 doi:10.20944/preprints202210.0024.v1

Structure-Decidable Structure-Examples of Decidable Structures With Proof-Some Examples Of Undecidable Structures

Zahra Sheikhaleslami
Zahra Sheikhaleslami,Ph.D. Student of Tabriz University. The University of Tabriz P.O.Box 51666-16471,Bahman 29thBoulevard; Tabriz; Iran.
zahra.sheikhaleslami@gmail.com

Abstract—In this paper, decidability of the structures of genuine, rational, integer and normal numbers will be examined in several languages. Decidability or undecidability of mathematical structures is one of the fundamental and sometimes very difficult problems of mathematical logic, where several examples of problems in this field are still open and unresolved even after decades. One of the goals of Mathematical Logic is the axiomatization of mathematical theoriesTarski has proved the decidability of the theory of real and complex numbers in the language of addition and multiplication, and it is proved that theories of natural, integer, and rational numbers, in the language of addition and multiplication, are undecidable (Theorems of Gödel and Robinson).

We will proof The following problemes:
The Main Problem 1: $\langle Q; \sqsubseteq\rangle$ is decidable?
Problem 2: an explicit axiomatization for $\langle Z; \times\rangle$?
and we will study boolean algebras. Boolean algebras are famous mathematical structures.Tarski showed the decidability of the elementary theory of Boolean algebras.n this paper, we conside the different kinds of Boolean algebras and their properties. And we present for the first-order theory of atomic Boolean algebras a quantifier elimination algorithm. The subset relation is a partial order and indeed a lattice order,And I will prove that the theory of atomic Boolean lattice orders is decidable, and furthermore admits elimination of quantifiers. So the theory of the subset relation is decidable. And we will study decidability of atomlss boolean algebra.

Keywords—Boolean algebras, Decidability,Model Theory, Quantifier-Elimination.

I. Introduction

Quantifier Elimination and Decidability:

Decision Procedures : The purpose is to produce an algorithm for determining whether or not a formula is valid . So, decision procedure is an algorithm that, given a decision problem, terminates with a yes or no answer.

Quantifier elimination: We say that a theory T has quantifier elimination if for every formula ϕ there is a quantifier-free formula ψ such that $\varphi \leftrightarrow \psi$.

Decidability: A class of questions is decidable if and only if there is a procedure such that, when given as input any question in the class, the procedure halts and says yes if the answer is positive and no if the answer is negative.
Example: For any natural number n, determining whether n is prime.

The mathematical structure consists of a specific set (usually a set of numbers, like natural, integer, rational, real or complex numbers) in a first-order language that contains some functions, predicates or constants. The theory of a structure is the set of all first-order sentences (in the language of that structure) which are true in that structure.

$$Structure : \mathcal{A} = \langle \mathbf{A}; \mathcal{L}\rangle \qquad \mathrm{Th}(\mathcal{A}) = \{\theta \in \mathcal{L} \mid \mathcal{A} \vdash \theta\}$$

For example, the sentence "any number is equal to the sum of another number with itself" is false in (the domain of) integer numbers, but it is true in (the domain of) rational numbers (e.g., 3 there are no integer n such that $n + n = 3$, but the sum of 3/2 with itself is 3).

$$\langle Z; +\rangle \nvdash \forall x\exists y(x = y + y) \qquad \langle Q; +\rangle \vdash \forall x\exists y(x = y + y)$$

Decidable: A theory T is decidable if there exists an effective procedure to determine whether $T \vdash \phi$ where ϕ is any sentence of the language.
Completenes: completeness of a theory T means that for any sentence ϕ in the language of the theory, we have either $T \vdash \phi$ or $T \vdash \neg\phi$. If this property does not hold (so if there is some ϕ that the theory says nothing about), we have an incomplete theory.

II. Structure-decidable structure-examples of decidable structures with proof-some examples of undecidable structures

lamma1.(The Lemma of Quantifier Elimination) A theory (or a structure) admits quantifier elimination if and

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

only if every formula of the form $\exists x(\wedge_i \alpha_i)$ is (recursively) equivalent with a quantifier -free formula , where each α_i is either an atomic formula or the negation of an atomic formula.

Proof:
Every formula ψ can be written (equivalency) in the prenex normal form, say

$$Q_1x_1Q_2x_2\cdots Q_nx_n\theta(x_1,x_2,\cdots,x_n)$$

where $Q_i^{,}s$ ara quantifiers and θ is quantifier-free. if $Q_n=\exists$, then let $\acute{\theta}=\theta$, and if $Q_n=\forall$, then $\acute{\theta}=\neg\theta$, (note that in the latter case $\forall x_n\equiv\neg\exists x_n\acute{\theta}$). Now, the quantifier- free formula $\acute{\theta}$ can be written in the disjunctive normal form,say $\bigvee_i\bigwedge_j\alpha_{i,j}$ where each $\alpha_{i,j}$ is a literal(i.e., an atomic or a negated atomic formula).Noting that $\exists x(\bigwedge_i\beta_i\equiv\bigvee_i\exists x\beta_i)$ we have

$$\psi\equiv Q_1x_1Q_2x_2\cdots Q_{n-1}x_{n-1}\bigcirc\bigvee\exists x_n(\bigwedge_j\alpha_{i,j})$$

where $\bigcirc$ is nothing (empty) when $Q_n=\exists$ and $\bigcirc=\neg$ when $Q_n=\forall$. Now,if $\exists x_n(\bigwedge_j\alpha_{i,j})$ is equivalent with a quantifier -free formula, ψ is equivalent with a formula with one less quantifier; counting this way one can show that ψ equivalent with a formula which has no quantifier.[1]

A. Decidability of Structure of Natural Numbers in Different Languages

Theorem1. the theory $Th\mathcal{N}_s$ where $\mathcal{N}_s=(N,0,s)$ admits elimination of quantifier.

Proof: [3].

Theorem2. The Theory $\mathcal{N}_L=(N,0,S,<)$ admits elimination quantifier, and so has a decidable theory and is finitely axiomatizable.

Proof: [3].

The additive theory of natural numbers:
Presburger proof decidability of the theory $\langle N;=,+\rangle$ with quantifier elimination. One common way of quantifier elimination is to extend the language, and we add Fixed symbols 0 and 1 and an infinite set of binary relations $<_n$ for $n\geq 1$. Which is defined as follows:

$$\forall x,y\in N, x<_n y\leftrightarrow(x<y\ \ \&\ \ x\equiv y\ \ (modn))$$

Theorem3. The following axioms at the Language $L=\{+,0,1,\leq,\{\equiv_m\}_{\geq 2}\}$, for the structure N allow quantifier elimination.

$(A1)$ $x+(y+z)=(x+y)+z$

$(A2)$ $x+y=y+x$

$(A3)$ $x+0=x$

$(A4)$ $x+z=y+z\rightarrow x=y$

$(A5)$ $x+y=0\rightarrow x=y=0$

$(O1)$ $x\leq y\leftrightarrow\exists z(x+z=y)$

$(O2)$ $x\leq y\vee y\leq x$

$(O3)$ $0\neq 1\wedge\forall y(0\leq y\leq 1\rightarrow y=0\vee y=1)$

$(D1)$ $\forall x\exists y,z(x=n\cdot y+t\wedge t<\bar{n})$

Proof:
Step 1: Identify the terms

In structure $\langle N;+,0,<,1,\{\equiv_n)\}_{n\geq 2}\rangle$, every term involving x is equal to,

$$n\cdot x+t\qquad(n\in N)$$

where x does not appear in t
Step 2: Identify Atomic Formulas and Delete $\neg$ if possible

All atomic formulas are,

$$u\leq v$$
$$u\equiv_k v$$

First, we eliminate the inequality behind the atoms. Because,

$x=y\leftrightarrow x\leq y\wedge y\leq x$
$x\neq y\leftrightarrow x+1\leq y\vee y+1\leq x$
$x\nleq y\leftrightarrow y+1\leq x$
$x\not\equiv_n y\leftrightarrow\bigvee_{0<i<n}x+i\equiv_n y$

So, the following formula admits quantifier elimination.

$$\exists x(\bigwedge_i n_i\cdot x+t_i\leq m_i\cdot x+s_i\wedge\bigwedge_j k_j\cdot x+u_j\equiv_{q_j}l_j\cdot x+v_j)$$

Step 3: Simplify atomic formulas

So the following formula must be eliminated quantifier.

$$\exists x(\bigwedge_i r_i\leq m_i\cdot x+s_i\wedge\bigwedge_j n_j\cdot x+t_j\leq u_j\wedge\bigwedge_l k_l+v_l\equiv_{q_l}w_l)$$

Step 4:Uniform the coefficients x
Let M is Multiply the coefficients by x

$M=\prod_i m_i\prod_j n_j\prod_l k_l$
$r_i\frac{M}{m_i}\leq Mx+\frac{M}{m_i}s_i$
$Mx+\frac{M}{n_j}t_j\leq\frac{M}{n_j}u_j$
$Mx+\frac{M}{k_l}v_l\equiv_{\frac{M}{k_l}q_l}\frac{M}{k_l}w_l$

So the following formula admits quantifier elimination

$$\exists x(\bigwedge_i r_i^{'}\leq Mx+s_i^{'}\wedge\bigwedge_j Mx+t_j^{'}\leq u_j^{'}\wedge\bigwedge_l Mx+v_l^{'}\equiv_{q_l}w_l^{'})$$

Step 5: Remove the coefficient x
$y=Mx$. So, we have

$$\exists y(\bigwedge_i r_i^{'}\leq y+s_i^{'}\wedge\bigwedge_j y+t_j^{'}\leq u_j^{'}\wedge\bigwedge_l y+v_l^{'}\equiv_{q_l}w_l^{'}\wedge y\equiv_M 0)$$

We use the following equations

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

$$t = s \leftrightarrow ct = cs$$
$$t < s \leftrightarrow ct < cs$$
$$t \equiv_m s \leftrightarrow ct \equiv_{cm} cs$$

so

$\exists x(\bigwedge_i r_i \leq x + s_i \wedge \bigwedge_j x + t_j \leq u_j \wedge \bigwedge_l x + v_l \equiv_{q_l} w_l)$

Step 6: Identification Phrases included x

$r_i \leq x + s_i \leftrightarrow r_i + t_j + v_l \leq x + s_i + t_j + v_l$
$x + t_j \leq u_j \leftrightarrow x + s_i + t_j + v_l \leq u_j + s_i + v_l$
$x + v_l \equiv_{q_l} w_l \leftrightarrow x + s_i + t_j + v_l \equiv_{q_l} s_i + t_j + w_l$
$P = s_i + t_j + v_l$

so

$\exists x(\bigwedge_i r_i^{'} \leq x + P \wedge \bigwedge_j x + P \leq u_j^{'} \wedge \bigwedge_l x + P \equiv_{q_l} w_l^{'})$

we put $y = x + P$

$\exists y(\bigwedge_i r_i^{'} \leq y \wedge \bigwedge_j y \leq u_j^{'} \wedge \bigwedge_l y \equiv_{q_l} w_l^{'} \wedge y \geq P)$

Therefore, it is enough to delete the quantifier in the following formula:

$\exists x(\bigwedge_{i=1}^m r_i \leq x \wedge \bigwedge_{j=1}^n x \leq u_j \wedge \bigwedge_{l=1}^k x \equiv_{q_l} w_l)$

Step 7: Reduce Boolen Combination

A: Reduce the order

$\exists x(r_0 \leq x \wedge r_1 \leq x \wedge \theta(x)) \equiv [r_0 \leq r_1 \wedge \exists x(r_1 \leq x \wedge \theta(x))] \vee [r_1 \leq r_0 \wedge \exists x(r_0 \leq x \wedge (r_0 \leq x \wedge \theta(x))]$

B: $\exists x(x \leq u_0 \wedge x \leq u_1 \wedge \theta(x)) \equiv [u_0 \leq u_1 \wedge \exists x(x \leq u_0 \wedge \theta(x))] \vee [u_1 \leq u_0 \wedge \exists x(x \leq u_1 \wedge \theta(x))]$

C: $\exists x(x \equiv_{q_0} w_0 \wedge x \equiv_{q_1} w_1 \wedge \theta(x)) \equiv \exists x(x \equiv_{q_0 \sqcup q_1} x_0 \wedge \theta(x))$

Step 8: Identify the states

$\exists x(r \leq x \wedge x \leq u \wedge x \equiv_q w) \equiv \bigvee_{i=0}^{q-1}(r + \bar{i} \leq u \wedge r + \bar{i} \equiv_q w)$
$\exists x(r \leq x \wedge x \leq u) \equiv r \leq u$
$\exists x(r \leq x \wedge x \equiv_q w) \equiv true$
$\exists x(x \leq u \wedge x \equiv_q w) \equiv \bigvee_{i=0}^{q-1}(\bar{i} \leq u \wedge \bar{i} \equiv_q w)$
$\exists x(r \leq x) \equiv true$
$\exists x(x \leq u) \equiv true$
$\exists x(x \equiv_q w) \equiv true$
$\exists x(\quad) \equiv true$

■

Introduction to Decidability of the Multiplication Theory of:
Natural Numbers

Skolem arithmetic : The theory of the structure $(N, \times)$ is decidable.

Mostowski deals with the notion of direct product in the theory of decision problems. This was well-known to Mostowski, who was able to prove decidability of Skolem Arithmetic through seeing it as a certain weak direct product of Presburger Arithmetic. Such that; This was well-known to Mostowski, who was able to prove decidability of Skolem Arithmetic through seeing it as a certain weak direct product of Presburger Arithmetic.

let L be a language with only constant 0. Let $(Q_i)_{i \in I}$ be non -empty family of L -structures such that for any $i \in I$ and any functional symbol F of L, we have: $F(0, \cdots, 0) = 0$. The direct sum of this family is the L -structure $\mathcal{A}$, denoted by $\bigoplus_{i \in I} Q_i$, that defined by:
$B = \{f \in \prod_{i \in I} A_i / f(i) = 0$ except for atmost finite number of $i\}$

For R is n -ary predicate of L: $R^{\mathcal{A}}(f_1, \cdots, f_n)$ iff for all i of I we have $R^{Q_i}(f_1(i), \cdots, f_n(i))$
For F is n -ary function symbol of L: $F^{\mathcal{A}}(f_1, \cdots, f_n) = F^{Q_i}(f_1(i), \cdots, f_n(i))$
and $\bigoplus_{i \in I} Q_i$ is an L -structure, and clsed for functions is provided by the conditions of the family of L -structures. If I is finite, the direct sum is the same as the direct product.We have $(\mathbf{N}^+, \cdot) = \bigoplus_{n \in N} Q_n$ where $Q_n = (\mathbf{N}, +)$ for any n. And we have $(\mathbf{N}^{>0}, |) = \bigoplus_{n \in N} Q_n$ where $Q_n = (\mathbf{N}, \leq)$ for any n.

p -adic numbers were first described by Kurt Hensel in 1897 though, with hindsight, some of Ernst Kummer's earlier work can be interpreted as implicitly using p -adic numbers.The p -adic numbers were motivated primarily by an attempt to bring the ideas and techniques of power series methods into number theory.

- p -adic number: p -adic number is sums of the form: $\sum_{i=k}^{\infty} a_i p^i$ where k is some (not necessarily positive) integer, and each coefficient a_i p -adic digit.and $0 \leq a_i \leq p - 1$
- Fundamental theorem of arithmetic
  every integer greater than 1 can be represented as the product of prime numbers and, moreover, this representation is unique.
- Euclid's theorem
  Euclid's theorem is a fundamental statement in number theory that asserts that there are infinitely many prime numbers. It was first proved by Euclid .
- p -adic valuation:We define p -adic valuation of x with V[1].If
  $$x = p_1^n p_2^m p_3^0 \cdots$$
  $$V(p_2, x) = p_2^m$$
- p is a prime number , and denoted by $\mathbf{P}(p)$ iff we have: $p \neq 1 \wedge \forall x(x \mid p \to (x = 1 \vee x = p))$
- p -primary number: x is a p -primary number, and denoted by $PR(p, x)$ iff we have $\mathbf{P}(p) \wedge \forall q((\mathbf{P}(q) \wedge q \neq p) \to q \mid x)$
- trunction:
  $$\forall x \forall y \exists z \forall p(P(p) \to (p \mid x \to V(p, z) = V(p, y)) \wedge (px \to V(p, z) = 1)))$$
  This z is unique, and denoted by $T(x, y) = \prod_{p|x} p^{\alpha}$. so we have :
  $$x = 2^u \cdot 3^v \cdot 5^w \cdot \cdots$$
  $$y = 2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma} \cdot \cdots$$

[1]If $n = \prod_p p^{v(n,p)}$ but we may not define, $v(n, p)$ in the theory. Hence $n = \prod_p V(n, p)$ (meaning $V(n, p) = p^{v(n,p)}$). For little v we have $v(p, x.y) = v(p, x) + v(p, y)$ But for big V we have $V(p, x.y) = V(p, x).V(p, y)$

 doi:10.20944/preprints202210.0024.v1

$$T(x,y) = \begin{cases} 2^{\alpha}\cdot 3^{\beta}\cdot 5^{\gamma} & u\neq 0, v\neq 0, w\neq 0 \\ 2^{0}\cdot 3^{0}\cdot 5^{0} & u=0, v=0, w=0 \end{cases}$$

$$T(2^1\cdot 3^0\cdot 5^2, 2^2\cdot 3^2\cdot 5^1) = 2^2\cdot 5^1$$

- Increment:
  $\forall x\exists y\forall p\ (P(p)\rightarrow (p\not| x\rightarrow V(p,y)=1)\wedge(p\mid x\rightarrow V(p,y)=p\cdot V(p,x))))$
- Separation:
  $\forall x\forall y\exists z\forall p((P(p)\rightarrow (p\mid x\cdot y\wedge V(p,x)\equiv_n V(p,y))\rightarrow V(p,z)=p)\wedge(p\not| x\cdot y\vee V(p,x)\not\equiv_n V(p,y)\rightarrow V(p,z)=1)\ \ n\in N$
  this z is unique, and denoted by $SP_n(x,y) = \prod_{(p^{\alpha}\equiv p^{\beta})} p, p\mid x\cdot y$ so we have
  $x=2^3\times 3^4 \& y=2^5\times 3^9\times 5^2$
  $SP_2(x,y)=2\times 5$
  $p=2,\ \ 2\mid 5-3$
  $p=3,\ \ 2\not| 9-4$
  $p=5,\ \ 2\mid 2-0$
- Divisability: $\forall x\exists y\exists z(x=y^n\cdot z\wedge\forall\acute{y}\forall\acute{z}(x=\acute{y}^n\cdot\acute{z}\rightarrow z\mid\acute{z}))$

  $360=2^3\times 3^2\times 5^1$

  $1,4,9,16,25,\cdots,x^2$

  $1,8,27,64,125,\cdots,x^3$

  $1^n,2^n,3^n,\cdots$
  $360=2^2\times 3^2\times 2^1\times 5^1$
  $=(2\times 3)^2\times 2\times 5=36\times 10$
  $360=8\times 45$
  $x=y^n\cdot z\rightarrow 360=6^2\times 10$

We must too begin that quantifiers can be eliminated.For any model $\mathcal{A}$ of the given axioms , and any prime p of the model , one can characterize

$$A_p=\{x\in A: x\, is\, p-primary\}$$

and consider the structure,

$$\mathcal{A}_p=(A_p,\cdot,1)$$

The axiomes given for $Th(\mathbf{N}^{>0};\cdot,1)$ guarantee each $\mathcal{A}_p$ to be a model of $Th(\mathbf{N};+,0)$. In fact, for the model $(\mathbf{N};\cdot,1)$, each $\mathcal{A}_p$ is isomorphic to $(\mathbf{N};+,0)$. $\mathcal{A}_p$ is a model of the theory of addition, and A_p is definable in $\mathcal{A}$ in terms of the parameter p:

$$v_0\in A_p; PR(\bar{p},v_0)$$

For any formula $\phi v_1\cdots v_n$ of the language of $\mathcal{A}$, we can find a formul ϕ^p such that , for all $x_0,\cdots,x_{n-1}\in A$,

$$\mathcal{A}\models\phi^p(\bar{x}_0,\cdots,\bar{x}_{n-1})\Leftrightarrow\mathcal{A}_p\models\phi(\bar{y}_0,\cdots,\bar{y}_{n-1})$$

where $y_i=V(p,x_i)$. To define ϕ^p, first relativise ϕ to A_p and then replace each free variable v_i in ϕ by $V(\bar{p},v_i)$. The construction of ϕ^p is uniform in the constant $\bar{p}$, i.e.,for each ϕ, there is a single formula $\phi^{v_0}v_1\cdots v_n$ from which each ϕ^p is obtained by substituting the constant $\bar{p}$ for the variable v_0.

so we use the additive notation (means $0,1,+,\leq,S,\cdots$) or use the multiplicative notion (means $1,p,+,\cdot,|,I,\cdots$) for the elements of A_p The results are the same.
Consider the additive formula

$$\bar{y_0}<\bar{y_1}, y_i=V(p,x_i)$$
$$\mathcal{A}_p\models\bar{y_0}<\bar{y_1}\leftrightarrow\mathcal{A}\models V(\bar{p},\bar{x_0})\mid V(\bar{p},\bar{x_1})$$
$$\leftrightarrow\mathcal{A}\models V(\bar{p},SP_1(\bar{x_0},\bar{x_1}))=\bar{p}$$

Mostowski observed was that $Th(\mathbf{N}^{>0};\cdot,1)$ is the weak direct power of $Th(\mathbf{N};+,0)$, The relevant theorem of Mostowski :
Let T be a decidable theory with a unique distinguished costant.The theory of weak direct powers of models of T is decidable.[10, Theorem 5.2.]
So, by The Feferman-Vaught Theorem every formula of the language $(\cdot,1)$ is equivalent to a propositional combination of formula of the form,

$$\exists p_1\cdots\exists p_k(\bigwedge_{1\leq i\leq j\leq k}p_i\neq p_j\wedge\bigwedge_{1\leq i\leq k}\mathbf{P}(p_i)\wedge\theta^{p_i})$$

θ is formula of the language $(\cdot,1)$
$Th(\mathbf{N}^{>0};\cdot,1)$ admites a quantifier elimination when language is augmented by the function symbols $I,T,SP_n(n\geq 0)$, and the unary symboles $E_n(n\geq 1)$.
Theorem4. the Theory $\langle N,\cdot\rangle$ admites quantifier - elimination. and so has decidable theory and is axiomatizable
Proof:
in the article [[13]] [2] has been proven. ■

Axiomatizing and decidability of the theory of $(\mathbf{N};\times)$:

(A_1) $\forall x\forall y\forall z\ (x\cdot(y\cdot z))=((x\cdot y)\cdot z)$

(A_2) $\exists x\forall y\ \ x\cdot y=y\cdot x=y$

(A_3) $\forall x\forall y\ \ x\cdot y=y\cdot x$

(A_4) $\forall x\forall y\forall z\ (x\cdot z=y\cdot z\rightarrow x=y)$

(A_5) $\forall x\forall y\ \ x\cdot y=1\rightarrow x=y=1$

$(A_{6,n})$ $\forall x\forall y x^n=y^n\rightarrow x=y\qquad(n\in N^*)$

$(A_{7,n})$ $\forall x\exists y\exists z\ (x=n\cdot y+z\wedge z\leq n\wedge z\neq n)\wedge\forall n\in N^*$

(A_8) $\forall x\exists y\exists z(x=y^n\cdot z\wedge\forall\acute{y}\forall\acute{z}(x=\acute{y}^n\cdot\acute{z}\rightarrow z\mid\acute{z}))$

(A_9) $\forall x\exists p\ (P(p)\wedge px)$

(A_{10}) $\forall p\forall x\forall y((PR(p,x)\wedge PR(p,y))\rightarrow x\mid y\vee y\mid x)$

(A_{11}) $\forall x\forall p\ (P(p)\rightarrow\exists y\ (y=V(p,x)\ )$

(A_{12}) $x=y\leftrightarrow\forall p\ (P(p)\rightarrow\exists y\ V(p,x)=V(p,y))$

(A_{13}) $\forall x\forall y\forall p(P(p)\rightarrow V(p,x\cdot y)=V(p,x)\cdot V(p,y))$

(A_{14}) $\forall x\forall y(\forall p(P(p)\rightarrow V(p,x)\mid V(p,y)\rightarrow x\mid y))$

(A_{15}) $\forall x\forall y\exists z\forall p(P(p)\rightarrow(p\mid x\rightarrow V(p,z)=V(p,y))\wedge(px\rightarrow V(p,z)=1)))$

(A_{16}) $\forall x\exists y\forall p\ (P(p)\rightarrow(p\not| x\rightarrow V(p,y)=1)\wedge(p\mid x\rightarrow V(p,y)=p\cdot V(p,x))))$

(A_{17}) $\forall x\forall y\exists z\forall p((P(p)\rightarrow(p\mid x\cdot y\wedge V(p,x)\equiv_n V(p,y))\rightarrow V(p,z)=p)\wedge(p\not| x\cdot y\vee V(p,x)\not\equiv_n V(p,y)\rightarrow V(p,z)=1)\ \ ,n\in N$

[2] Cégielski

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

The $Th(\mathbf{N}^{>0};\cdot,1)$ is complete and decidable.
because:
Let $L^{'}=(\cdot,V)$ which is th language obtained from L by adding a binary function sign V, for any formula φ in L, there is the associated formula φ^{p} in $L^{'}$ whose set of free variables is the ones in φ plus the variables p ,that is φ^{p} is obtained by replacing each free variable x in φ with the term $V(p,x)$. we let $M=Th(\mathbf{N}^{>0};\cdot,1)$ and $\mathcal{A}$ be a model of M, and φ be an $n-$ formula in L ,and p be a prime number and $\overrightarrow{f}\in A^{n}$

$$\mathcal{A}\models\varphi^{p}(\overrightarrow{f})\leftrightarrow\mathcal{A}_{p}\models\varphi(V(p,\overrightarrow{f}))$$

and $M^{''}$ denoted to a theory in the language $L^{''}=(\cdot,V,p)$. Let θ be a formula of L and $k\in N>0$,then we denoted $R_{k}(\theta)$ to the formula in L as follows:

$$\exists p_{1}\cdots\exists p_{k}(\bigwedge_{1\leq i\leq j\leq k}p_{i}\neq p_{j}\wedge\bigwedge_{1\leq i\leq k}\mathbf{P}(p_{i})\wedge\theta^{p_{i}})$$

Any formula ϕ of L is $M^{''}$ -equivalent to a combination to a boolean formula of the form $R_{k}(\theta)$; because it is $M^{''}$ -equivalent to a formula of L. and $M^{''}$ is an extension of M .so it is sufficient to prove complete and decidable for $M^{''}$. When φ is a statement we can effectively reduce it to a boolean combination of formulas of the type $R_{k}(\theta)$ where is a statement, $R_{k}(\theta)$ is true if and only if, it is true in the theory of addiction, since addition theory is complete and decidable, so The theory of multiplication of natural numbers is complete and decidable. And The multiplication theory of natural numbers is not finitely axiomatizable, because the theory of addition is not finitely axiomatizable.
Skolem claimed the decidablity of the theory $(\mathbf{N};\times,=)$ by using the quantifier elimination. The first decidability proof appeared in the work of Mostowski. Cegielski axiomatized multiplication theory and proved quantifier elimination.[13]

1) Peano Arithmetic: Peano's Axiomatic System:

- 1. $\forall x\,\neg\big(S(x)=0\big)$
- 2. $\forall x\forall y\big(S(x)=S(y)\longrightarrow x=y\big)$
- 3. $\forall x\big(x+0=x\big)$
- 4. $\forall x\forall y\big(x+S(y)=S(x+y)\big)$
- 5. $\forall x\big(x\cdot 0=0\big)$
- 6. $\forall x\forall y\big(x\cdot S(y)=x\cdot y+x\big)$
- 7. $\forall x\big[\neg(x<0)\big]$
- 8. $\forall x\forall y\big(x<S(y)\longleftrightarrow x<y\vee x=y\big)$
- 9. $\forall x\forall y\big(x<y\vee y<x\vee x=y\big)$.
- 10. $\varphi(0)\wedge\forall x\big[\varphi(x)\longrightarrow\varphi\big(S(x)\big)\big]\longrightarrow\forall x\varphi(x)$.
Peano Arithmetic **PA** is undecidable.

The decidablity of the structures of natural numbers in different languages is shown in the following tables so that the theories that admit QE by $\surd$ and, the theories do not admit QE by $\times$ is shown.

	N
$\{<\}$	$\langle N,<\rangle$
$\{+\}$	$\langle N,+\rangle$
$\{\times\}$	$\langle N,\times\rangle$
$\{<,+\}$	$\langle N,<+\rangle$
$\{<,\times\}$	$\langle N,<,\times\rangle$
$\{+,\times\}$	$\langle N,+,\times\rangle$
$\{+,\times,<\}$	$\langle N,+,\times,<\rangle$

TableI

Structures	The decidablity of the structures
$\langle\mathbf{N};<\rangle$	$\surd$
$\langle\mathbf{N};+\rangle$	$\surd$
$\langle\mathbf{N};\times\rangle$	$\surd$
$\langle\mathbf{N};<,+\rangle$	$\surd$
$\langle\mathbf{N};<,\times\rangle$	$\times$
$\langle\mathbf{N};+,\times\rangle$	$\times$
$\langle\mathbf{N};+,\times,<\rangle$	$\times$

TableII

Decidability of The theory of $\langle N;\sqsubseteq\rangle$:
Theorem5.The following completely axiomatizes the structure $\langle N;\sqsubseteq\rangle$ and, moreover , its theory admits quanrifier elimination, and so is decidable.[6]

- [1]$\forall x(x\sqsubseteq x)$
- [2]$\forall x,y(x\sqsubseteq y\sqsubseteq x\rightarrow x=y)$
- [3]$\forall x,y,z(x\sqsubseteq y\sqsubseteq z\rightarrow x\sqsubseteq z)$
- [4]$\forall x,y\exists z(z\sqsubseteq x,y\wedge\forall t[t\sqsubseteq x,y\rightarrow t\sqsubseteq z]),z=x\sqcap y$
- [5]$\forall x,y\exists z(x,y\sqsubseteq z\wedge\forall t[x,y\sqsubseteq t\rightarrow z\sqsubseteq t]),z=x\sqcup y$
- [6]$\forall x(1\sqsubseteq x)$
Definition 1. An element x of a lattice is join-irreducible iff it satisfies:
$\forall a,b(x=a\vee b\rightarrow(x=aorx=b))$ This is denoted by $SI(x)$ (or $SI^{\star}(x)$ if x is not zero).
- [7]$\forall x,y[\forall z(SI(z)\rightarrow[z\sqsubseteq x\rightarrow z\sqsubseteq y])\rightarrow x\sqsubseteq y]$
- [8]$\forall x,y,z(SI^{*}(x)\wedge SI^{*}(y)\wedge SI^{*}(z)\wedge[(x\sqsubseteq z\sqsubseteq z)\wedge(z\sqsubseteq x\sqsubseteq y)]\rightarrow x\sqsubseteq y\vee y\sqsubseteq x)$
- [9]$\forall x,a([SI^{*}(a)\wedge a\sqsubseteq x]\rightarrow\exists bSI(b)\wedge a\sqsubseteq b\sqsubseteq x\wedge\forall c(SI(c)\wedge c\sqsubseteq x,a)\rightarrow c\sqsubseteq b])$
b is called a valuation of x
- [10]$VAL(x,a)\wedge VAL(y,b)\wedge[(a=b=1)\vee(a=\wedge b\neq 1\wedge\forall x[SI^{*}(c)\wedge b\sim c]\rightarrow c\not\sqsubseteq x)\vee(1a\sqsubseteq b)]\Rightarrow VAL(x\sqcap y,a)\&VAL(x\sqcup y,a)$
- [11]$\forall x(x\neq 0\rightarrow\exists a(\mathbf{P}(\mathbf{a})\&a\sqsubseteq x))$
- [12]$\forall x(x\neq 0\rightarrow\exists a(\mathbf{P}(\mathbf{a})\&a\not\sqsubseteq x)$
- [13]$\forall x\exists s\forall a(\mathbf{P}(\mathbf{a})\rightarrow(V(a,x)\neq 0\rightarrow V(a,s)\neq a)\&(V(a,x)=0\rightarrow V(a,s)=0)))$
this s which is unique, is denoted by $\mathbf{SUPP}(x)$
- [14]$\forall x\forall y\exists z\forall a(\mathbf{P}(\mathbf{a})\rightarrow((a\not\sqsubseteq x\rightarrow V(a,z)=$

$V(a,y))\&a \sqsubseteq x \to V(a,z) = 0)))$
this z which is unique, is denoted by $\bar{\mathbf{T}}(x,y)$

- $[15-1]\forall a,x(SI(a,x) \to \exists y(SI(a,y)\&x \sqsubseteq y\&y \neq x\&\forall z((SI(a,z)\&xz) \to y \sqsubseteq z)))$
this y which is unique, is denoted by $\mathbf{S}_a(x)$
- $[15-2]\forall a,x(SI(a,x)\wedge x \neq 0) \to \exists y(SI(a,y)\&\mathbf{S}_a(y) = x))$.
this y which is unique, is denoted by $\mathbf{P}_a(x)$
- $[16]\forall x\exists y\forall a(\mathbf{P}(\mathbf{a}) \to ((a \not\sqsubseteq x \to V(a,y) = 0)\&(a \sqsubseteq x \to V(a,y) = \mathbf{S}_a V(a,x))))$
this y which is unique, is denoted by $\mathbf{I}(x)$
- $[17]\forall x\forall y\exists z\forall a(\mathbf{P}(\mathbf{a}) \to (V(a,z) = 0 or a\&V(a,z) = a \leftrightarrow ((a \sqsubseteq x or a \sqsubseteq y)\&V(a,x) \sqsubseteq V(a,y)))$

proof: [6].

The Quantifier Elimination of the structure of natural numbers in different languages is shown in the following tables so that the theories that admit QE by $\surd$ and, the theories do not admit QE by $\times$ is shown.

Table (III) A Quantifier Elimination Procedure for the Natural numbers at different Language:

Theory of	admit QE
$(\mathbf{N},<)$	$\times$
$(\mathbf{N},0,<)$	$\times$
$(\mathbf{N},0,S,<)$	$\surd$
$(\mathbf{N},0,1,+,\leq)$	$\times$
$(\mathbf{N},0,1,+,\leq,\{\equiv_n\}_{n\geq 2})$	$\surd$
$(\mathbf{N},\times)$	$\surd$
$(\mathbf{N},\sqsubseteq)$	$\surd$

B. Decidability of Structure of Integer Numbers in Different Languages

Theorem6. The structure $(Z;0,s)$ admits elimination of quantifier, and it has decidable theory.
Proof:
It suffices to consider a formula,

$$\exists x(\alpha_0 \wedge \cdots \wedge \alpha_q)$$

where each α_i is atomic or is the negation of an atomic formula. In the language of $\mathcal{Z}_s$ the only terms are of the from $S^k u$ where u is 0 or a variable. we may suppose that the variable x occurs in each α_i For if x does not occur in α then,

$$\exists x(\alpha \wedge \beta) \leftrightarrow \alpha \wedge \exists x\beta$$

Thus each α_i has the form $S^m x = S^n u$ or the negation of this equation, where u is 0 or a variable. We may further suppose u is different from x since $S^m x = S^n x$ could be replaced by $0 = 0$. if $m = n$ and by $0 \neq 0$ if $m \neq n$.
Case 1: Each α_i is the negation of an equation. Then the formula may be replaced by $0 = 0$.

Case 2: There is at least one α_i not negated;say α_0 is,

$$S^m x = t$$

where the term t does not contain x. Since the solution for x must be non-negative, we replace α_0 by,

$$t \neq 0 \wedge \cdots \wedge t \neq S^{m-1}0$$

Then in each other $\alpha_j$we replace, say, $S^k x = u$ first by $S^{k+m}x = S^m u$ which in turn becomes $S^k t = S^m u$ We now have a formula in which x no longer occurs,so the quantifier may be omitted .[3] ■

Theorem7. The Theory $(Z,0,S,<)$ admits elimination quantifier, and so has a decidable theory and is finitely axiomatizable.

S3.	$\forall y(y \neq 0 \to \exists xy = Sx)$
L1.	$\forall x\forall y(x < Sy \leftrightarrow x < y)$
L2.	$x \nless 0$
L3.	$\forall x\forall y(x < y \vee y < x \vee x = y)$
L4.	$\forall x\forall y(x < y \to y \nless x)$
L5.	$\forall x\forall y\forall z(x < y \to y < z \to x < z)$

Proof:
We consider a formula,

$$\exists x(\beta_0 \wedge \cdots \wedge \beta_n)$$

where each β_i is atomic or the negation of an atomic formula. The terms are of the form $S^k u$ Where u is 0 or a variable.There are two possibilities for atomic formula,

$$S^k u = S^l t, S^k u < S^l t$$

1. We can eliminate the negation symbol.Replace $t_1 t_2$ by $t_1 = t_2 \vee t_2 < t_1$ and replace $t_1 \neq t_2$ by $t_1 < t_2 \vee t_2 < t_1$ By regrouping the atomic formulas and noting that

$$\exists x(\phi \vee \psi) \leftrightarrow \exists x\phi \vee \exists x\psi$$

we may again reach formulas of the form,

$$\exists x(\alpha_0 \wedge \cdots \wedge \alpha_q)$$

where now, each α_i is atomic
2. We may suppose that the variable x occur in each α_i This is because if x does not occure in α_i then

$$\exists x(\alpha \wedge \beta) \leftrightarrow \alpha \wedge \exists x\beta$$

Furthermore,we may suppose that x occurs on only one side of the equality or inequality α_i
Case1: Suppose that some α_i is an equality.Then we can proceed as in case 2 of the quantifier-elimination proof Previous theory'
Case2: Otherwise each α_i is an inequality.Then the formula can be rewritten

$$\exists x(\bigwedge_i t_i < S^{m_i}x \wedge \bigwedge_j S^{n_j}x < u_j)$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

we have lower bounds on x
If the second conjuction is empty (i.e., if there are no upper bounds on x) then we can replace the formula by $0 = 0$ If the second conjunction is empty (i.e., if there are no upper bounds on x) then we an replace the formula by $\bigwedge_j S^{n_j} 0 < u_j$ which asserts that zero satisfies the upper bounds. Otherwise, we rewrite the formula successively as,

$$\exists x \bigwedge_{i,j} (t_i < S^{m_i} x \wedge S^{n_j} x < u_j)(1)$$
$$\exists x \bigwedge_{i,j} (S^{n_j} t_i < S^{m_i+n_j} x < S^{m_i} u_j)(2)$$
$$(\bigwedge_{i,j} S^{n_j+1} t_i < S^{m_i} u_j) \wedge \bigwedge_j S^{n_j} 0 < u_j$$

In each case, we have arrived at a quantifier-free version of the given formula.[3] ■

The additive theory of Integer numbers:
Theorem8.The theory of the structure $Z = \{+, 0, 1, \leq, \{\equiv_m\}_{\geq 2}\}$, admits quantifier elimination, and this theory is decidablr theory.

$$(A1) \quad x + (y + z) = (x + y) + z$$

$$(A2) \quad x + y = y + x$$

$$(A3) \quad x + 0 = x$$

$$(A4) \quad x + z = y + z \to x = y$$

$$(A5) \quad x + y = 0 \to x = y = 0$$

$$(O1) \quad x \leq y \leftrightarrow \exists z(x + z = y)$$

$$(O2) \quad x \leq y \vee y \leq x$$

$$(O3) \quad 0 \neq 1 \wedge \forall y(0 \leq y \leq 1 \to y = 0 \vee y = 1)$$

$$(D1) \quad \forall x \exists y, z(x = n \cdot y + t \wedge t < \bar{n})$$

Proof: at structure $\langle Z; +, 0, <, 1, \{\equiv_n)\}_{n\geq 2}\rangle$ every term involving x is equal to

$$n \cdot x + t \qquad (n \in N)$$

for some x -free term t and $n \geq 1$. Therefore, every atomic formula involving x is equall to the following formulas:

$$u = v$$
$$u < v$$
$$u \equiv_k v$$

whence ϕ is an atomic formula and x is a variable. ϕ is of the form: $t_0 = s_0$ or $t_0 R s_0$ such that $R \in (<, (\equiv_n)_{n\geq 2})$.
If ϕ is L atomic formula with variables x then ϕ at $\langle Z; +, 0, 1, \leq, \equiv_n)_{n\geq 2}\rangle$ is equivalent to one of the following formulas

$ax + t = s; ax + t < s; ax + t > s; ax + t \equiv_n s, n > 1$

because:

$$n \cdot x + t = m \cdot x + k \Rightarrow a \cdot x + t = s$$
$$n \cdot x + t < m \cdot x + k \Rightarrow a \cdot x + t < s \vee s < a \cdot x + t$$
$$n \cdot x + t \equiv_p m \cdot x + k \Rightarrow a \cdot x + t \equiv_p s$$

first we remove the following negation signs. Because:
$t \neq s \leftrightarrow t < s \vee s < t$;
$ts \leftrightarrow t = s \vee s < t$;
$\neg(t \equiv_n s) \leftrightarrow (t \equiv_n s + 1) \vee \cdots \vee (t \equiv_n s + n - 1)$.
We have $\phi = (\alpha_1 \wedge \cdots \wedge \alpha_k) \vee \cdots \vee (\beta_1 \wedge \cdots \wedge \beta_k)$ so
$\exists x \phi(x, x_1, \cdots, x_n) \quad \leftrightarrow \exists x((\alpha_1 \wedge \cdots \wedge \alpha_k) \vee \cdots \vee (\beta_1 \wedge \cdots \wedge \beta_k))$
$\leftrightarrow (\exists x(\alpha_1 \wedge \cdots \wedge \alpha_k) \vee \cdots \vee \exists x(\beta_1 \wedge \cdots \wedge \beta_k))$.
Then,we can assume,

$$\exists x(\alpha_1 \wedge \cdots \wedge \alpha_k)$$

α_i is of the form $ax + t\Delta s$;$\Delta \in \{=, <, >, \equiv_n\}$ It can be assumed that any α_i is of the form,

$ax + t = s; ax + t < s; ax + t > s; ax + t \equiv_n s, n > 1$

Thus, by the Main Lemma of Quantifier Elimination it suffices to show that every formula of the form

$$\exists x(\bigwedge_{i<h} N_i x + a_i = b_i \wedge \bigwedge_{j<p} c_j < L_j x + d_j \wedge \bigwedge_{k<q} K_k x + s_k < t_k \wedge \bigwedge_{l<r} M_l x + u_l \equiv_{m_l} v_l)$$

is equivalent with a quantifier -free formula, Step 1: Unification of coefficients x.
We assume A be the least common multiple of the coefficients of x.

$$\exists x(\bigwedge_{i<h} Ax + \acute{a_i} = \acute{b_i} \wedge \bigwedge_{j<p} \acute{c_j} < Ax + \acute{d_j} \wedge \bigwedge_{k<q} Ax + \acute{s_k} < \acute{t_k} \wedge \bigwedge_{l<r} Ax + \acute{u_l} \equiv_{m_l} \acute{v_l})$$

Step 2 : substituting $A \cdot x$ with y :

$$\exists y(\bigwedge_{i<h} y + \acute{a_i} = \acute{b_i} \wedge \bigwedge_{j<p} \acute{c_j} < y + \acute{d_j} \wedge \bigwedge_{k<q} y + \acute{s_k} < \acute{t_k} \wedge \bigwedge_{l<r} y + \acute{u_l} \equiv_{m_l} \acute{v_l} \wedge y \equiv_A 0).$$

By the equivalences

$$t = s \leftrightarrow ct = cs$$
$$t < s \leftrightarrow ct < cs$$
$$t \equiv_m s \leftrightarrow ct \equiv_{cm} cs$$

which are provable, where $c > 0$ is an integer and s, t both L are terms. By adding $x \equiv_A 0$, It suffices to eliminate the quantifier of

$$\exists x(\bigwedge_{i<h} x + a_i = b_i \wedge \bigwedge_{j<p} c_j < x + d_j \wedge \bigwedge_{k<q} x + s_k < t_k \wedge \bigwedge_{l<r} x + u_l \equiv_{m_l} v_l).$$

We can assume $h = 0$ Because if $h \neq 0$ then

$\phi \equiv a_0 < b_0 \wedge \bigwedge_{i<h} b_0 + a_i = a_0 + b_i \wedge \bigwedge_{j<p} c_j + a_0 < b_0 + d_j \wedge \bigwedge_{k<q} b_0 + s_k < a_0 + t_k \wedge \bigwedge_{l<r} b_0 + u_l \equiv_{m_l} v_l + a_0$

quantifier was removed.
Now that $h = 0$ we have:

$$\exists x(\bigwedge_{j<p} c_j < x + d_j \wedge \bigwedge_{k<q} x + s_k < t_k \wedge \bigwedge_{l<r} x + u_l \equiv_{m_l} v_l).$$

Now we can assume that $p \leq 1$ because for $p > 1$, we have ,

$$\exists x(c_0 < x + d_0 \wedge c_1 < x + d_1 \wedge \psi(x))$$
$$x > 0, c_0 - d_0 < x, c_1 - d_1 < 0$$
$$0 \leq c_0 - d_0 \leq c_1 - d_1$$
$$0 \leq c_1 - d_1 \leq c_0 - d_0$$
$$\equiv c_0 < d_0 \wedge (c_0 + d_1 \leq d_0 + c_1 \wedge \exists x(c_1 < x + d_1 \wedge \psi(x)) \vee (c_1 + d_0 \leq c_0 + d_1 \wedge \exists x(c_0 < x + d_0 \wedge \psi(x)).$$

And we can assume $q \leq 1$ because for $q > 1$

$$\exists x(x + s_0 < t_0 \wedge x + s_1 < t_1 \wedge \psi(x))$$
$$x < t_0 - s_0 \geq 0, x < t_1 - s_1 > 0$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

$$x < t_0 - s_0 \leq t_1 - s_1$$
$$x < t_1 - s_1 \leq t_0 - s_0$$
$\Leftrightarrow (s_0 \leq t_0 \wedge t_0 + s_1 \leq s_0 + t_1 \wedge \exists x(x + s_0 < t_0 \wedge \psi(x))$
$\vee(s_1 \leq t_1 \wedge t_1 + s_0 \leq t_0 + s_1 \wedge \exists x(x + s_1 < t_1 \wedge \psi(x)).$

And most importantly we can assume that $r \leq 1$ because for $r \geq 2$ we have

$\exists x(x + u_0 \equiv_{m_0} v_0 \wedge x + u_1 \equiv_{m_1} v_1 \wedge \psi(x))$
$x \equiv_{m_0} v_0 - u_0 \Leftrightarrow x = m_0 y + (v_0 - u_0)$
$x \equiv_{m_1} v_1 - u_1 \Leftrightarrow m_0 y + (v_0 - u_0) \equiv_{m_1} v_1 - u_1$
$m_0 y \equiv_{m_1} (v_1 - u_1 - v_0 + u_0)$
$d = m_0 \sqcap m_1$
$1 = \frac{m_0}{d} \sqcap \frac{m_1}{d}$
$v_1 - u_1 - v_0 + u_0 \equiv_d 0 \Leftrightarrow u_0 + v_1 \equiv_{m_0 \sqcap m_1} u_1 + v_0$
$m_0 y \equiv_{m_1} (v_1 - u_1 - v_0 + u_0) \Leftrightarrow \frac{m_0}{d} y \equiv_{\frac{m_1}{d}} \frac{v_1 - u_1 - v_0 + u_0}{d}$
$1 = \frac{m_0}{d} \sqcap \frac{m_1}{d} \Leftrightarrow \exists n n \cdot \frac{m_0}{d} \equiv_{\frac{m_1}{d}} 1$
$n \cdot \frac{m_0}{d} \equiv_{\frac{m_1}{d}} 1 \Leftrightarrow n \cdot \frac{m_0}{d} y \equiv_{\frac{m_1}{d}} y$
$y \equiv_{m_1} n \cdot v_1 - u_1 - v_0 + u_0 \Leftrightarrow y = n \cdot \frac{v_1 - u_1 - v_0 + u_0}{d} + \frac{m_1}{d} z$
$x = \frac{m_0 m_1}{d} z + \frac{n \cdot m_0}{d}(v_1 - u_1 - v_0 + u_0) + (v_0 - u_0)$
$m_0 \sqcup m_1 = \frac{m_0 m_1}{d}$
$x \equiv_{m_0 \sqcup m_1} n \cdot \frac{m_0}{d}(v_1 - u_1) + (1 - \frac{n \cdot m_0}{d})(v_0 - u_0)$
$\equiv (u_0 + v_1 \equiv_{m_0 \sqcap m_1} u_1 + v_0) \wedge \exists x(x + u_0 + \frac{n m_0}{m_0 \sqcap m_1}(u_1 + v_0) \equiv_{m_0 \sqcup m_1} \frac{n m_0}{m_0 \sqcap m_1}(v_1 + u_0) + v_0 \wedge \psi(x)).$

That n is a natural number and $n \cdot \frac{m_0}{m_0 \sqcap m_1} \equiv_{\frac{m_1}{m_0 \sqcap m_1}} 1$. Finally, we can assume that $h = 0, p, q, r \leq 1$. Check the available cases: if $p = q = r = 0$ is equivalent true. And the rest of the cases are as follows:

- p=q=0 ,r$\neq 0 \exists x(x + u_0 \equiv_{m_0} v_0) \equiv 0 = 0 \equiv true$
- p=0,q$\neq 0, r = 0 \exists x(x + s_0 < t_0) \leftrightarrow s_0 < t_0 \equiv true$
- p=0,q$\neq 0, r \neq 0 \exists x(x + s_0 < t_0 \wedge x + u_0 \equiv_{m_0} v_0) \leftrightarrow \bigvee_{i<m_0}(\bar{i} + s_0 < t_0 \wedge \bar{i} + u_0 \equiv_{m_0} v_0) \equiv true$
- p$\neq 0, q = r = 0 \exists x(c_0 < x + d_0) \equiv 0 = 0 \equiv true$
- p$\neq 0, q = 0, r \neq 0 \exists x(c_0 < x + d_0 \wedge x + u_0 \equiv_{m_0} v_0)$
  $x = m_0 c_0 + (m_0 - 1)u_0 + v_0 > c_0$
  $\equiv 0 = 0 \equiv true$
- p$\neq 0, q \neq 0, r = 0 \exists x(c_0 < x + d_0 \wedge x + s_0 < t_0) \leftrightarrow c_0 + s_0 + 1 < t_0 + d_0 \equiv true$
- p$\neq 0, q \neq 0, r \neq 0 \exists x(c_0 < x + d_0 \wedge x + s_0 < t_0 \wedge x + u_0 \equiv_{m_0} v_0)$
  $\bigvee_{i=1}^{m_0 - 1}(c_0 + \bar{i} < d_0 \wedge \bar{i} + s_0 < t_0 \wedge \bar{i} + u_0 \equiv_{m_0} v_0) \equiv true$

So we showed the theory of the addition of the integer numbers of the language $(+, 0, 1, \leq, (\equiv_n)_{n \geq 2})$ admits quantifier elimination.[2,3] ■

The Decidability of The multiplicative theory of integers:

Theorem9.The following theory completely axiomatizes the structure $(\mathbf{Z}^{>0}; \cdot, 1)$ and,moreover , its theory admits quantifier - elimination. and so the Theory $\langle Z, \cdot \rangle$ is decidable.

(A_1) $\forall x \forall y \forall z \ (x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)$

(A_2) $\exists x \forall y \ x \cdot y = y \cdot x = y$

(A_3) $\forall x \forall y \ x \cdot y = y \cdot x$

(A_4) $\forall x \forall y \forall z \ (x \cdot z = y \cdot z \rightarrow x = y)$

(A_5) $\forall x \forall y \ x \cdot y = 1 \rightarrow x = y = 1$

$(A_{6,n})$ $\forall x \forall y x^n = y^n \rightarrow x = y \qquad (n \in N^*)$

$(A_{7,n})$ $\forall x \exists y \exists z \ (x = n \cdot y + z \wedge z \leq n \wedge z \neq n) \wedge \forall n \in N^*$

(A_8) $\forall x \exists y \exists z (x = y^n \cdot z \wedge \forall \acute{y} \forall \acute{z}(x = \acute{y}^n \cdot \acute{z} \rightarrow z \mid \acute{z}))$

(A_9) $\forall x \exists p \ (P(p) \wedge px)$

(A_{10}) $\forall p \forall x \forall y((PR(p, x) \wedge PR(p, y)) \rightarrow x \mid y \vee y \mid x)$

(A_{11}) $\forall x \forall p \ (P(p) \rightarrow \exists y \ (y = V(p, x) \ )$

(A_{12}) $x = y \leftrightarrow \forall p \ (P(p) \rightarrow \exists y \ V(p, x) = V(p, y))$

(A_{13}) $\forall x \forall y \forall p(P(p) \rightarrow V(p, x \cdot y) = V(p, x) \cdot V(p, y))$

(A_{14}) $\forall x \forall y(\forall p(P(p) \rightarrow V(p, x) \mid V(p, y) \rightarrow x \mid y))$

(A_{15}) $\forall x \forall y \exists z \forall p(P(p) \rightarrow (p \mid x \rightarrow V(p, z) = V(p, y)) \wedge (px \rightarrow V(p, z) = 1)))$

(A_{16}) $\forall x \exists y \forall p \ (P(p) \rightarrow (p \not| x \rightarrow V(p, y) = 1) \wedge (p \mid x \rightarrow V(p, y) = p \cdot V(p, x))))$

(A_{17}) $\forall x \forall y \exists z \forall p((P(p) \rightarrow (p \mid x \cdot y \wedge V(p, x) \equiv_n V(p, y)) \rightarrow V(p, z) = p) \wedge (p \not| x \cdot y \vee V(p, x) \not\equiv_n V(p, y) \rightarrow V(p, z) = 1) \ ,n \in N$

(A_{18}) $\forall x \exists y x + y = 0$

Proof:[13]. ■

Table (IV) :A Quantifier Elimination Procedure for the integers:

Theory of	Language	admit QE
$(\mathbf{Z}, +)$	$L = (+)$	×
$(\mathbf{Z}, +, ; -; <)$	$L = (0; 1; +; -; <)$	×
$(\mathbf{Z}, 0, 1, +, \leq, \{\equiv_n\}_{n \geq 2})$	$L = (0; 1, +, \leq, \{\equiv_{n_{n \geq 2}}\})$	√
$(\mathbf{Z}, \times)$	$L = (\cdot, v, p)$	√

C. Decidability of Structure of rational Numbers in Different Languages

Theorem10.Theory $(Q, <)$ admits elimination of quantifier.

Proof:

1: Identify the terms

In structure $\langle Q; < \rangle$, every term involving x is equal to,

$$n \cdot x + t \qquad (n \in N)$$

where x does not appear in t

2: Identify Atomic Formulas and Delete ¬ if possible

All atomic formulas are,

$$u < v$$
$$u = v$$

First, we eliminate the inequality behind the atoms. Because,

$x \neq y \leftrightarrow x < y \wedge y < x$
$x \not< y \leftrightarrow x = y \vee y < x$

3: Simplify atomic formulas

So the following formula must be eliminated quantifier.

$$\exists x(\bigwedge_i r_i < m_i \cdot x + s_i \wedge \bigwedge_j n_j \cdot x + t_j < u_j \wedge \bigwedge_l k_l \cdot x + v_l = w_l)$$

4:Uniform the coefficients x
Let M is Multiply the coefficients by x

$M = \prod_i m_i \prod_j n_j \prod_l k_l$
$r_i \frac{M}{m_i} < Mx + \frac{M}{m_i} s_i$
$Mx + \frac{M}{n_j} t_j < \frac{M}{n_j} u_j$
$Mx + \frac{M}{k_l} v_l = \frac{M}{k_l} w_l$
So the following formula admits quantifier elimination

$$\exists x(\bigwedge_i r_i^{'} < Mx + s_i^{'} \wedge \bigwedge_j Mx + t_j^{'} < u_j^{'} \wedge \bigwedge_l Mx + v_l^{'} = w_l^{'})$$

5: Remove the coefficient x
$y = Mx$. So, we have
$\exists y(\bigwedge_i r_i^{'} < y + s_i^{'} \wedge \bigwedge_j y + t_j^{'} < u_j^{'} \wedge \bigwedge_l y + v_l^{'} = w_l^{'})$
We use the following equations

$$t = s \leftrightarrow ct = cs$$
$$t < s \leftrightarrow ct < cs$$

so
$\exists x(\bigwedge_i r_i < x + s_i \wedge \bigwedge_j x + t_j < u_j \wedge \bigwedge_l x + v_l = w_l)$
6: Identification Phrases included x

$r_i < x + s_i \leftrightarrow r_i + t_j + v_l < x + s_i + t_j + v_l$
$x + t_j < u_j \leftrightarrow x + s_i + t_j + v_l < u_j + s_i + v_l$
$x + v_l = w_l \leftrightarrow x + s_i + t_j + v_l = s_i + t_j + w_l$
$P = s_i + t_j + v_l$
so
$\exists x(\bigwedge_i r_i^{'} < x + P \wedge \bigwedge_j x + P < u_j^{'} \wedge \bigwedge_l x + P = w_l^{'})$
we put $y = x + P$

$$\exists y(\bigwedge_i r_i^{'} < y \wedge \bigwedge_j y < u_j^{'} \wedge \bigwedge_l y = w_l^{'})$$

Therefore, it is enough to delete the quantifier in the following formula:

$$\exists x(\bigwedge_i r_i < x \wedge \bigwedge_j x < u_j \wedge \bigwedge_l x = w_l)$$

7: Identify the states
$l \neq 0 \equiv \bigwedge_i r_i < w_0 \wedge \bigwedge_j w_0 < u_j \wedge \bigwedge_l w_0 = w_l \equiv True$
$l = 0 \equiv \exists x(\bigwedge_i r_i < x \wedge \bigwedge_j x < u_j)$
$l = j = 0 \equiv \exists x(\bigwedge_i r_i < x) \equiv True$
$l = i = 0 \equiv \exists x(\bigwedge_j x < u_j) \equiv True$
$l = 0, i, j \neq 0 \equiv \exists x(\bigwedge_i r_i < x \wedge \bigwedge_j x < u_j) \equiv \bigwedge_i \bigwedge_j r_i < u_j \equiv True$

■

Decidability Mathematical Structures: Structures The Theory of Addition $(Q, +)$:
Theorem11.The Theory of Addition $(Q, +)$ admits elimination of quantifier.
Proof:
Step 1: Identify the terms

In structure $(Q, +)$, every term involving x is equal to,

$$n \cdot x + t \qquad (n \in N)$$

where x does not appear in t

Step 2: Identify Atomic Formulas

All atomic formulas are,

$$u = v$$
$$u \neq v$$

Step 3: Simplify atomic formulas

So the following formula must be eliminated quantifier.

$$\exists x(\bigwedge_i k_i \cdot x + v_i = w_i \wedge \bigwedge_j m_j \cdot x + n_j \neq s_j)$$
$$\equiv \exists x(\bigwedge_i k_i \cdot x = u_i \wedge \bigwedge_j m_j \cdot x \neq t_j)$$

Step 4:Uniform the coefficients x
Let M is Multiply the coefficients by x

$M = \prod_i k_i \prod_j m_j$
So the following formula admits quantifier elimination

$$\exists x(\bigwedge_i M \cdot x = u_i^{'} \wedge \bigwedge_j M \cdot x \neq t_j^{'})$$

Step 5: Remove the coefficient x
$y = Mx$. So, we have

$$\exists y(\bigwedge_i y = u_i^{'} \wedge \bigwedge_j y \neq t_j^{'})$$

We use the following equations

$$t = s \leftrightarrow ct = cs$$
$$t \neq s \leftrightarrow ct \neq cs$$

so

$$\exists x(\bigwedge_i x = u_i \wedge \bigwedge_j x \neq t_j)$$

Step 6: Identify the states

$i \neq 0 \equiv \bigwedge_i u_0 = u_i \wedge \bigwedge_j u_0 \neq t_j$
$i = 0, j \neq 0 \equiv True$
Theorem12. the Theory $\langle Q; +, -.0, < \rangle$ admites quantifier - elimination. and so has decidable theory . Proof:
The following formula must be eliminated quantifier.

$$\exists x(\bigwedge_i n_i \cdot x = t_i \wedge \bigwedge_j 0 < m_j \cdot x + s_j)$$

Similar to previous proofs, admites quantifier - elimination. and so has decidable theory .

■

Theorem13. the Theory $\langle Q^+; \times, 1.0^{-1}, \{R_n\}_{n \geq 2} < \rangle$ admites quantifier - elimination. and so has decidable theory .
Proof:[1]
Similar to previous proofs, admites quantifier - elimination. and so has decidable theory .
Theorem14: The theory of the rational numbers $(\mathbf{Q}, \sqsubseteq)$ is decidable, and moreover axiomatizable.
Proof:
quantifier elimination for The theory of the rational numbers $(\mathbf{Q}^+, \sqsubseteq)$:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

$$p \sqsubseteq q \leftrightarrow \exists m \in \mathbf{N}^+(p \cdot m = q)$$

Structure $(\mathbf{Q}^+, \sqsubseteq)$ Is equivalent With structure $(\mathbf{Q}^+, \times)$ First,We conclude decidablity $(\mathbf{Q}^+, \times)$ of paper [1] so, the structure $(\mathbf{Q}^+, \sqsubseteq)$ Based on the article [1] is decidable .

We will express the axioms of rational numbers as follows:

Positive rational numbers are formed from two parts, the integer part whose denominator is one, and the Intrevel Algebra of rational numbers. The positive part of all the properties of natural numbers .So we have the axioms of $(\mathbf{N}, \sqsubseteq)$ and atomless Boolean Algebra and the axioms of $(\mathbf{Q}^+, \times)$.

so we have the following axioms for $\langle \mathbf{Q}^+, \sqsubseteq \rangle$:

[1]$\forall x(x \sqsubseteq x)$
[2]$\forall x, y(x \sqsubseteq y \sqsubseteq x \to x = y)$
[3]$\forall x, y, z(x \sqsubseteq y \sqsubseteq z \to x \sqsubseteq z)$
[4]$\forall x, y\exists z(z \sqsubseteq x, y \wedge \forall t[t \sqsubseteq x, y \to t \sqsubseteq z]), z = x \sqcap y$
[5]$\forall x, y\exists z(x, y \sqsubseteq z \wedge \forall t[x, y \sqsubseteq t \to z \sqsubseteq t]), z = x \sqcup y$
[6]$\forall x(1 \sqsubseteq x)$
[7]$\forall x, y[\forall z(SI(z)[z \sqsubseteq x \to z \sqsubseteq y]) \to x \sqsubseteq y]$
[8]$\forall x, y, z(SI^*(x) \wedge SI^*(y) \wedge SI^*(z) \wedge [(x, y \sqsubseteq z \vee z \sqsubseteq x, y)] \to x \sqsubseteq y \vee y \sqsubseteq x)$
[9]$\forall x, a([SI^*(a) \wedge a \sqsubseteq x] \to \exists b(SI(b)^* \wedge a \sqsubseteq b \sqsubseteq x \wedge \forall c(SI(c) \wedge c \sqsubseteq x, a) \to c \sqsubseteq b])$
[10]$VAL(x, a) \wedge VAL(y, b) \wedge [(a = b = 1) \vee (a = \wedge b \neq 1 \wedge \forall x[SI^*(c) \wedge b \sim c] \to c \not\sqsubseteq x) \vee (1 \sqsubseteq a \sqsubseteq b)] \Rightarrow VAL(x \sqcap y, a) \& VAL(x \sqcup y, a)$
[11]$\forall x(x \neq 0 \to \exists a(\mathbf{P(a)} \& a \sqsubseteq x))$
[12]$\forall x(x \neq 0 \to \exists a(\mathbf{P(a)} \& a \not\sqsubseteq x)$
[13]$\forall x \exists s \forall a(\mathbf{P(a)} \to (V(a, x) \neq 0 \to V(a, s) \neq a) \& (V(a, x) = 0 \to V(a, s) = 0)))$
$s = \mathbf{SUPP}(x)$
[14]$\forall x \forall y \exists z \forall a(\mathbf{P(a)} \to ((a \not\sqsubseteq x \to V(a, z) = V(a, y)) \& a \sqsubseteq x \to V(a, z) = 0)))$
$z = \bar{\mathbf{T}}(x, y)$
[15 − 1]$\forall a, x(SI(a, x) \to \exists y(SI(a, y) \& x \sqsubseteq y \& y \neq x \& \forall z((SI(a, z) \& x \sqsubseteq z) \to y \sqsubseteq z)))$
$y = \mathbf{S}_a(x)$
[15−2]$\forall a, x(SI(a, x) \wedge x \neq 0) \to \exists y(SI(a, y) \& \mathbf{S}_a(y) = x))$.
$y = \mathbf{P}_a(x)$
[16]$\forall x \exists y \forall a(\mathbf{P(a)} \to ((a \not\sqsubseteq x \to V(a, y) = 0) \& (a \sqsubseteq x \to V(a, y) = \mathbf{S}_a V(a, x))))$
$y = \mathbf{I}(x)$
[17]$\forall x \forall y \exists z \forall a(\mathbf{P(a)} \to (V(a, z) = 0 ora \& V(a, z) = a \leftrightarrow ((a \sqsubseteq xora \sqsubseteq y) \& V(a, x) \sqsubseteq V(a, y)))$
[18]$x \sqcap y = y \sqcap x \qquad x \sqcup y = y \sqcup x$
[19]$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z \qquad x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
[20]$(x \sqcap y) \sqcup y = y \qquad (x \sqcup y) \sqcap y = y$
[21]$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z) \qquad x \sqcup (y) = (x \sqcup y) \sqcap (x \sqcup z)$
[22]$x \sqcap x^{-1} = 1 \qquad x \sqcup x^{-1} = x$
[23]$\neg P(a)$
[24]$\forall x, y, z(x \cdot (y \cdot z)(x \cdot y) \cdot z)$
[25]$\forall x, y(0 < x < y \to \exists z(x^{2n} < z < y^{2n})), n \geq 1$
[26]$\forall v_1, \cdots, v_l \exists x \forall z \bigwedge_{k=1}^{l} (x^n \cdot v_k \neq z^{m_k})$

The axiomatization is a modeling in the first order language L, $\langle \mathbf{Q}^+, \sqsubseteq \rangle$, will be a model of this language.

[1]$\forall x(x \sqsubseteq x)$
[2]$\forall x, y(x \sqsubseteq y \sqsubseteq x \to x = y)$
[3]$\forall x, y, z(x \sqsubseteq y \sqsubseteq z \to x \sqsubseteq z)$

$\sqsubseteq$ is a strict partial order.

number theory	$\sqsubseteq$	$\sqcap$	$\sqcup$			
set theory	$\subseteq$	$\cap$	$\cup$	ϕ	X	$'$
logic	$\leq$	$\wedge$	$\vee$	0	1	$\neg$

[4]$\forall x, y\exists z(z \sqsubseteq x, y \wedge \forall t[t \sqsubseteq x, y \to t \sqsubseteq z]), z = x \sqcap y$
[5]$\forall x, y\exists z(x, y \sqsubseteq z \wedge \forall t[x, y \sqsubseteq t \to z \sqsubseteq t]), z = x \sqcup y$

Axioms 4 and 5 are equivalent to the following axioms in set theory:

[4]$\forall A, B\exists C(C \subseteq A, B \wedge \forall T[T \subseteq A, B \to T \subseteq C]), C = A \cap B$
[5]$\forall A, B\exists C(A, B \subseteq C \wedge \forall T[A, B \subseteq T \to C \sqsubseteq T]), C = A \cup B$

Definition: x is p-primary and denoted by $PR(p, x)$ iff we have $P(p) \wedge \forall q((P(q) \wedge p \neq q) \longrightarrow qx)x = p^n$
Definition:An element x is join-irreducible iff it satisfies $\forall a, b(x = a \sqcup b \longrightarrow (x = a) \vee (x = b))$. This is denoted by $SI(x)$ or $SI^*(x)$ if $x \neq 1$

lemma 2: x is p-primary number why?
If x is not p-primary number then we have:

$$x = \prod_i p_i^{\alpha_i} = p_i^{\alpha_i} \prod_j p_j^{\alpha_j} = p_i^{\alpha_i} {}_j^{\alpha_j}$$

then x is not join-irreducible .

[7]$\forall x, y[\forall z(SI(z)[z \sqsubseteq x \to z \sqsubseteq y]) \to x \sqsubseteq y]$

Propostion 1: $\forall x, y \Leftrightarrow \forall z(SI(z)[z \sqsubseteq x \leftrightarrow z \sqsubseteq y])$

[8]$\forall x, y, z(SI^*(x) \wedge SI^*(y) \wedge SI^*(z) \wedge [(x, y \sqsubseteq z \vee z \sqsubseteq x, y)] \to x \sqsubseteq y \vee y \sqsubseteq x)$

Propostion 2: $xy \Leftrightarrow x \sqsubseteq y \vee y \sqsubseteq x$

[9]$\forall x, a([SI^*(a) \wedge a \sqsubseteq x] \to \exists b(SI(b)^* \wedge a \sqsubseteq b \sqsubseteq x \wedge \forall c(SI(c) \wedge c \sqsubseteq x, a) \to c \sqsubseteq b])$

Propostion 3: (1)$\forall x, y(x \sqsubseteq y \leftrightarrow \forall a VAL(x, a) \to a \sqsubseteq y)$
(2)$\forall x, y(x = y \leftrightarrow \forall a VAL(x, a) \to VAL(y, a))$

Proof:
(1)$x \sqsubseteq y \leftrightarrow \forall a VAL(x, a) \to a \sqsubseteq y$
$\to$:
It is obvious.Because:

$$x \sqsubseteq y \to \forall a(VAL(x, a) \to a \sqsubseteq x) \to a \sqsubseteq y)$$

$\leftarrow$?

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

We consider that $\forall a(VAL(x,a))$ then by axiom (7) we show: $\forall z(SI(z)[z \sqsubseteq x \to z \sqsubseteq y])$
so,
for the z arbitrary, we suppose that $SI(z), z \sqsubseteq x$ by (9), $\exists b, SI(b), z \sqsubseteq b \wedge \forall c(SI(c) \wedge z \sqsubseteq c \sqsubseteq x \to c \sqsubseteq b)$
then we have, $VAL(x,b)$. so, $b \sqsubseteq y$ then $z \sqsubseteq b \Rightarrow z \sqsubseteq y$ so, $x \sqsubseteq y$.

[10]$VAL(x,a) \wedge VAL(y,b) \wedge [(a=b=1) \vee (a = \wedge b \neq 1 \wedge \forall x[SI^*(c) \wedge b \sim c] \to c \not\sqsubseteq x) \vee (1 \sqsubseteq a \sqsubseteq b)] \Rightarrow VAL(x \sqcap y, a) \& VAL(x \sqcup y, a)$
Propostion 4:

$$\forall x,y,z(x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z))$$

Atom: $a \neq 0 \forall x(x \leq a \to (x = 0 \vee x = a))$, we denotes by $A(a)$.

[11]$\forall x(x \neq 0 \to \exists a(\mathbf{P(a)} \& a \sqsubseteq x))$
[12]$\forall x(x \neq 0 \to \exists a(\mathbf{P(a)} \& a \not\sqsubseteq x)$

Propostion 5: $\forall x(SI^*(x) \to \exists! a(P(a)a \sqsubseteq x))$

lemma3. $\exists! a(P(a)a \sqsubseteq x) \to SI^*(a)$
Proof:
$a = b \sqcup c \Rightarrow b \sqsubseteq a \Rightarrow b = a\surd \vee b = 1$
$b = 1 \Rightarrow a = 1 = c\surd$

The decidability of the structure of rational numbers in different languages is shown in the following tables so that the theories of decidable by $\surd$and, undecidable theories by $\times$ is shown.

L	Q
$\{<\}$	$\langle Q,<\rangle$
$\{+\}$	$\langle Q,+\rangle$
$\{\times\}$	$\langle Q,\times\rangle$
$\{<,+,-,0\}$	$\langle Q,<,+\rangle$
$\{\sqsubseteq\}$	$\langle Q,\sqsubseteq\rangle$

TableV

Structures	The decidablity of the structures
$\langle \mathbf{Q};<\rangle$	$\surd$
$\langle \mathbf{Q};+\rangle$	$\surd$
$\langle \mathbf{Q};\times\rangle$	$\surd$
$\langle \mathbf{Q};<,+\rangle$	$\surd$
$\langle \mathbf{Q};\sqsubseteq\rangle$	$\surd$

TableVI

D. Deciability of structures of real numbers at different Language

Theorem14.The structure $(\mathbf{R};<)$ admites quantifier elimination , so has a decidable theory.
Proof:
Quantifier Elimination Procedure for $(\mathbf{R};<)$

$$\underbrace{\exists x[\bigwedge_{i=1}^{m} x = x_i \wedge \bigwedge_{j=1}^{n} z_j < x \wedge \bigwedge_{k=1}^{p} x < u_k]}_{\varphi}$$

If $m > 0$:

$$R \models \varphi \iff \bigwedge_{i=2}^{m} x_1 = x_i \wedge \bigwedge_{j=1}^{n} z_j < x_1 \wedge \bigwedge_{k=1}^{p} x_1 < u_k$$

If $m = 0$ then we distinguish 3 subcases:

If $n = 0$, then $R \models \varphi \iff$ true, because R has no minimum. If $p = 0$, then $R \models \varphi \iff$ true, because R has no maximum.If $n > 0$ and $p > 0$, then
$R \models \bigwedge_{j=1}^{n} \bigwedge_{k=1}^{p} z_j < u_k$
Proof:
$'' \to '' <$ is transitive.
$'' \leftarrow ''$ there exists $x \in R$ with $max_j z_j < x < min_k u_k$.

Theorem15.The structure $\langle \mathbf{R};<,+\rangle$ admits quantifier elimination,and so is decidable. [1]
Proof:
It suffices to prove that the following formula is equivalent to a formula without quantifier.

$$\exists x(\bigwedge_{i<l} t_i p_i \cdot x \wedge \bigwedge_{j<m} q_j \cdot x < s_j \wedge \bigwedge_{k<n} r_k \cdot x = u_k)$$

Consider the coefficients p_i, q_j, r_k are equal. As a result, we have the following equivalence

$$\exists y(\bigwedge_{i<l} t_i < y \wedge \bigwedge_{j<m} y < s_j \wedge \bigwedge_{k<n} y = u_k)$$

Now the quantifier of this formula is easily removed.
Theorem16.The structure $\langle \mathbf{R};+\rangle$ admits quantifier elimination,and so is decidable. [1]
Proof:
Each term included x is equivalent to $k \cdot x + t$. So each atomic formula contains x equal to $k \cdot x = t$. For term t without x and positive integers k It is enough to delete the suras of the following formula:

$$\exists x(\bigwedge_{i<l} n_i \cdot x = t_i \wedge \bigwedge_{j<k} m_j \cdot x \neq s_j)$$

We can assume all of $n_i^{,}s$ and $m_j^{,}s$ are equal to each other. As a result, we show the following equivalence

$$\exists x(\bigwedge_{i<l} q \cdot x = t_i \wedge \bigwedge_{j<k} q \cdot x \neq s_j)$$

We consider $y = q \cdot x$ so, we have

$$\exists y(\bigwedge_{i<l} y = t_i \wedge \bigwedge_{j<k} y \neq s_j)$$

If $l > 0$ so we have

$$\bigwedge_{i<l} t_0 = t_i \wedge \bigwedge_{j<k} t_0 \neq s_j$$

If $l = 0$ so we have $\bigwedge_{j<k} y \neq s_j$ and so is equivalent with the quantifier-free formula $0 = 0$

Theorem17.The structure $(\mathbf{R};\times)$ admits quantifier elimination,and so is decidable. [1]

Theorem18.The structure $(\mathbf{R};\times,0^{-1},0,-1,P^3)$ admits quantifier elimination,and so is decidable. [1]

Table (VII) :A Quantifier Elimination Procedure for the Reals Numbers at different Language:

Theory of	Language	admit QE
$(\mathbf{R},<)$	$L = (<)$	$\surd$
$(\mathbf{R},0,+,-)$	$L = (0;+;-)$	$\surd$
$(\mathbf{R},0,+,-,<)$	$L = (0,+,-,<)$	$\surd$
$(\mathbf{R},\times)$	$L = (\times,0^{-1},0,1,-1,P)$	$\surd$

E. Deciability of structures of complex numbers at different Language

The additive of theory of the complex number is similiar to the additive theory of real and rational number, and

[3]postivity property

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

so has a decidable theory. It is interesting , we know that the proof desidability of thr theory of $(\mathbf{C};+)$ and $(\mathbf{R};+)$ and $(\mathbf{Q};+)$ is easire than $(\mathbf{Z};+),(\mathbf{N};+)$.

Theorem19.The theoty of $(\mathbf{C};\times)$ admits quantifier elimination , and so has a decidable theory.[1]

Table $(VIII)$:A Quantifier Elimination Procedure for the complex Numbers at different Language:

Theory of	admit QE
$(\mathbf{C},+)$	$\surd$
$(\mathbf{C},\times)$	$\surd$

F. Deciding Boolean Algebras:

Boolean algebras was to begin with intoduced by Boole in an effort to automate reasoning.Since that they have been extensively studied, and have proved fundemental in numerous application areas. At the consider of Boolean algebras, we show decidability and undecidability questions for the class of Boolean algebras,And We describe an algorithm for deciding the Boolean algebras. A basic result of Tarski is that the elementary theory of Boolean algebras is decidable. Even the theory of Boolean algebras with a distinguished ideal is decidable. On the other hand, the theory of a Boolean algebra with a distinguished subalgebra is undecidable. Both the decidability results and undecidablity results extend in various ways to Boolean algebras in extensions of first-order logic.

Definition:Atoms are exactly the minimal nonzero elements, i.e. a is an atom iff $0 \leq a$ and $0 < x \leq a \Longrightarrow x = a$.

An algorithm for deciding the theory Atomic Boolean algebras: We present an algorithm and show how decide.We have some definitions:

- $L = \{\subseteq, \cap, \cup, A\setminus B, =, \emptyset, C_n, E_n, n \in \mathbf{N}^+\}$
- $\mathbf{A}(a) \leftrightarrow \forall x[x \subseteq a \to x = \emptyset \vee x = a] \wedge a \neq \emptyset$
- $C_n(x) \equiv \exists a_1 \cdots a_n(\bigwedge_{i<j} a_i \neq a_j \wedge \bigwedge_{i=1}^n \mathbf{A}(a_i) \wedge \bigwedge_{i=1}^n a_i \subseteq X)$
- $E_{n(x)} \equiv C_n(x) \wedge \neg C_{n+1}(x)$
- The next step of the algorithm is eliminate =:
  Because: $a = b \Longleftrightarrow a \subseteq b \wedge b \subseteq a$
- eliminate $\subseteq$
  Because: $a \subseteq b \Longleftrightarrow a\setminus b = \emptyset \leftrightarrow E_0(a-b)$
- And eliminat: $\neg$:
  Because: $\neg C_n(x) \Longleftrightarrow \bigvee_i E_i(x)$
  $\neg E_n(x) \Longleftrightarrow C_{n+1}(x) \vee \bigvee_i E_i(x)$

Quantifier-Elimination for Boolean formulas is as follows:

- $L = \{\cap, \cup, \setminus, =, \{C_n\}, \{E_n\}, n \in \mathbf{N}^+\}$
  We have the following
- $R = \{= |\{C_n\}_{n\geq 0}| \subseteq |\{E_n\}_{n\geq 0}\}$
  $F = \{A|F_1 \wedge F_2|F_1 \vee F_2|\neg F|\exists F|\forall F\}$
  $A = \{B_1 = B_2|B_1 \subseteq B_2|C_n(B), E_n(B)\}$
  $B = \{x|\emptyset|I|B_1 \cap B_2|B_1 \cup B_2|B^c\}$
  $n = \{0|1|2|\cdots\}$

So it is enough to consider only the following formulas: $C_n(x) = |x| \geq n, E_n(x) = |x| = n$. Contradictions of liters are eliminated according to the above definitions.

$\neg|x| = n \leftrightarrow |x| = 0 \vee \cdots \vee |x| = n-1 \vee |x| \geq n+1$
$\neg|x| \geq n \leftrightarrow |x| = 0 \vee \cdots \vee |x| = n-1$

So at this step we've removed some of the relationships as follow :

1.Eliminate equality

$a = b \leftrightarrow a \subseteq b \wedge b \subseteq a$

2. Delete inclusion

$a \subseteq b \leftrightarrow |a \cap b^c| = 0$

3.Eliminate contradictions

$\neg C_n(x) \leftrightarrow \bigvee_{i<n} E_i(x)$
$\neg E_n(x) \leftrightarrow C_{n+1}(x) \vee \bigvee_{i<n} E_i(x)$

Language to Quantifier-Elimination

$$\cap, \cup, ^c, \emptyset, \{C_n\}_{n\geq 0}, \{E_n\}_{n\geq 0}$$

term:

$$x, \emptyset, \cap, \cup, ^c$$

Quantifier Elimination:

In the resulting formula, each set variable x occurs in some term $|t(x)|$. each set expression $|t(x)|$ as a union of cubes (regions in the Venn diagram). The cubes have the form $\bigcap_{i=1}^n x_i^{\alpha_i}$ where $x_i^{\alpha_i}$ is either x_i or x_i^c ; there are $m = 2^n$ cubes . The resulting formula is then equivalent to

$$\exists x(\wedge_i C_{n_i}(t_i(x)) \wedge (\wedge_j E_{n_j}(t_j'(x)))$$

for example:

$$\exists x(|x \cap c| \geq 3 \wedge |x \cap c| \geq 7 \wedge |c - x| = 2)$$
$$\exists x(C_3(x \cap c) \ \wedge C_7(x \cap c) \wedge E_2(c-x)) \equiv C_9(c)$$
$$\exists x(C_5(x \cap c) \ \wedge C_7(x \cap d) \wedge E_6(c-x)) \equiv C_{11}(c) \wedge C_7(d)$$

More explained in the table below

The main formula	Deleted form
$\exists z \cdots |x \cap z| \geq k \wedge |x \cap z^c| \geq l \cdots$	$|x| \geq k+l$
$\exists z \cdots |x \cap z| = k \wedge |x \cap z^c| \geq l \cdots$	$|x| \geq k+l$
$\exists z \cdots |x \cap z| \geq k \wedge |x \cap z^c| = l \cdots$	$|x| \geq k+l$
$\exists z \cdots |x \cap z| = k \wedge |x \cap z^c| = l \cdots$	$|x| = k+l$

TABLE IX

A Boolean Algebra is atomless if it has no atoms.Every atomless Boolean algebras with more than one element must be infinite.Indeed,the unit 1 is different from zero, so there is a non-zero element p_1 strictly below 1;otherwise, 1 would be an atom. Because p_1 is not zero,there must be a non-zero element p_2 strictly below p_1; otherwise, p_1 decreasing sequence of elements

$$1 > p_1 > p_2 > \cdots.$$

atomless boolean algebra: we have the interval algebra of rational number . the intrval algebra of the rational number is aomless.

lemma4. We have in every Boolean algebra:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

$$p \subset P, q \subset Q, P \cap Q = \emptyset, P \cdot Q = 0 \rightarrow p + q + Q$$

Proof: $p + q \subseteq P + Q$.
To show:
$p + q \neq P + Q$ We assume
$p + q = P + Q$ so.
$(P + Q) \cdot \bar{p} = P \cdot \bar{p} + Q \cdot \bar{p}$
Because $Q \cap p = \emptyset$ we have $= P \cdot \bar{p} + Q = (p + q) \cdot \bar{p}$
$p \cdot \bar{p} + q \cdot \bar{p}$
$= q \qquad (q \cap p = \emptyset)$
$= q$
$(P + Q) \cdot \bar{p} \cdot \bar{q} = q \cdot \bar{q} = \emptyset$
$P \cdot \bar{p} \cdot \bar{q} + Q \cdot \bar{p} \cdot \bar{q} = 0 \Longleftrightarrow P \cdot \bar{q} + Q \cdot \bar{p} = 0$
$\Longleftrightarrow P \cdot \bar{q} = Q \cdot \bar{p} = 0$
Which contradicts with the assumption
lemma5. The following formulas are equivalent:

$\exists x(rx = 0 \wedge s\bar{x} = 0 \wedge \bigwedge_{i=1}^{m} u_i x \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{x} \neq 0)$
$rs = 0 \wedge \exists y(\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{s} \bar{y} \neq 0)$

Proof:
$\Longrightarrow$
If there is x such that,

$$rx = 0 \wedge s\bar{x} = 0 \wedge \bigwedge_{i=1}^{m} u_i x \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{x} \neq 0)$$
$$rx = 0 \wedge rs\bar{x} = 0 \Rightarrow rs(x + \bar{x}) = 0 \Rightarrow rs(1) = 0 \Rightarrow rs = 0$$
$$u_i x \neq 0 \Rightarrow u_i x(r + \bar{r}) \neq 0$$
$$\Rightarrow u_i xr + u_i x\bar{r} \neq 0$$
$$u_i x\bar{r} \neq 0$$
$$\Rightarrow \exists x(\bigwedge_{i=1}^{m} u_i \bar{r} x \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{s} \bar{x} \neq 0)$$

$\Longleftarrow$
Suppose $rs = 0$, there is y such that ,

$$\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{s} \bar{y} \neq 0$$

We put,

$$x = \bar{r} \cdot (s + y)$$
$$\bar{x} = r + \bar{s}\bar{y} = (r + \bar{s})(r + \bar{y})$$
$$\bar{s} \cdot (r + \bar{y})$$

We show,

$$\bigwedge_{i=1}^{m} u_i x \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{x} \neq 0$$
$$u_i x = u_i \bar{r}(s + y) = u_i \bar{r} s + u_i \bar{r} y \supseteq u_i \bar{r} y \neq 0$$
$$v_j \bar{x} = v_j \bar{s}(r + \bar{y}) = v_j \bar{s} r + v_j \bar{s} \bar{y} \supseteq v_j \bar{s} \bar{y} \neq 0$$

■

Theorem20. The theory of atomless Boolean algebra in the language $L = \langle 0, 1, \wedge, \vee, \neg, = \rangle$ accepts the quantifier elimination.
Proof:

$F = \{A | F_1 \wedge F_2 | F_1 \vee F_2 | \neg F | \exists F | \forall F\}$
$A = \{t_1 = t_2\}$
$T = \{x | 0 | 1 | t_1 \vee t_2 | t_1 \wedge t_2 | \neg t\}$

we have:
$t = s$ and $t \neq s$ so

$$t = \bigcup_{i \in I} (\bigcap_{j \in J} i, j)$$

such that i, j Is variable or complement variable.

. Terms included x:

$$x \cdot r + \bar{x} \cdot s$$

Atomic formulas:

$t = s \leftrightarrow t \cdot \bar{s} + \bar{t} \cdot s = 0$
$t \neq s \leftrightarrow t \cdot \bar{s} + \bar{t} \cdot s \neq 00$

Atomic formulas include x:

$r \cdot x + s \cdot \bar{x} = 0$
$\leftrightarrow r \cdot x = 0 \wedge s \cdot \bar{x} = 0$
$\leftrightarrow x \subseteq \bar{r} \wedge s \subseteq x \leftrightarrow sr = \emptyset$

Contradiction of atomic formulas include x:

$r \cdot x + s \cdot \bar{x} \neq 0 \leftrightarrow r \cdot x \neq 0 \vee s \cdot \bar{x} \neq 0$

so it is enough to eliminate quantifiers of the folloeing formulas:

$\exists x(rx = 0 \wedge s\bar{x} = 0 \wedge \bigwedge_{i=1}^{m} u_i x \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{x} \neq 0)$
$\equiv rs = 0 \wedge \exists y(\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{s} \bar{y} \neq 0)$

becuase:

$\Rightarrow$:

if there is x such that

$$rs = 0 \wedge \exists y(\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{s} \bar{y} \neq 0)$$

so

$rsx = 0, rs\bar{x} = 0 \rightarrow rs(x + \bar{x}) = 0 \rightarrow rs = 0$
$u_i x \neq 0 \Rightarrow u_i x(r + \bar{r}) \neq 0$
$\Rightarrow u_i xr + u_i x\bar{r} \neq 0$
$\Rightarrow u_i x\bar{r} \neq 0$
$v_j \bar{x} \neq 0 \rightarrow v_j \bar{x}(s + \bar{s}) \neq 0$
$\rightarrow v_j \bar{x} s + v_j \bar{x} \bar{s} \neq 0$
$\rightarrow v_j \bar{x} \bar{s} \neq 0$
$\rightarrow \exists x(\bigwedge_{i=1}^{m} u_i \bar{r} x \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{s} \bar{x} \neq 0)$

$\Leftarrow$:

we assume there was $rs \neq 0$ and y such that

$$\bigwedge_{i=1}^{m} u_i \bar{r} y \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{s} \bar{y} \neq 0$$

we put

$x = \bar{r}(s + y)$
$\bar{x} = r + \bar{s}\bar{y} = (r + \bar{s}) \cdot (r + \bar{y})$
$= \bar{s} \cdot (r + \bar{y})$

so $rx = 0, s\bar{x} = 0$ it is enough to show

$\bigwedge_{i=1}^{m} u_i x \neq 0 \wedge \bigwedge_{j=}^{n} v_j \bar{x} \neq 0$
$u_i x = u_i \bar{r}(s + \bar{y}) = u_i \bar{r} s + u_i \bar{r} y \supseteq u_i \bar{r} y$
$v_j \bar{x} = v_j \bar{s}(r + \bar{y}) = v_j \bar{s} r + v_j \bar{s} \bar{y} \supseteq v_j \bar{s} \bar{y} \neq 0$

.
so it suffices to eliminate the quantifier of the formula

$\exists y(\bigwedge_{i=1}^{m} a_i y \neq 0 \wedge \bigwedge_{j=}^{n} b_j \bar{y} \neq 0)$
$\equiv \bigwedge_{i=1}^{m} a_i \neq 0 \wedge \bigwedge_{j=}^{n} b_j \neq 0$

$\Longrightarrow$ it isobviously.
$\Longleftarrow$:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

we consider all cells C_α includin a_i, b_j
both cells are distinctly distinct $C_\alpha \cap C_\beta = \emptyset, C_\alpha \cdot C_\beta = 0$
each set is equall tocommunity of cells contained in it
$Z = \sum_{C_\alpha \subseteq Z} C_\alpha$ for all Z and any cell C we have $C \subseteq Z$ with $C \subseteq \bar{Z}$.

$$Z \neq 0 \leftrightarrow \exists \alpha (C_\alpha \subseteq Z \wedge C_\alpha \neq 0)$$

for any cells is not equall zero C_α from being atomless d_α there is such that

$$0 \neq d_\alpha C_\alpha \neq 0$$

if $C_\alpha = 0$ put $d_\alpha = 0$ we put it now $y = \sum_{C_\alpha \neq 0} d_\alpha$.

$a_i y \neq 0 : a_i y = a_i \sum_{C_\alpha \neq 0} d_\alpha = \sum_{C_\alpha \neq 0} a_i d_\alpha$
$\supseteq a_i d_\beta \supseteq c_\beta d_\beta = d_\beta \neq 0$
$0 \neq a_i = \sum_{C_\beta a_i} C_\beta$
$\exists \beta : C_\beta a_i \wedge C_\beta \neq 0$
$0 \neq d_\beta C_\beta \neq 0$
$b_j \bar{y} \neq 0 : b_j \bar{y} = b_j \prod_{C_\alpha \neq 0} \bar{d_\alpha} = \prod_{C_\alpha \neq 0} b_j \bar{d_\alpha}$

forall C_α we have $C_\alpha \subseteq b_j$ with $C_\alpha \subseteq \bar{b_j} \to \bar{C_\alpha} \supseteq b_j$
if $C_\alpha \neq 0$ then $d_\alpha C_\alpha \to \bar{d_\alpha} \supseteq \bar{C_\alpha} \supseteq b_j$ so $b_j \bar{d_\alpha} = b_j$

$b_j \bar{y} = \prod_{C_\alpha \subseteq b_j} b_j \bar{d_\alpha} = \prod_{0 \neq C_\alpha \subseteq b_j} b_j \bar{d_\alpha}$
$C_\alpha = 0 \to \bar{d_\alpha} = 1$
$C_\alpha \neq 0 \to 0 \neq d_\alpha C_\alpha \subseteq b_j$
$\sum_{0 \neq C_\alpha \subseteq b_j} d_\alpha \sum_{0 \neq C_\alpha \subseteq b_j} C_\alpha = b_j$
$b_j \sum \bar{d}_{apha} \neq 0$
$b_j \prod_{0 \neq C_\alpha \subseteq b_j} d_{alpha} \neq 0$
$b_j \bar{y} \neq 0$

The above proof we proved theory of Boolean algebras by the quantifier-elimination is decidable.
The first-order theory of Boolean algebras, established by Alfred Tarski in 1940 (found in 1940 but announced in 1949).

Theory of	Proved by	A method of proof
Boolean algebras	by Tarski in 1949	model completeness

$Table(X)$

G. Mereological structures

Mereology is sometimes understood as a formal, or logical, analysis of the part-to-whole relation.A theory of parts and wholes should tell us what items can be parts. Since something is a part only if it is a part of a whole, a mereology will tell us what items can be wholes.
Classic extensial mereology was intoduced by S.Lesniewskien 1916 and developed by him in the years thereafter.It was reformulatedas calculus of individualas by H.Leonard and N.Goodman in 1940.
Classic mereology is equivalent (isomorphic) to Boolean Algebra complete (without0) [4] .Mereology is about the relation of part to whole between objects or individuals.Any whole is a part of itself .And the empty object is a pat of every whole. Any part of a whole different from the empty part and from the whole itself is a proper part.
In section , we consider a set-theoretic version of mereology based on the inclusion relation $\subseteq$. We consider a mereological perspective in set theory.
Mereology, in set theory by means of the inclusion relation $\subseteq$, so that one set x is a part of another y , just in case x is a subset of y, written $x \subseteq y$.
The axioms of mereology are these of complete Boolean algebra , provided with the following interpretation:
$x \sqsubseteq y$:x is a part of y
$x \sqcup y$: Mereological sum or union of x and y
$x \sqcap y$: Mereological product or overlap of x and y
0: the empty individual
1:the universal individual
$1 - x$: complement of x, the universal individual minus x
The mereological sum coresponds to the join or union of Boolean algebra, which is supremum or least upper bound of two members of the algebra.Classical mereology accepts the mereological sum of any number of objects, without any restriction .It is because of this generosity that it costitues a complete Boolean algebra(i.e. a Boolean algebra in which every subset has a supermum).
An atom (in the mereological sense)is an object lacking proper parts.Classical mereology can be atomistic or atomless.[14,6]
Theorem 21.If $\langle W, \in^W \rangle$ is a model of set theory with the corresponding inclusion relation $\subseteq$, then $\langle W, \subseteq \rangle$ is an atomic unbounded relatively complemented distributive lattice, and this theory satisfies the elimination of quantifiers in the language containing the Boolean operations of intersection $x \cap y$, union $x \cup y$, relative complement xy and the unary size relations $|x| = n$, for each natural number n.[14]
Theorem. 22. Set-theoretic mereology, considered as the theory of $\langle V, \subseteq \rangle$, where V is the universe of all sets; is precisely the theory of an atomic unbounded relatively complemented distributive lattice, and furthermore, this theory is finitely axiomatizable, complete and decidable.[14]
Mereology is often contrasted with set theory and its membership relation, the relation of element to set.
Teorem 23. Let $\mathcal{A} = \langle A, +, \bullet, -, 0, 1 \rangle$, be a complete Boolean algebra. Assume $\sqsubseteq = \leq |_{A \setminus \{0\}}$,where the relation $\leq$ is defined by (def $\leq$). Then $\langle A \setminus \{0\}, \sqsubseteq \rangle$ is a mereological structure. After «adding» zero element to some mereological structure we will «turn» it into a complete Boolean algebra.
And the particular formulation of set-theoretic mereology via the inclusion relation $\subseteq$ is a decidable theory.

III. Conclusion

We could completey axiomatize the theoty $\langle Q^+; \sqsubseteq \rangle$, Indeed , the theory of the structure $\langle Q^+; \sqsubseteq \rangle$ is decidable. But we result decidability theorical, we leave open the

[4]It has been stated by Tarski and proved by Grzegorczyk tat: The models of mereology and models of complete Boolean algebra with zero deleted are identical

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 October 2022 doi:10.20944/preprints202210.0024.v1

problem of finding a $\langle Q^{+}; \sqsubseteq\rangle$ such that admits quantifier elimnation.The theory $\langle N; \times\rangle$ is decidable and axiomatizable.So the theory $\langle Z; \times\rangle$ we proved by mthods o decidability of $\langle N; \times\rangle$ is decidable and axiomatizable, and this paper we present an explicit axioomatization for the theory $\langle Z; \times\rangle$. And , in this paper, decidability (i.e., there exists an algorithm that decides whether a given sentence is derivable from the theory) of the structures study in different languages and introduce ways that it allows quantifier elimination (for the theory) and review some classical theorems and give for some of old results ,new proofs.

The Quantifier Elimination of the structure and decidability of them in different languages is shown in the following tables so that the theories that admit QE by $\surd$ and, the theories do not admit QE by $\times$ is shown.

	N	**Z**	**Q**	**R**	**C**
$\{+\}$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\{\times\}$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\{+, \times\}$	$\times$	$\times$	$\times$	$\surd$	$\surd$
$\{\sqsubseteq\}$	$\surd$	?	$\surd$	?	?

$Table(XI)$

References

[1] Salehi, Saeed, On Axiomatizability of the Multiplicative Theory of Numbers, Fundamenta Informaticae ,159:3 (2018) 279-296. DOI: 10.3233/FI-2018-1665

[2] Boolos, George S. Burgess, John P. Jeffrey, Richard C, Computability and Logic,Cambridge University Press (5th ed. 2007), ISBN: 9780521701464.

[3] Enderton, Herbert B, A Mathematical Introduction to Logic, Academic Press (2nd ed. 2001) , ISBN: 9780122384523.

[4] Salehi,Saeed,Axiomatizing Mathematical Theories :Multiplication, in:A.Kamali-Nejad (ed). Proceedings of Frontiers in Mathematical Sciences ,Sharif University of Technology .Tehran , Iran 2012 , pp.165-176.

[5] Cégielski, Patrick; The Elementary Theory of the Natural Lattice is Finitely Axiomatizable, Notre Dame Journal of Formal Logic 30:11:988138–150.

[6] D. Prawitz, Dag Westerståhl, Logic and Philosophy of Science in Uppsala, Springer Science Business Media, Tir 8, 1392 AP - Philosophy - 614 pages. .

[7] Givant Steven and Halmos Paul , Introduction to Boolean algebras, Undergraduate Texts in Mathematics. Springer, 2009

[8] Dutilh Novaes, Catarina; Formal Languages in Logic,November 2012, ISBN: 9781107020917.

[9] Christopher C Leary, Lars Kristiansen; A Friendly Introduction to Mathematical Logic, Milne Library, 2019.

[10] Smorynski, Craig; Logical Number Theory I, Springer-Verlag (1991).

[11] Clara Trullenque Ortiz, COMPLETE THEORIES OF BOOLEAN ALGEBRAS,Departament de Matemàtiques Informàtica,June 27, 2018.

[12] Burris, Stanley N., and H.P. Sankappanavar, CA Course in Universal Algebra,1981. Springer-Verlag. ISBN 3-540-90578-2 Free online edition.

[13] Cégielski, Patrick, Théorie Elémentair de La Multiplication des Entiers Naturals, Model Theory and Arithmetic, Proc., Paris 1979/1980, Lect.Notes Math .890,44-89(1981).

[14] M. E. Habič, J. D. Hamkins, L. D. Klausner, J. Verner, and K. J. Williams, “Set-theoretic blockchains,” Archive for Mathematical Logic, 2019.

Zahra Sheikhaleslami Ph.D. Student of Tabriz University. The University of Tabriz P.O.Box 51666-16471,Bahman 29thBoulevard; Tabriz; Iran.
zahra.sheikhaleslami@gmail.com