
Algebraic Andô Dilation

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Abstract: We solve the Andô dilation problem for linear maps on vector space asked by Krishna and Johnson in [Oper. Matrices, 2022]. More precisely, we show that any commuting linear maps on vector space can be dilated to commuting injective linear maps.

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1. INTRODUCTION

After a decade of work of Sz.-Nagy [13, 14], Andô [1] made a breakthrough result in the dilation theory of contractions on Hilbert space which states as follows.

Theorem 1.1. [1] (*Andô Dilation*) *Let \mathcal{H} be a Hilbert space and $T, S : \mathcal{H} \rightarrow \mathcal{H}$ be commuting contractions. Then there exists a Hilbert space \mathcal{K} which contains \mathcal{H} isometrically and a pair of commuting unitaries $U, V : \mathcal{K} \rightarrow \mathcal{K}$ such that*

$$T^n S^m = P_{\mathcal{H}} U^n V^m|_{\mathcal{H}}, \quad \forall n, m \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\},$$

where $P_{\mathcal{H}} : \mathcal{K} \rightarrow \mathcal{H}$ is the orthogonal projection onto \mathcal{H} .

After the work of Andô, Parrott [10] showed that it is not possible to improve Theorem 1.1 for more than two commuting contractions. Later, Andô dilation is derived for commuting contractions on Banach spaces [12]. In 2021 (arXiv version), in the paper [7], while continuing the work of Bhat, De and Rakshit [2] on dilations of linear maps on vector spaces, Krishna and Johnson [7] asked following problem.

Question 1.2. [7] *Whether there is an Andô dilation for linear maps on vector spaces? More precisely, whether commuting linear maps on vector space can be dilated to commuting bijective linear maps?*

In this paper, we solve Question 1.2 partially by showing that we can go upto injective linear maps.

2. ALGEBRAIC ANDÔ DILATION

We first give a different proof Theorem 2.1 than given in [2] which helps us to give a proof of algebraic version of Andô dilation.

Theorem 2.1. [2] (*Algebraic Sz.-Nagy Dilation or Bhat-De-Rakshit Dilation*) *Let \mathcal{V} be a vector space and $T : \mathcal{V} \rightarrow \mathcal{V}$ be a linear map. Then there is a vector space \mathcal{W} containing \mathcal{V} through a natural coordinate*

injective map and an injective linear map $U : \mathcal{W} \rightarrow \mathcal{W}$ such that

$$(Dilation\ equation) \quad T^n = P_{\mathcal{V}} U^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{Z}_+,$$

where $P_{\mathcal{V}} : \mathcal{W} \rightarrow \mathcal{V}$ is a coordinate projection (idempotent) onto \mathcal{V} .

Proof. Our construction is motivated from the construction of Sz.-Nagy dilation of a contraction on a Hilbert space given in Chapter 1 of [14]. Given a vector space \mathcal{V} , let $I_{\mathcal{V}}$ be the identity operator on \mathcal{V} and $\oplus_{n=0}^{\infty} \mathcal{V}$ be the vector space defined by

$$\oplus_{n=0}^{\infty} \mathcal{V} := \{(x_n)_{n=0}^{\infty}, x_n \in \mathcal{V}, \forall n \in \mathbb{Z}_+, x_n \neq 0 \text{ only for finitely many } n's\}.$$

Let $T : \mathcal{V} \rightarrow \mathcal{V}$ be a linear map. Define $\mathcal{W} := \oplus_{n=0}^{\infty} \mathcal{V}$ and

$$\begin{aligned} I : \mathcal{V} \ni x &\mapsto (x, 0, \dots) \in \mathcal{W}, \\ U : \mathcal{W} \ni (x_n)_{n=0}^{\infty} &\mapsto (Tx_0, (I_{\mathcal{V}} - T)x_0, x_1, x_2, \dots) \in \mathcal{W}, \\ P : \mathcal{W} \ni (x_n)_{n=0}^{\infty} &\mapsto x_0 \in \mathcal{V}. \end{aligned}$$

Then clearly the dilation equation is satisfied. The proof is complete if we show that U is injective. Let $(x_n)_{n=0}^{\infty} \in \mathcal{W}$ be such that $U(x_n)_{n=0}^{\infty} = 0$. then

$$(Tx_0, (I_{\mathcal{V}} - T)x_0, x_1, x_2, \dots) = (0, 0, 0, \dots).$$

We then have $x_1 = x_2 = \dots = 0$ and $Tx_0 = (I_{\mathcal{V}} - T)x_0 = 0$. Rewriting

$$x_0 = Tx_0 = 0.$$

Thus (\mathcal{W}, U) is an injective linear dilation of T . □

Following is the most important result of this paper which we call algebraic Andô dilation.

Theorem 2.2. (Algebraic Andô Dilation) *Let \mathcal{V} be a vector space and $T, S : \mathcal{V} \rightarrow \mathcal{V}$ be commuting linear maps. Then (T, S) admits an injective linear dilation.*

Proof. Our arguments are motivated from original argument for Andô dilation for commuting contractions on Hilbert spaces by Andô [1]. Define $\mathcal{W} := \oplus_{n=0}^{\infty} \mathcal{V}$ and

$$\begin{aligned} W_1 : \mathcal{W} \ni (x_n)_{n=0}^{\infty} &\mapsto (Tx_0, (I_{\mathcal{V}} - T)x_0, 0, x_1, x_2, \dots) \in \mathcal{W}, \\ W_2 : \mathcal{W} \ni (x_n)_{n=0}^{\infty} &\mapsto (Sx_0, (I_{\mathcal{V}} - S)x_0, 0, x_1, x_2, \dots) \in \mathcal{W}, \\ P : \mathcal{W} \ni (x_n)_{n=0}^{\infty} &\mapsto x_0 \in \mathcal{V}. \end{aligned}$$

Let $x \in \mathcal{V}$ be such that $(I_{\mathcal{V}} - T)Sx = 0 = (I_{\mathcal{V}} - S)x$. Then

$$(I_{\mathcal{V}} - S)Tx = Tx - STx = Tx - TSx = T(I_{\mathcal{V}} - S)x = T0 = 0$$

and

$$(I_{\mathcal{V}} - T)x = x - Tx = x - T(Sx) = Sx - TSx = (I_{\mathcal{V}} - T)Sx = 0.$$

This observation says that the map

$$v : \{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\} \rightarrow \{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}$$

defined by

$$v((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) := ((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0)$$

is a well-defined injective linear map. Clearly v is surjective. We now claim that v can be extended as a bijective linear map (which we again denote by v) from $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ to $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. We get two cases.

Case (i): $\dim(\mathcal{V}) < \infty$.

Let \mathcal{Y} be any vector space complement of $\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ and \mathcal{Z} be any vector space complement of $\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. From the dimension formula for vector spaces, we then get

$$\begin{aligned} \dim(\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}) &= \dim(\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}) + \dim(\mathcal{Y}) \\ &= \dim(\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}) + \dim(\mathcal{Z}) \end{aligned}$$

Since $\dim(\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}) = \dim(\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\})$,

$$\dim(\mathcal{Y}) = \dim(\mathcal{Z}).$$

Thus v can be extended bijectively and linearly from $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ to $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$.

Case (i): $\dim(\mathcal{V}) = \infty$.

Let \mathcal{Y} be any vector space complement of $\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}$ containing the space $\{(0, x, 0, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ and \mathcal{Z} be any vector space complement of $\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}$ containing the space $\{(0, x, 0, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. Then

$$\dim(\mathcal{V}) = \dim(\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}) \geq \dim(\mathcal{Y}) \geq \dim(\mathcal{V})$$

and

$$\dim(\mathcal{V}) = \dim(\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}) \geq \dim(\mathcal{Z}) \geq \dim(\mathcal{V}).$$

Therefore $\dim(\mathcal{Y}) = \dim(\mathcal{Z})$ and hence v can be extended bijectively and linearly from $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ to $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$.

Define $\mathcal{V}^{(4)} := \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. We identify \mathcal{W} and $\mathcal{V} \oplus (\oplus_{n=1}^{\infty} \mathcal{V}^{(4)})$ by the map

$$(x_n)_{n=0}^{\infty} \mapsto (x_0, (x_1, x_2, x_3, x_4), (x_5, x_6, x_7, x_8), \dots)$$

Now we define $W : \mathcal{W} \rightarrow \mathcal{W}$ by

$$W(x_n)_{n=0}^{\infty} := (x_0, v(x_1, x_2, x_3, x_4), v(x_5, x_6, x_7, x_8), \dots)$$

which becomes bijective linear map with inverse

$$W^{-1}(x_n)_{n=0}^{\infty} := (x_0, v^{-1}(x_1, x_2, x_3, x_4), v^{-1}(x_5, x_6, x_7, x_8), \dots)$$

We finally define $U := WW_1$, $V := W_2W^{-1}$ and show that $(\mathcal{W}, (U, V))$ is the required injective linear dilation of (T, S) . Clearly U and V are injective. By induction, we also have the multivariate dilation equation

$$T^n S^m x = P_{\mathcal{V}} U^n V^m x, \quad \forall n, m \in \mathbb{Z}_+, \forall x \in \mathcal{V}.$$

Now we are left only with proving that U and V commute. Let $(x_n)_{n=0}^\infty \in \mathcal{W}$. Then

$$\begin{aligned}
 UV(x_n)_{n=0}^\infty &= WW_1W_2W^{-1}(x_n)_{n=0}^\infty \\
 &= WW_1W_2(x_0, v^{-1}(x_1, x_2, x_3, x_4), v^{-1}(x_5, x_6, x_7, x_8), \dots) \\
 &= WW_1(Sx_0, (I_V - S)x_0, 0, v^{-1}(x_1, x_2, x_3, x_4), v^{-1}(x_5, x_6, x_7, x_8), \dots) \\
 &= W(TSx_0, (I_V - T)Sx_0, 0, (I_V - S)x_0, 0, v^{-1}(x_1, x_2, x_3, x_4), v^{-1}(x_5, x_6, x_7, x_8), \dots) \\
 &= (TSx_0, v((I_V - T)Sx_0, 0, (I_V - S)x_0, 0), (x_1, x_2, x_3, x_4), (x_5, x_6, x_7, x_8), \dots) \\
 &= (STx_0, (I_V - S)Tx_0, 0, (I_V - T)x_0, 0), (x_1, x_2, x_3, x_4), (x_5, x_6, x_7, x_8), \dots)
 \end{aligned}$$

and

$$\begin{aligned}
 VU(x_n)_{n=0}^\infty &= W_2W^{-1}WW_1(x_n)_{n=0}^\infty = W_2W_1(x_n)_{n=0}^\infty \\
 &= W_2(Tx_0, (I_V - T)x_0, 0, x_1, x_2, \dots) \\
 &= (STx_0, (I_V - S)Tx_0, 0, (I_V - T)x_0, 0), x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \dots.
 \end{aligned}$$

Therefore $VU = UV$. □

Theorem 2.2 and the works presented in [3, 4, 8, 9, 11] gives the following problem.

- Question 2.3.** (i) *Whether there is an explicit (matrix) construction of algebraic Andô dilation?*
(ii) *Whether there is a Halmos dilation for commuting linear maps on vector spaces?*
(iii) *Whether there is an Egerváry N -dilation for commuting linear maps on vector spaces?*
(iv) *Does Theorem 2.2 holds for more than two commuting linear maps?*
(v) *Can the dilated injective linear maps U, V in Theorem 2.2 be improved to bijective linear maps?*

Remark 2.4. *Andô dilation for p -adic magic contractions and self-adjoint morphisms on indefinite inner product modules over $*$ -rings of characteristic 2 are still open [5, 6].*

3. CONCLUSIONS

- (1) In 1950, Halmos showed that every contraction on a Hilbert space can be lifted to a unitary [4].
- (2) In 1953, Sz.-Nagy derived his dilation theorem [13].
- (3) In 1955, Schaffer gave simple proof of Sz.-Nagy dilation result [11].
- (4) In 1963, Andô showed that Sz.-Nagy dilation holds for two commuting contractions [1].
- (5) In 1973, Stroescu derived Andô dilation for contractions on Banach spaces [12].
- (6) In 2021, Bhat, De and Rakshit introduced set theoretic and vector space approach to dilation theory [2]. Later, Krishna and Johnson continued this study in 2022 [7].
- (7) In this paper, we derived Andô dilation for linear maps on vector spaces.

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