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## Indefinite Halmos, Egervary and Sz.-Nagy Dilations

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**Abstract:** Dilation of contractions on Hilbert space and Banach space is classical. Recently, dilations theory has been put in the setting of sets, vector spaces, p-adic Hilbert spaces and modules. In this paper, we derive important dilation results for self-adjoint morphisms on indefinite inner product modules over  $\ast$ -rings of characteristic 2. More precisely, we prove indefinite inner product versions of Halmos dilation, Egervary N-dilation and Sz.-Nagy dilation.

**Keywords:** Dilation; Indefinite inner product space; Module

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### 1. INTRODUCTION

In 1950, Halmos [22] made a deep insight into structure theory of operators on Hilbert space by exhibiting any contraction as a part of a unitary. In 1953, Sz.-Nagy [39] showed that Halmos result can be extended to powers of contractions using a unitary operator. In 1963, T. Ando [5] showed that there is a version of Sz.-Nagy dilation for commuting contractions. Combined with spectral theory and theory of (several) complex variables, today, dilation theory of contractions is a rapidly evolving area of research and for a comprehensive look, we refer [1, 4–7, 9–16, 19–21, 27, 28, 31–37, 40–43]. Started in 1970's, dilations of contractions acting on Lebesgue spaces and Banach spaces followed Hilbert space developments [2, 3, 17, 18, 24, 30, 38].

In 2021, by identifying essential mechanisms of dilation theory, Bhat, De and Rakshit [8] obtained surprising results in the set theory context and vector spaces. In 2022, further study in the context of vector spaces was carried by Krishna and Johnson [26]. We note that another vector space variant is also studied by Han, Larson, Liu and Liu [23]. Recently Krishna introduced the notion of magic contractions and derived Sz.-Nagy dilation for p-adic Hilbert spaces and modules [25].

In this paper, we derive indefinite inner product module versions of Halmos dilation (Theorem 2.2), Egervary N-dilation (Theorem 2.3), Sz.-Nagy dilation (Theorem 2.4). Our article is highly motivated from the paper of Halmos [22], Egervary [16], Schaffer [36], Sz.-Nagy [39], Bhat, De and Rakshit [8], Krishna and Johnson [26] and Krishna [25].

### 2. INDEFINITE HALMOS, EGERVARY AND SZ.-NAGY DILATIONS

We are going to use the following notions. A ring  $\mathcal{R}$  with an automorphism  $\ast$  which is either identity or of order 2 is called as an  $\ast$ -ring. Throughout the paper we assume that characteristic of ring is 2.

**Definition 2.1.** [29] Let  $\mathcal{V}$  be a module over  $\mathcal{R}$ . We say that  $\mathcal{V}$  is an indefinite inner product module (we write IIPM) if there is a map (called as indefinite inner product)  $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{R}$  satisfying following.

- (i) If  $x \in \mathcal{V}$  is such that  $\langle x, y \rangle = 0$  for all  $y \in \mathcal{V}$ , then  $x = 0$ .
- (ii)  $\langle x, y \rangle = \langle y, x \rangle^*$  for all  $x, y \in \mathcal{V}$ .
- (iii)  $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$  for all  $\alpha \in \mathcal{R}$ , for all  $x, y, z \in \mathcal{V}$ .

Let  $\mathcal{V}$  be a IIPM and  $T : \mathcal{V} \rightarrow \mathcal{V}$  be a morphism. We say that  $T$  is adjointable if there is a morphism, denoted by  $T^* : \mathcal{V} \rightarrow \mathcal{V}$  such that  $\langle Tx, y \rangle = \langle x, T^*y \rangle$ ,  $\forall x, y \in \mathcal{V}$ . Note that (i) in Definition 2.1 says that adjoint, if exists, is unique. An adjointable morphism  $U$  is said to be a unitary if  $UU^* = U^*U = I_{\mathcal{V}}$ , the identity operator on  $\mathcal{V}$ . An adjointable morphism  $P$  is said to be projection if  $P^2 = P^* = P$ . An adjointable morphism  $T$  is said to be an isometry if  $T^*T = I_{\mathcal{V}}$ . An adjointable morphism  $T$  is said to be self-adjoint if  $T^* = T$ . We denote the identity operator on  $\mathcal{V}$  by  $I_{\mathcal{V}}$ .

Our first result is the indefinite Halmos dilation.

**Theorem 2.2. (Indefinite Halmos dilation)** *Let  $\mathcal{V}$  be a IIPM over a \*-ring of characteristic 2 and  $T : \mathcal{V} \rightarrow \mathcal{V}$  be a self-adjoint morphism. Then the morphism*

$$U := \begin{pmatrix} T & I_{\mathcal{V}} + T \\ I_{\mathcal{V}} + T & T \end{pmatrix}$$

is unitary on  $\mathcal{V} \oplus \mathcal{V}$ . In other words,

$$T = P_{\mathcal{V}}U|_{\mathcal{V}}, \quad T^* = P_{\mathcal{V}}U^*|_{\mathcal{V}},$$

where  $P_{\mathcal{V}} : \mathcal{V} \oplus \mathcal{V} \ni (x, y) \mapsto x \in \mathcal{V}$ .

*Proof.* A direct calculation says that

$$V := \begin{pmatrix} T & I_{\mathcal{V}} + T \\ I_{\mathcal{V}} + T & T \end{pmatrix}$$

is the inverse and adjoint of  $U$ . □

Our second result is the indefinite Egervary N-dilation.

**Theorem 2.3. (Indefinite Egervary N-dilation)** *Let  $\mathcal{V}$  be a IIPM over a \*-ring of characteristic 2 and  $T : \mathcal{V} \rightarrow \mathcal{V}$  be a self-adjoint morphism. Let  $N$  be a natural number. Then the morphism*

$$U := \begin{pmatrix} T & 0 & 0 & \cdots & 0 & 0 & I_{\mathcal{V}} + T \\ I_{\mathcal{V}} + T & 0 & 0 & \cdots & 0 & 0 & T \\ 0 & I_{\mathcal{V}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_{\mathcal{V}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & I_{\mathcal{V}} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_{\mathcal{V}} & 0 \end{pmatrix}_{(N+1) \times (N+1)}$$

is unitary on  $\oplus_{k=1}^{N+1} \mathcal{V}$  and

$$(1) \quad T^k = P_{\mathcal{V}}U^k|_{\mathcal{V}}, \quad \forall k = 1, \dots, N, \quad (T^*)^k = P_{\mathcal{V}}(U^*)^k|_{\mathcal{V}}, \quad \forall k = 1, \dots, N,$$

where  $P_{\mathcal{V}} : \oplus_{k=1}^{N+1} \mathcal{V} \ni (x_k)_{k=1}^{N+1} \mapsto x_1 \in \mathcal{V}$ .

*Proof.* A direct calculation of power of  $U$  gives Equation (1). To complete the proof, now we need show that  $U$  is unitary. Define

$$V := \begin{pmatrix} T & I_{\mathcal{V}} + T & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_{\mathcal{V}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{\mathcal{V}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & I_{\mathcal{V}} \\ I_{\mathcal{V}} + T & T & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}_{(N+1) \times (N+1)}.$$

Then  $UV = VU = I_{\oplus_{k=1}^{N+1} \mathcal{V}}$  and  $U^* = V$ .  $\square$

Note that the Equation (1) holds only upto  $N$  and not for  $N + 1$  and higher natural numbers. In the following theorem, given a IIPM  $\mathcal{V}$ ,  $\oplus_{n=-\infty}^{\infty} \mathcal{V}$  is the IIPM defined by

$$\oplus_{n=-\infty}^{\infty} \mathcal{V} := \{ \{x_n\}_{n=-\infty}^{\infty}, x_n \in \mathcal{V}, \forall n \in \mathbb{Z}, x_n \neq 0 \text{ only for finitely many } n's \}$$

equipped with inner product

$$\langle \{x_n\}_{n=-\infty}^{\infty}, \{y_n\}_{n=-\infty}^{\infty} \rangle := \sum_{n=-\infty}^{\infty} \langle x_n, y_n \rangle, \quad \forall \{x_n\}_{n=-\infty}^{\infty}, \{y_n\}_{n=-\infty}^{\infty} \in \oplus_{n=-\infty}^{\infty} \mathcal{V}.$$

Our third result is the indefinite Sz.-Nagy dilation.

**Theorem 2.4. (Indefinite Sz.-Nagy dilation)** Let  $\mathcal{V}$  be a IIPM over a  $*$ -ring of characteristic 2 and  $T : \mathcal{V} \rightarrow \mathcal{V}$  be a self-adjoint morphism. Let  $U := (u_{n,m})_{-\infty \leq n, m \leq \infty}$  be the morphism defined on  $\oplus_{n=-\infty}^{\infty} \mathcal{V}$  given by the infinite matrix defined as follows:

$$\begin{aligned} u_{0,0} &:= T, & u_{0,1} &:= I_{\mathcal{V}} + T, & u_{-1,0} &:= I_{\mathcal{V}} + T, & u_{-1,1} &:= T, \\ u_{n,n+1} &:= I_{\mathcal{V}}, & \forall n \in \mathbb{Z}, n \neq 0, 1, & & u_{n,m} &:= 0 & \text{otherwise,} \end{aligned}$$

i.e.,

$$U = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & I_{\mathcal{V}} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & I_{\mathcal{V}} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & I_{\mathcal{V}} + T & T & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{T} & I_{\mathcal{V}} + T & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I_{\mathcal{V}} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & I_{\mathcal{V}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{\infty \times \infty}$$

where  $T$  is in the  $(0,0)$  position (which is boxed), is unitary on  $\oplus_{n=-\infty}^{\infty} \mathcal{V}$  and

$$(2) \quad T^n = P_{\mathcal{V}} U^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{N}, \quad (T^*)^n = P_{\mathcal{V}} (U^*)^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{N},$$

where  $P_{\mathcal{V}} : \oplus_{n=-\infty}^{\infty} \mathcal{V} \ni (x_n)_{n=-\infty}^{\infty} \mapsto x_0 \in \mathcal{V}$ .

*Proof.* We get Equation (2) by calculation of powers of  $U$ . The matrix  $V := (v_{n,m})_{-\infty \leq n,m \leq \infty}$  defined by

$$\begin{aligned} v_{0,0} &:= T, & v_{0,-1} &:= I_{\mathcal{V}} + T, & v_{1,0} &:= I_{\mathcal{V}} + T, & v_{1,-1} &:= T, \\ v_{n,n-1} &:= I_{\mathcal{V}}, & \forall n \in \mathbb{Z}, n \neq 0, 1, & & v_{n,m} &:= 0 & \text{otherwise,} \end{aligned}$$

i.e.,

$$V = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & I_{\mathcal{V}} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & I_{\mathcal{V}} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & I_{\mathcal{V}} + T & \boxed{T} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & T & I_{\mathcal{V}} + T & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I_{\mathcal{V}} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & I_{\mathcal{V}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{\infty \times \infty}$$

where  $T$  is in the  $(0,0)$  position (which is boxed), satisfies  $UV = VU = I_{\oplus_{n=-\infty}^{\infty} \mathcal{V}}$  and  $U^* = V$ .  $\square$

We note that explicit sequential form of  $U$  is

$$U(x_n)_{n=-\infty}^{\infty} = (\dots, x_{-2}, x_{-1}, (I_{\mathcal{V}} + T)x_0 + Tx_1, \boxed{Tx_0 + (I_{\mathcal{V}} + T)x_1}, x_2, x_2, \dots)$$

where  $Tx_0 + (I_{\mathcal{V}} + T)x_1$  is in the 0 position (which is boxed) and  $U^*$  is

$$U^*(x_n)_{n=-\infty}^{\infty} = (\dots, x_{-3}, x_{-2}, \boxed{(I_{\mathcal{V}} + T)x_{-1} + Tx_0}, Tx_{-1} + (I_{\mathcal{V}} + T)x_0, x_1, \dots),$$

where  $(I_{\mathcal{V}} + T)x_{-1} + Tx_0$  is in the 0 position (which is boxed). We next wish to derive indefinite isometric Sz.-Nagy dilation.

**Theorem 2.5. (Indefinite isometric Sz.-Nagy dilation)** Let  $\mathcal{V}$  be a IIPM over a  $*$ -ring of characteristic 2 and  $T : \mathcal{V} \rightarrow \mathcal{V}$  be a self-adjoint morphism. Let  $U := (u_{n,m})_{0 \leq n,m \leq \infty}$  be the morphism defined on  $\oplus_{n=0}^{\infty} \mathcal{V}$  given by the infinite matrix defined as follows:

$$u_{0,0} := T, \quad u_{2,1} := I_{\mathcal{V}} + T, \quad u_{n+1,n} := I_{\mathcal{V}}, \quad \forall n \geq 2, \quad u_{n,m} := 0 \quad \text{otherwise,}$$

i.e.,

$$U = \begin{pmatrix} \boxed{T} & 0 & 0 & 0 & 0 & 0 & \cdots \\ I_{\mathcal{V}} + T & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & I_{\mathcal{V}} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & I_{\mathcal{V}} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & I_{\mathcal{V}} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & I_{\mathcal{V}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{\infty \times \infty}$$

where  $T$  is in the  $(0,0)$  position (which is boxed), is isometry on  $\oplus_{n=0}^{\infty} \mathcal{V}$  and

$$(3) \quad T^n = P_{\mathcal{V}} U^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{N}, \quad (T^*)^n = P_{\mathcal{V}} (U^*)^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{N},$$

where  $P_{\mathcal{V}} : \oplus_{n=0}^{\infty} \mathcal{V} \ni (x_n)_{n=0}^{\infty} \mapsto x_0 \in \mathcal{V}$ .

*Proof.* It suffices to note the adjoint of  $U$  is

$$U^* = \begin{pmatrix} \boxed{T} & I_{\mathcal{V}} + T & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & I_{\mathcal{V}} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & I_{\mathcal{V}} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & I_{\mathcal{V}} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & I_{\mathcal{V}} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\infty \times \infty}$$

where  $T$  is in the  $(0, 0)$  position (which is boxed). □

We now formulate following problems.

**Problem 2.6.**

- (i) *Whether there is an indefinite Ando dilation? If yes, whether one can dilate commuting three, four, ... commuting self-adjoint morphisms to commuting unitaries?*
- (ii) *Whether there is (a kind of) uniqueness of indefinite Halmos dilation?*
- (iii) *Whether there is a indefinite intertwining-lifting theorem (commutant lifting theorem)?*

### 3. CONCLUSIONS

- (1) In 1950, Halmos showed that every contraction on a Hilbert space can be lifted to a unitary [22].
- (2) In 1953, Sz.-Nagy derived his dilation theorem [39].
- (3) In 1955, Schaffer gave simple proof of Sz.-Nagy dilation result [36].
- (4) In 1963, Ando showed that Sz.-Nagy dilation holds for two commuting contractions [5].
- (5) In 1973, Stroescu studied dilations of contractions on Banach spaces [38].
- (6) In 2021, Bhat, De and Rakshit introduced set theoretic and vector space approach to dilation theory [8]. Later, Krishna and Johnson continued this study in 2022 [26].
- (7) In 2022, Krishna derived Halmos, Egervary and Sz.-Nagy dilation results for magic contractions on p-adic Hilbert spaces and modules [25].
- (8) In this paper, we derived Halmos, Egervary and Sz.-Nagy dilation results for indefinite inner product modules over \*-rings of characteristic 2.

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