*Article*

# Geometric method: a novel fast and accurate solution for the inverse problem in Risley prisms

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**Abstract:** Today, mechanical tracking systems have been downsized to allow them to be used in the field of airborne laser communications and in the military domain. Risley systems are used for this purpose, which work by directing a beam of light to a given target point, this procedure is commonly known as the inverse problem. In this paper, an analytical method, the geometric method, has been designed and developed to determine the beam steering in a Risley system and solve the inverse problem. The method focuses on different geometric shapes, like circumference or ellipse, that are described when the beam passes through the second prism. The accuracy and efficiency of the geometric method has been analysed and found to be faster than the two-step method. Furthermore, the geometric method has been implemented in an iterative process and an accuracy of up to 1 pm has been achieved. This high accuracy would allow the geometric method to be applied in fields such as lithography, stereolithography or 3D printers.

**Keywords:** Geometrical optics, Risley prism, inverse solution, rotational wedges.

## 1. Introduction

Beam scanners make it possible to obtain the geometry of the scanned environment from massive data storage. Several beam scanners are available like gimbal scanners, galvanometers scanners and Risley prisms scanner. Traditional carried-axis gimbal scanners can be heavy and subsequently require more driving energy, non-carried-axis galvanometers have limited aperture size while Risley scanners can offer a lighter, more compact, vibration-insensitive scanning option. This paper is based on the latest laser scanner.

The Risley configuration consists of a combination of two or more Wedge prisms [1] that can rotate independently, **Figure 1**. The system allows to set the direction of the beam $\vec{S}$ to a specific point in space, and to generate different patterns by setting the relative velocity between the prisms.

Some of the applications of the Risley configuration are in LiDAR technology [2,3] to scan Earth's surface with different scanning patterns or to collect wind field data [4] from an aircraft. In addition, the Risley configuration allows the development of systems with a wide field of view, multi-target tracking and angular scanning of the DMD (Digital Micro-mirror Device) [5].

Two problems arise in the Risley configuration, the direct and the inverse problem. The direct problem is to determine the deviation of the laser beam direction given a set of prism orientation angles. This problem is commonly solved by the Forward Vector Refraction Theorem [6]. However, the inverse problem is more complex, based on calculating the rotation angles needed to reach any region of space within the allowed limits of the system. Some research groups have solved it using an iterative process, which presents a high precision solution [6,7], the third-order solution [8,9], the two-step method [10–12], the look-up table method [13], the control system based on Newton's method [14] and the error compensation method based on the paraxial approximation [15], among others.

This paper proposes a fast and direct inverse solution for applications that require a quick response and high accuracy. Two approximations of a mathematical method based on geometry are developed and compared with the two-step method. Later, the developed mathematical method has been implemented in an iterative method.

The organization of the paper is as follows: Section 2 expounds the method that offers an inverse solution for the Risley prisms. Section 3 shows the results obtained from a time and accuracy analysis of the developed method and compares the two approximations of the method with each other and with other methods developed by other authors. Conclusions are drawn in Section 4.

## 2. Inverse solution for Risley prism

Given the generic Risley configuration system shown in **Figure 1**, the two prisms $\pi_1$ and $\pi_2$ are rotated independently about the Z axis. The refractive index of the prisms has been denoted $n_1$ and $n_2$ and the wedge angle of the prisms, $\alpha_1$ and $\alpha_2$. Orientations of the prisms are specified by the respective rotation angles $\theta_1$ and $\theta_2$. The distance between both prisms is D1 and the distance between the emergent surface of prism $\pi_2$ and the receiving screen is D2.

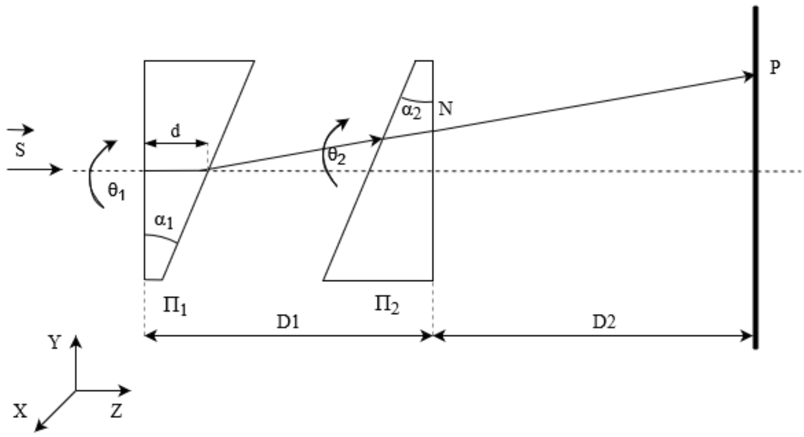


**Figure 1.** Risley system diagram.

Assuming the prisms are identical and have the same characteristics:

$$n_1 = n_2 = n$$
$$\alpha_1 = \alpha_2 = \alpha \quad (1)$$

The proposed method consists of modelling the pattern described by a beam when the set of prisms rotate.

When a beam passes through the first prism $\pi_1$ of the Risley system a circumference is described and when the beam passes through the second prism $\pi_2$ different geometric shapes can be approximated depending on the wedge angle of the prisms, **Figure 2**.

For low wedge angles of the prisms the geometric shape can be approximated to a circumference while for high wedge angles the geometric shape is closer to an ellipse.

Knowing the equation that define the geometric shape described by the beam when it passes through the second prism $\pi_2$, the intersection points with the circumference described by the beam when it passes through the first prism $\pi_1$ of the system can be calculated. In this way, by obtaining intersection points, the angles of rotation of the desired prisms can be calculated. This process has been called the 'geometric method'.

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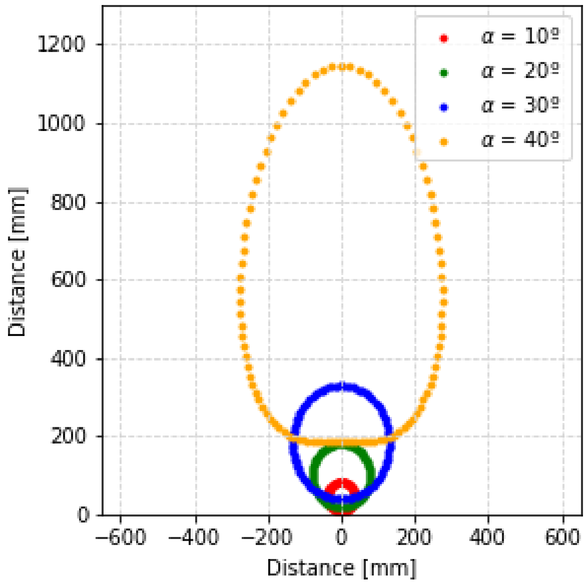


**Figure 2.** Representation of the different geometric shapes that depend on the wedge angle of the prisms.

In this article, two approximations to the geometric shapes described by the beam are proposed. A fast but less accuracy approximation (the circumference) and a slow but higher accuracy approximation (the ellipse). Then, the geometric method has been implemented into an iterative process to improve accuracy.

*2.1 Circumference approximation*

In this approximation, **Figure 3**, the geometric shape that describes the beam when the second prism $\pi_2$ rotates approximates a circumference $c_2$ with centre at any point of the circle $c_1$. The circumference $c_1$ is described by the beam when the first prism $\pi_1$ rotates and there is no second prism $\pi_2$, with primary radius $r_1$ and centre at the point of origin.

Using the Forward Vector Refraction Theorem, the maximum and minimum radius that the system can reach at a given distance D2 can be calculated. The maximum radius is obtained when both prisms are oriented with the same angle of rotation and the minimum radius is obtained when the prisms are out of phase 180º. Once both radii have been calculated, primary and secondary radius can be obtained:

$$r_1 = r_2 + min\_radius$$
$$r_2 = (max\_radius - min\_radius)/2 \tag{2}$$

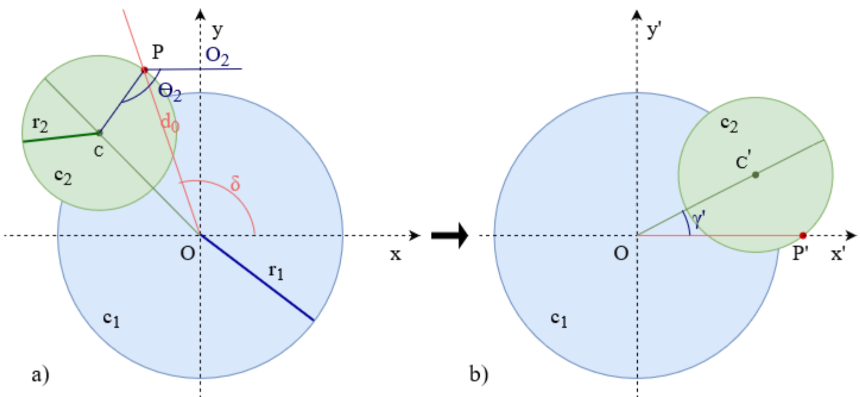

**Figure 3.** Circumference approximation diagram.

Given the target point P to be reached with coordinates $(x_0, y_0)$, a change of axes through the angle δ is made, **Figure 3**, so that the target point P is on the X' axis, now becoming P'$(x_0', y_0')$.

Considering $y_0' = 0$ and solving the system of equations formed by the equations (3) that define the radius $(r_1, r_2)$, the centre point $C'(x_c', y_c')$ is obtained.

$$r_1 = \sqrt{x_c'^2 + y_c'^2}$$
$$r_2 = \sqrt{(x_c' - x_0')^2 + y_c'^2} \quad (3)$$

Knowing the centre point C′, it is possible to calculate the angle $\gamma'$:

$$\gamma' = acos(x_c'/r_1) = acos\big((r_1^2 - r_2^2 + x_0'^2)/2r_1x_0'\big) \quad (4)$$

Finally, undoing the change of coordinate axes, the rotation angle of the first prism $\theta_1$ is:

$$\theta_1 = \gamma' + \delta \quad (5)$$

Where $\delta$ is:

$$\delta = acos\left(x_0/\sqrt{x_0^2 + y_0^2}\right) \quad (6)$$

The centre point of the circumference $C'(x_c', y_c')$ on the X' and Y' coordinate axes is isolated from (4) to obtain the rotation angle of the second prism $\theta_2$.

$$x_c' = r_1 \cdot cos\,(\gamma')$$
$$y_c' = r_1 \cdot sin\,(\gamma') \quad (7)$$

Translating the central point C′ to the X and Y coordinate axes:

$$x_c = x_c' \cdot cos(\delta) - y_c' \cdot \sin(\delta)$$
$$y_c = x_c' \cdot sin(\delta) + y_c' \cdot cos(\delta) \quad (8)$$

The rotation angle of the second prism $\theta_2$ is the angle of the section formed by the points $(C, P, O_2)$ in **Figure 3.**

$$\theta_2 = acos\left((x_0 - x_c)/\sqrt{-x_0 + x_c)^2 + (-y_0 + y_c)^2}\right) \quad (9)$$

*2.2 Ellipse approximation*

The second approximation is based on the geometric shape described by the beam is an ellipse. Using the properties of eccentricity and the focal distance of the ellipse it is possible to obtain the rotation angles of the prisms. To obtain the rotation angles of the prisms is necessary to calculate the value of the semi-axis $a$, which depends on the wedge angle of the prisms $\alpha$ and the distance D2.

2.2.1 Calculation of the semi-axis a

The value of the semi-axis *b* is the value of the secondary radius $r_2$. On the other hand, the value of the semi-axis *a* must be optimised so that the error between the points calculated by the Forward Vector Refraction Theorem and by the parametric equations of the ellipse for a specific distance D2 must be minimised. The value of the semi-axis *a* can be optimised with different optimisation methods such as: Nelder-Mead, Powell…

Once the value of the semi-axis *a* has been optimised for the specific distance D2, the incident angle $\varepsilon$ of the laser beam on the screen can be obtained:

$$\varepsilon = atan(a/D2) \tag{10}$$

The incident angle $\varepsilon$ is constant for the optic system configuration and it does not depend on the distance D2′, so it is not necessary to repeat the optimization process. It is possible to calculate *a′* for any distance D2′ as:

$$a' = D2' \cdot tan\,(\varepsilon) \tag{11}$$

Once the value of the semi-axis *a* has been obtained, the angles of rotation of the prisms can be calculated.

2.2.2 Calculation of the rotation angles of the prisms

In this approximation, the geometric shape that describe the beam, when the second prism $\pi_2$ rotates, approximates an ellipse $e_1$ with centre at any point of the circle $c_1$.

Since the sum of the focal distance of any point P on an ellipse is constant and equal to the length of the major axis of the ellipse (*b*):

$$PF1 + PF2 = 2b \rightarrow d_1 + d_2 = 2b \tag{12}$$

For simplicity, a rotation of the coordinate axes through the angle δ is made, **Figure 4,** so that the target point P is on the X' axis, being now the point P'$(x_0', y_0')$.

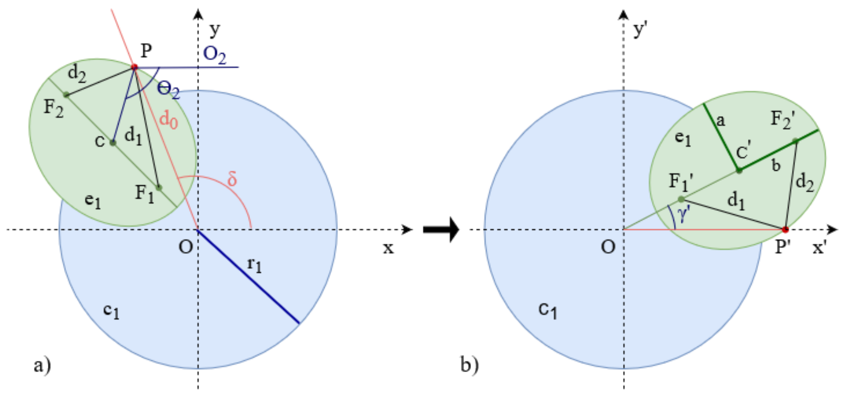


**Figure 4.** Rotation of the coordinate axes. a) The diagram is represented on the original axes (X, Y). b) The diagram is represented on the rotated axes (X′, Y′).

Knowing the target point *P′* and the points of the focal F1′$(x_1', y_1')$ and F2′$(x_2', y_2')$, the distance between them is calculated as:

$$d_1 = \sqrt{(x_1' - x_0')^2 + (y_1' - y_0')^2}$$
$$d_2 = \sqrt{(x_2' - x_0')^2 + (y_2' - y_0')^2} \tag{13}$$

Where,

$$x_1' = r_{F1} \cdot cos(\gamma')$$
$$y_1' = r_{F1} \cdot sin(\gamma') \tag{14}$$

$$x_2{}' = r_{F2} \cdot cos(\gamma')$$
$$y_2{}' = r_{F2} \cdot sin(\gamma')$$

The radius $r_{F1}$ and $r_{F2}$ is the radius of the circle formed with centre at the origin and passing through the focus point F1 and the focus point F2, respectively. The focus points (F1 and F2) are at the same distance from the primary radius. This distance is $\sqrt{b^2 - a^2}$.

Considering $y_0' = 0$ and with the equations (12), (13) and (14), it is possible to obtain the solution of the angle $\gamma'$ between the X′-axis and the line that pass through the centre point C′ of the ellipse:

$$\gamma' = -acos(-(c_4 - 4b \cdot c_7 + c_5)/x_0' \cdot c_3) \tag{15}$$

Where,

$$c_1 = -4b^2 \cdot r_{F1} - 4b^2 \cdot r_{F2}$$
$$c_2 = r_{F1}{}^3 - r_{F1}{}^2 \cdot r_{F2} - r_{F1} r_{F2}{}^2 + r_{F2}{}^3$$
$$c_3 = 2 \cdot (r_{F1} - r_{F2})^2$$
$$c_4 = -c_1$$
$$c_5 = -c_2$$
$$c_6 = 4b^2 r_{F1} r_{F2} - r_{F1}{}^3 r_{F2} + 2 r_{F1}{}^2 r_{F2}{}^2 - r_{F1} r_{F2}{}^3$$
$$c_7 = \sqrt{c_6 + r_{F1}{}^2 \cdot x_0{}'^2 - 2 r_{F1} \cdot r_{F2} \cdot x_0{}'^2 + r_{F2}{}^2 \cdot x_0{}'^2}$$

Finally, the rotation angle of the first prism $\theta_1$ and the second prism $\theta_2$ can be following the same process as the circumference approximation (undoing the change of coordinate axes) with (5), (6), (7), (8) and (9).

*2.2 Iterative method with geometric method*

To improve the accuracy to reach the target point, Anhu Li developed an iterative method [6] consisting of a combination of the two-step method [10] and the Forward Vector Refraction Theorem. In this paper, the two-step method has been replaced by the geometric method in the iterative method.

In the first iteration, the exiting point of the system is assumed to be the centre $N\ (0, 0, D1)$ of the second surface of the prism $\Pi_2$. Given a target point $P\ (x_p, y_p, z_p)$, it is possible to calculate the rotation angles $(\theta_1, \theta_2)$ of the prisms by the geometric method. Then, substituting the rotation angles into the Forward Vector Refraction Theorem, the corresponding exiting point $N = (x_n, y_n, z_n)$ and the tracking point $P_r = (x_{rp}, y_{rp}, z_{rp})$ is obtained. The tracking point $P_r$ has a deviation $(\varphi_x, \varphi_y)$ from the target point P, being $\varphi_x = x_p - x_{rp}$ y $\varphi_y = y_p - y_{rp}$.

In the second iteration, taking the point $P_i = (x_i, y_i, z_i) = (x_p + \varphi_x, y_p + \varphi_y, D2)$ as the target point of the system, it is possible to calculate the rotation angles $(\theta_1, \theta_2)$ of the prisms by the geometric method. This process is iterated until the separation distance between tracking point $P_r$ and target point $P$ meets the given accuracy requirement $\delta$.

The steps to implement the geometric method in the iterative method are:

**Step 1.** Given a target point $P\ (x_p, y_p, z_p)$, assume that the initial point is $N\ (x_n, y_n, z_n) = (0, 0, D1)$ on prism $\Pi_2$. Also, it is assumed that the deviation $(\varphi_x, \varphi_y)$ is zero.

**Step 2.** Calculate the rotation angles $(\theta_1, \theta_2)$ of the two prisms using the geometric method by considering the initial point $P_i = (x_i, y_i, z_i) = (x_p + \varphi_x, y_p + \varphi_y, D2)$ as the point to be reached.

**Step 3.** Update the current exiting point $N = (x_n, y_n, z_n)$ and the current target point $P_r = (x_{rp}, y_{rp}, z_{rp})$ substituting the rotation angles into the Forward Vector Refraction Theorem.

**Step 4.** Calculate the error $\Delta$ between the tracking point and the current target point on the receiving screen as: $\Delta = \sqrt{(x_{rp} - x_p)^2 + (y_{rp} - y_p)^2 + (z_{rp} - z_p)^2}$. If the error is less than the error threshold ($\Delta < \delta$) the rotation angle ($\theta_1, \theta_2$) are the final rotation angle. If not, update deviation $\varphi_x = x_i - x_{rp}$, $\varphi_y = y_i - y_{rp}$ and go to step 2.

This procedure is the same that the iterative method developed by Anhu Li except for the step 2.

## 3. Results of the geometric method

In order to evaluate how good a method is, it is necessary to evaluate its accuracy and efficiency. In this section, an analysis of these two aspects of the geometric method and the two-step method will be made to compare them. In addition, an analysis of time and number of iterations of the iterative method with the geometric method and with the two-step method will be made to compare them.

The parameters of two identical prisms used to obtain the results are: refractive index n = 1.517; the diameter of the prisms is 25.4 mm; the distance between two prisms D1 = 40 mm; the thickness of the thinnest end $d_0$ = 2.75 mm.

There are three parameters that have been modified to analyse the behaviour of the approximations under different conditions. The distance between the emergent surface of prism $\pi_2$ and the receiving screen D2 has taken values between 500 mm and 16.384 km, the wedge angle $\alpha$ has been changed between 2º and 40º and the error threshold δ for the iterative method has taken values between $0.1\ \mu m$ and $1\ pm$.

The computer used to obtain the following results has an AMD Ryzen 5 CPU and 8 GB memory, and the code has been developed in C++.

### *3.1. Comparative of the error obtained by geometric method and the two-step method*

The accuracy in the direction of the laser beam to any point in space varies radially and also depends on the wedge angle $\alpha$. To analyse the accuracy of the method the error is calculated, where the error is the difference between the target point and the point that has been reached with the method used.

The **Figure 5** (a)(c)(e) shows that the error increases when the wedge angle is larger for the three methods. In addition, three wedge angle values have been selected to observe their profile as a function of the radius, **Figure 5**(b)(d)(f). The two-step method (**Figure 5**(e)(f)) has a radial behaviour, the error increases as the radius increases. However, in the geometric method (**Figure 5**(a-d)) the minimum error is obtained at the minimum and maximum radius. This is because the laser beam does not actually describe a circumference, it is closer to an ellipse and the radius of the circumference is the same to the semi-axis $b$ of the ellipse. Moreover, in the ellipse approximation (**Figure 5**(c)(d)) the error has been minimized so that when the radius is 65% of the maximum radius, a minimum error occurs.

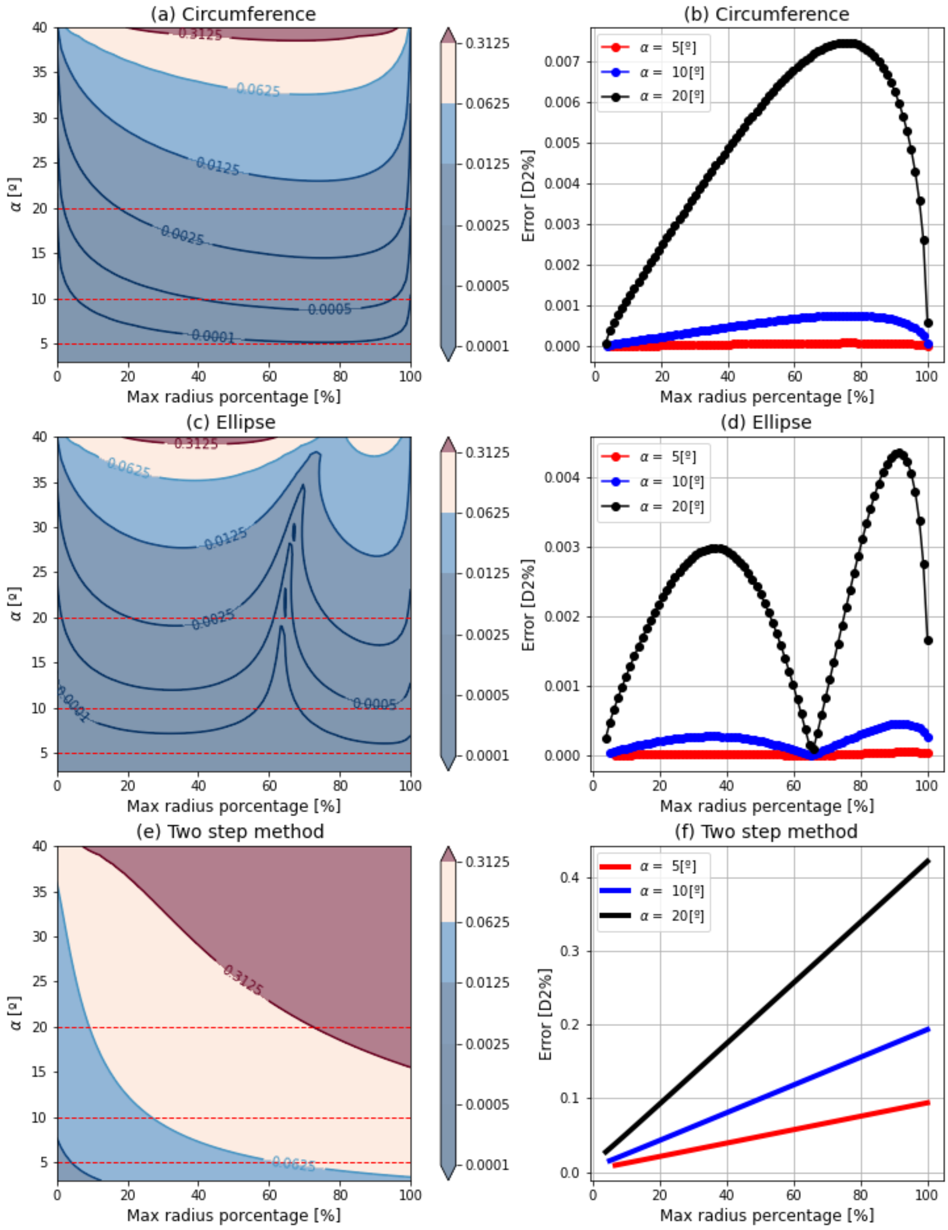


**Figure 5.** Error of the two approximations for different values of wedge angle $\alpha$ and radius of the target point. (a) Represents a heat map with the error obtained by the circumference approximation. (b) The graph represents the error profile obtained by the circumference approximation. (c) Represents a heat map with the error obtained by the ellipse approximation. (d) The graph represents the error profile obtained by the ellipse approximation. (e) Represents a heat map with the error obtained by the two-step method. (f) The graph represents the error profile obtained by the two-step method.

In the three methods the error depends on the wedge angle of the prisms $\alpha$, however, the error as a percentage of the distance D2 does not change even if the distance increases. Considering a target distance (D2) of 1 m, the **Figure 6** shows the mean error as a percentage of the distance D2 and the shadow regions correspond to the standard deviation of the measurements for each wedge angle of the prisms. The mean error has been obtained from a set of eighty points. This set of points has been obtained on a straight line for an azimuth angle of 45°, as the error is radial it does not matter which angle is considered.

It can be seen from **Figure 6** that the error obtained by the ellipse approximation is smaller than the error obtained by the circumference approximation. The error of the circumference approximation is approximately twice the error of the ellipse approximation.

However, both approximation of the geometric method have a smaller error than the two-step method.

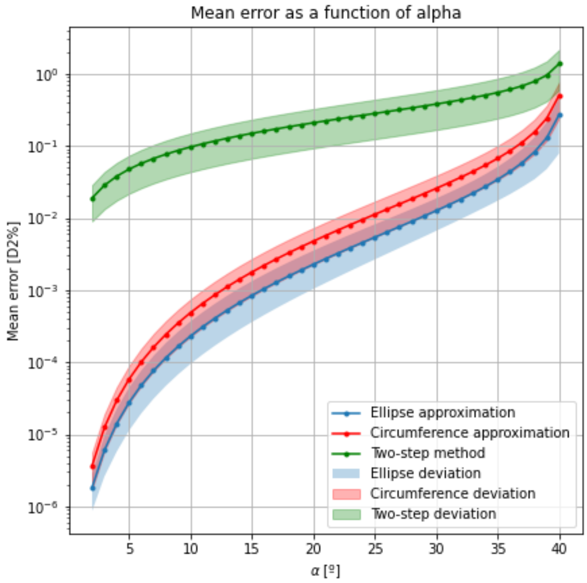


**Figure 6.** Mean error as percentage of D2 for each value of the prism angle $\alpha$.

Another important aspect to consider is the time needed to obtain the desired data, the results of time obtained are shown in **Table 1**. It has been obtained from a set of 4900 points, which contain points for seventy different angles of the whole possible region and seventy different radii between the minimum radius and the maximum radius. This set of points has been repeated sixteen times giving a total of 78,400 points to calculate the mean time.

The ellipse approximation has a longer computation mean time than the circumference approximation. The circumference approximation is about 1.6 times faster than the ellipse approximation. However, both approximations of the geometric method are faster than the two-step method.

**Table 1.** Result of time of the geometric method and the two-step method.

| | Geometric method | | Two-step method |
|---|---|---|---|
| | Circumference approximation | Ellipse approximation | |
| **Mean time ($\mu s$)** | 2.72571 | 4.33776 | 9.67367 |
| **Deviation ($\mu s$)** | 0.475435 | 0.501481 | 7.39166 |
| **Number of point** | 78,400 | | |

*3.2. Comparative resulst of the error with the iterative method.*

The iterative method proposed by Anhu Li [6] has been implemented with the two-step method and compared with the circumference and ellipse approximation of the geometric method for a wedge angle $\alpha$ of 10º. However, the ellipse approximation has been performing worse than the circumference approximation due to it takes almost twice as long as the circumference approximation. Therefore, only the results of the circumference of the ellipse will be shown.

The mean computation time that each method takes to reach the required error threshold have been calculated, the result obtained are shown in **Figure 7**. Each point represented in the graphs in **Figure 7** is the mean time obtained for a set of 40,000 points.

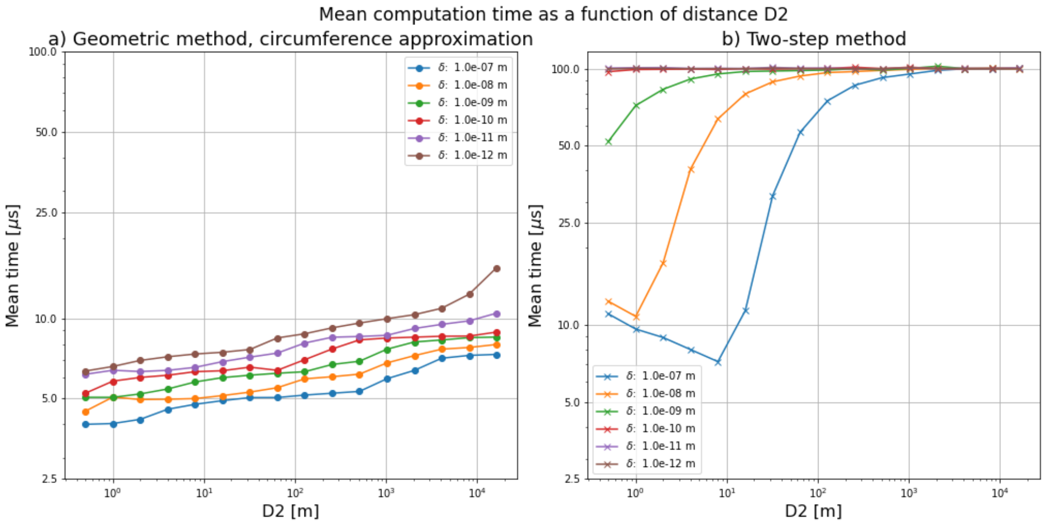


**Figure 7.** Mean computation time as a function of distance D2 for each error threshold. a) Represents the mean computation time obtained by the iterative method with the geometric method (circumference approximation) for each error threshold. b) Represents the mean computation time obtained by the iterative method with the two-step method for each error threshold.

**Figure 7** shows that the iterative method has a shorter computation time with the geometric method than with the two-step method for any distance D2 and error threshold. The shortest time taken by the iterative method with the geometric method to obtain the final rotation angles is 3 μs while with the two-step method it is 7 μs.

The maximum number of iterations that the iterative method can reach is fifty iterations, this maximum value has been established because after several tests it has been found that if the method is not able to reach the required threshold error in fifty iterations it will never get there.

The graph in **Figure 7** b) shows that with the two-step method for large distances (more than 100 m) and for very small error thresholds $(1\ nm)$ the results are obtained in $100\ \mu s$ on average. This is because the method has reached the maximum number of iterations, i.e., the two-step method does not achieve error thresholds smaller than $1\ nm$. On the other hand, an error threshold up to $1\ pm$ can be achieved for any distances up to 16.384 $km$ with the geometric method (**Figure 7** a).

The iterative method with the geometric method achieves an accuracy three orders of magnitude lower than the iterative method with the two-step. Moreover, the iterative method with the geometric method achieves more accurate results for larger D2 distances than the iterative method with the two-step method.

## 4. Conclusion

Many studies on the inverse problem have been performed, such as the third-order theory [8], the inverse solution [4], the two-step method [10–12] and the iterative method [6]. In this paper, a geometric method is proposed to solve the rotation angles for a given target trajectory in less computing time and with high accuracy.

Two approximations have been made for the geometric method and the accuracy and time obtained by both approximations have been analysed. The error of the circumference approximation is approximately twice the error of the ellipse approximation. However, the ellipse approximation has a longer computation time than the circumference approximation.

The geometric method has been compared with the two-step method and it has been concluded that the geometric method is faster and more accurate than the two-step method.

Until now, the most accurate method was the iterative method using the two-step method. Therefore, the geometric method has been implemented in the iterative method

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to analyse if it improves time and accuracy. It has been concluded that the iterative method with the two-step method obtains good time and accuracy results for small D2 distances while the iterative method with the geometric method obtains good results for small and large D2 distances. In addition, the iterative method with the two-step method takes more iterations to reach the established error threshold and, therefore, the computation time is longer than the iterative method with the geometric method.

Therefore, the geometric method can solve the inverse problem of Risley systems that prevent the laser beam from being steered to any region of space. Solving the inverse problem enables applications involving the control of dual-prism rotary motions such as multimode beam tracking in the field of airborne laser communications, the monitoring of containers in harbour operations and target tracking in the military domain. Moreover, the geometric method could be used in applications that require a higher accuracy like lithography, stereolithography or 3D printers.

Future research will apply the geometric method in various real applications.

## 6. Patents

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