u_n + u_{n-1}}{d^2} + \frac{1}{t_n}\frac{u_n - u_{n-1}}{d} + \frac{1}{t_n^2}\frac{\partial^2 u_n}{\partial \theta^2}\right) + \alpha u_n^3 - \beta u_n = f_n,$$

$\forall n \in \{1, \cdots, N-1\}$, with the boundary conditions

$$u_0 = 0,$$

and

$$u_N = 0.$$

Here N is the number of lines and $d = 1/N$.

In particular, for $n = 1$ we have

$$-\varepsilon\left(\frac{u_2 - 2u_1 + u_0}{d^2} + \frac{1}{t_1}\frac{(u_1 - u_0)}{d} + \frac{1}{t_1^2}\frac{\partial^2 u_1}{\partial\theta^2}\right) + \alpha u_1^3 - \beta u_1 = f_1,$$

so that

$$u_1 = \left(u_2 + u_1 + u_0 + \frac{1}{t_1}(u_1 - u_0)\, d + \frac{1}{t_1^2}\frac{\partial^2 u_1}{\partial\theta^2} d^2 + (-\alpha u_1^3 + \beta u_1 - f_1)\frac{d^2}{\varepsilon}\right)/3.0,$$

We solve this last equation through the Banach fixed point theorem, obtaining u_1 as a function of u_2.

Indeed, we may set

$$u_1^0 = u_2$$

and

$$\begin{aligned} u_1^{k+1} &= \left(u_2 + u_1^k + u_0 + \frac{1}{t_1}(u_1^k - u_0)\, d + \frac{1}{t_1^2}\frac{\partial^2 u_1^k}{\partial\theta^2} d^2\right. \\ &\left. + (-\alpha(u_1^k)^3 + \beta u_1^k - f_1)\frac{d^2}{\varepsilon}\right)/3.0, \end{aligned} \tag{37}$$

$\forall k \in \mathbb{N}$.

Thus, we may obtain

$$u_1 = \lim_{k\to\infty} u_1^k \equiv H_1(u_2, u_0).$$

Similarly, for $n = 2$, we have

$$\begin{aligned} u_2 &= \left(u_3 + u_2 + H_1(u_2, u_0) + \frac{1}{t_1}(u_2 - H_1(u_2, u_0))\, d + \frac{1}{t_1^2}\frac{\partial^2 u_2}{\partial\theta^2} d^2\right. \\ &\left. + (-\alpha u_2^3 + \beta u_2 - f_2)\frac{d^2}{\varepsilon}\right)/3.0, \end{aligned} \tag{38}$$

We solve this last equation through the Banach fixed point theorem, obtaining u_2 as a function of u_3 and u_0.

Indeed, we may set

$$u_2^0 = u_3$$

and

$$\begin{aligned} u_2^{k+1} &= \left(u_3 + u_2^k + H_1(u_2^k, u_0) + \frac{1}{t_2}(u_2^k - H_1(u_2^k, u_0))\, d + \frac{1}{t_2^2}\frac{\partial^2 u_2^k}{\partial\theta^2} d^2\right. \\ &\left. + (-\alpha(u_2^k)^3 + \beta u_2^k - f_2)\frac{d^2}{\varepsilon}\right)/3.0, \end{aligned} \tag{39}$$

$\forall k \in \mathbb{N}$.

Thus, we may obtain

$$u_2 = \lim_{k \to \infty} u_2^k \equiv H_2(u_3, u_0).$$

Now reasoning inductively, having

$$u_{n-1} = H_{n-1}(u_n, u_0),$$

we may get

$$\begin{aligned} u_n &= \left(u_{n+1} + u_n + H_{n-1}(u_n, u_0) + \frac{1}{t_n}(u_n - H_{n-1}(u_n, u_0))\, d + \frac{1}{t_n^2}\frac{\partial^2 u_n}{\partial \theta^2} d^2 \right. \\ &\quad \left. + (-\alpha u_n^3 + \beta u_n - f_n)\frac{d^2}{\varepsilon} \right) /3.0, \end{aligned} \tag{40}$$

We solve this last equation through the Banach fixed point theorem, obtaining u_n as a function of u_{n+1} and u_0.

Indeed, we may set

$$u_n^0 = u_{n+1}$$

and

$$\begin{aligned} u_n^{k+1} &= \left(u_{n+1} + u_n^k + H_{n-1}(u_n^k, u_0) + \frac{1}{t_n}(u_n^k - H_{n-1}(u_n^k, u_0))\, d + \frac{1}{t_n^2}\frac{\partial^2 u_n^k}{\partial \theta^2} d^2 \right. \\ &\quad \left. + (-\alpha (u_n^k)^3 + \beta u_n^k - f_n)\frac{d^2}{\varepsilon} \right) /3.0, \end{aligned} \tag{41}$$

$\forall k \in \mathbb{N}$.

Thus, we may obtain

$$u_n = \lim_{k \to \infty} u_n^k \equiv H_n(u_{n+1}, u_0).$$

We have obtained $u_n = H_n(u_{n+1}, u_0)$, $\forall n \in \{1, \cdots, N-1\}$.

In particular, $u_N = u_f(\theta)$, so that we may obtain

$$u_{N-1} = H_{N-1}(u_N, u_0) = H_{N-1}(0) \equiv F_{N-1}(u_N, u_0) = F_{N-1}(u_f(\theta), u_0(\theta)).$$

Similarly,

$$u_{N-2} = H_{N-2}(u_{N-1}, u_0) = H_{N-2}(H_{N-1}(u_N, u_0)) = F_{N-2}(u_N, u_0) = F_{N-1}(u_f(\theta), u_0(\theta)),$$

an so on, up to obtaining

$$u_1 = H_1(u_2) \equiv F_1(u_N, u_0) = F_1(u_f(\theta), u_0(\theta)).$$

The problem is then approximately solved.

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7.2 Software in Mathematica for solving such an equation

We recall that the equation to be solved is a Ginzburg-Landau type one, where

$$\begin{cases} -\varepsilon\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)+\alpha u^3-\beta u=f, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega_1, \\ u=u_f(\theta), & \text{on } \partial\Omega_2. \end{cases} \tag{42}$$

Here

$$\Omega=\{(r,\theta)\in\mathbb{R}^2\ :\ 1\leq r\leq 2,\ 0\leq\theta\leq 2\pi\},$$

$$\partial\Omega_1=\{(1,\theta)\in\mathbb{R}^2\ :\ 0\leq\theta\leq 2\pi\},$$

$$\partial\Omega_2=\{(2,\theta)\in\mathbb{R}^2\ :\ 0\leq\theta\leq 2\pi\},$$

$\varepsilon>0,\ \alpha>0, \beta>0$, and $f\equiv 1$, on Ω. In a partial finite differences scheme, such a system stands for

$$-\varepsilon\left(\frac{u_{n+1}-2u_n+u_{n-1}}{d^2}+\frac{1}{t_n}\frac{u_n-u_{n-1}}{d}+\frac{1}{t_n^2}\frac{\partial^2 u_n}{\partial\theta^2}\right)+\alpha u_n^3-\beta u_n=f_n,$$

$\forall n\in\{1,\cdots,N-1\}$, with the boundary conditions

$$u_0=0,$$

and

$$u_N=u_f[x].$$

Here N is the number of lines and $d=1/N$.

At this point we present the concerning software for an approximate solution.

Such a software is for $N=10$ (10 lines) and $u_0[x]=0.$.

****

1. $m_8=10;\ (N=10\ lines)$
2. $d=1/m8;$
3. $e_1=0.1;\ (\varepsilon=0.1)$
4. $A=1.0;$
5. $B=1.0;$
6. $For[i=1,i<m8,i++,f[i]=1.0];\ \ (f\equiv 1,\ \text{on }\Omega)$
7. $a=0.0;$
8. $For[i=1,i<m8,i++,$
   $Clear[b,u];$
   $t[i]=1+i*d;$
   $b[x\_]=u[i+1][x];$

9. $For[k = 1, k < 30, k++,$ (we have fixed the number of iterations)
$z = \Big(u[i+1][x] + b[x] + a + \frac{1}{t[i]}(b[x] - a) * d$
$+\frac{1}{t[i]^2} D[b[x], \{x, 2\}] * d^2 + (-A * b[x]^3 + B * u[x] + f[i]) * \frac{d^2}{e_1}\Big) /3.0;$
$z =$
$Series[z, \{u[i+1][x], 0, 3\}, \{u[i+1]'[x], 0, 1\}, \{u[i+1]''[x], 0, 1\},$
$\{u[i+1]'''[x], 0, 0\}, \{u[i+1]''''[x], 0, 0\}];$
$z = Normal[z],$
$z = Expand[z];$
$b[x\_] = z];$
10. $a_1[i] = z;$
11. $Clear[b];$
12. $u[i+1][x\_] = b[x];$
13. $a = a_1[i]\ ];$
14. $b[x\_] = u_f[x];$
15. $For[i = 1, i < m8, i++,$
$A_1 = a_1[m8 - i];$
$A_1 = Series[A_1, \{u_f[x], 0, 3\}, \{u_f'[x], 0, 1\}, \{u_f''[x], 0, 1\}, \{u_f'''[x], 0, 0\}, \{u_f''''[x], 0, 0\}];$
$A_1 = Normal[A_1];$
$A_1 = Expand[A_1];$
$u[m8 - i][x\_] = A_1;$
$b[x\_] = A_1];$
$Print[u[m8/2][x]];$

****

The numerical expressions for the solutions of the concerning $N = 10$ lines are given by

$$\begin{aligned} u[1][x] &= 0.47352 + 0.00691u_f[x] - 0.00459u_f[x]^2 + 0.00265u_f[x]^3 + 0.00039(u_f'')[x] \\ &\quad -0.00058u_f[x](u_f'')[x] + 0.00050u_f[x]^2(u_f'')[x] - 0.000181213u_f[x]^3(u_f'')[x] \end{aligned} \tag{43}$$

$$\begin{aligned} u[2][x] &= 0.76763 + 0.01301u_f[x] - 0.00863u_f[x]^2 + 0.00497u_f[x]^3 + 0.00068(u_f'')[x] \\ &\quad -0.00103u_f[x](u_f'')[x] + 0.00088u_f[x]^2(u_f'')[x] - 0.00034u_f[x]^3(u_f'')[x] \end{aligned} \tag{44}$$

$$\begin{aligned} u[3][x] &= 0.91329 + 0.02034u_f[x] - 0.01342u_f[x]^2 + 0.00768u_f[x]^3 + 0.00095(u_f'')[x] \\ &\quad -0.00144u_f[x](u_f'')[x] + 0.00122u_f[x]^2(u_f'')[x] - 0.00051u_f[x]^3(u_f'')[x] \end{aligned} \tag{45}$$

$$\begin{aligned} u[4][x] &= 0.97125 + 0.03623u_f[x] - 0.02328u_f[x]^2 + 0.01289u_f[x]^3 + 0.00147331(u_f'')[x] \\ &\quad -0.00223u_f[x](u_f'')[x] + 0.00182uf[x]^2(u_f'')[x] - 0.00074u_f[x]^3(u_f'')[x] \end{aligned} \tag{46}$$

$$\begin{aligned} u[5][x] &= 1.01736 + 0.09242u_f[x] - 0.05110u_f[x]^2 + 0.02387u_f[x]^3 + 0.00211(u_f'')[x] \\ &-0.00378u_f[x](u_f'')[x] + 0.00292u_f[x]^2(u_f'')[x] - 0.00132u_f[x]^3(u_f'')[x] \end{aligned} \quad (47)$$

$$\begin{aligned} u[6][x] &= 1.02549 + 0.21039u_f[x] - 0.09374u_f[x]^2 + 0.03422u_f[x]^3 + 0.00147(u_f'')[x] \\ &-0.00634u_f[x](u_f'')[x] + 0.00467u_f[x]^2(u_f'')[x] - 0.00200u_f[x]^3(u_f'')[x] \end{aligned} \quad (48)$$

$$\begin{aligned} u[7][x] &= 0.93854 + 0.36459u_f[x] - 0.14232u_f[x]^2 + 0.04058u_f[x]^3 + 0.00259(u_f'')[x] \\ &-0.00747373u_f[x](u_f'')[x] + 0.0047969u_f[x]^2(u_f'')[x] - 0.00194u_f[x]^3(u_f'')[x] \end{aligned} \quad (49)$$

$$\begin{aligned} u[8][x] &= 0.74649 + 0.57201u_f[x] - 0.17293u_f[x]^2 + 0.02791u_f[x]^3 + 0.00353(u_f'')[x] \\ &-0.00658u_f[x](u_f'')[x] + 0.00407u_f[x]^2(u_f'')[x] - 0.00172u_f[x]^3(u_f'')[x] \end{aligned} \quad (50)$$

$$\begin{aligned} u[9][x] &= 0.43257 + 0.81004u_f[x] - 0.13080u_f[x]^2 + 0.00042u_f[x]^3 + 0.00294(u_f'')[x] \\ &-0.00398u_f[x](u_f'')[x] + 0.00222u_f[x]^2(u_f'')[x] - 0.00066u_f[x]^3(u_f'')[x] \end{aligned} \quad (51)$$

8 Conclusion

In the first part of this article we develop duality principles for non-convex variational optimization. In the final concerning sections we propose dual convex formulations suitable for a large class of models in physics and engineering. In the last article section, we present an advance concerning the computation of a solution for a partial differential equation through the generalized method of lines. In particular, in its previous versions, we used to truncate the series in d^2 however, we have realized the results are much better by taking line solutions in series for $u_f[x]$ and its derivatives, as it is indicated in the present software.

This is a little difference concerning the previous procedure, but with a great result improvement as the parameter $\varepsilon > 0$ is small.

Indeed, with a sufficiently large N (number of lines), we may obtain very good results even as $\varepsilon > 0$ is very small.

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