
Continuous Non-Archimedean and p-adic Welch Bounds

K. MAHESH KRISHNA

Post Doctoral Fellow

Statistics and Mathematics Unit

Indian Statistical Institute, Bangalore Centre

Karnataka 560 059, India

Email: kmaheshak@gmail.com

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Abstract: We prove the continuous non-Archimedean (resp. p-adic) Banach space and Hilbert space versions of non-Archimedean (resp. p-adic) Welch bounds proved by M. Krishna. We formulate continuous non-Archimedean and p-adic functional Zauner conjectures.

Keywords: Non-Archimedean valued field, Non-Archimedean Banach space, p-adic number field, p-adic Banach space, Welch bound, Zauner conjecture.

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1. INTRODUCTION

Prof. L. Welch proved the following bounds in 1974 which appears in everyday life [84].

Theorem 1.1. [84] (**Welch Bounds**) Let $n > d$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in \mathbb{C}^d , then

$$\sum_{1 \leq j, k \leq n} |\langle \tau_j, \tau_k \rangle|^{2m} = \sum_{j=1}^n \sum_{k=1}^n |\langle \tau_j, \tau_k \rangle|^{2m} \geq \frac{n^2}{\binom{d+m-1}{m}}, \quad \forall m \in \mathbb{N}.$$

In particular,

$$\sum_{1 \leq j, k \leq n} |\langle \tau_j, \tau_k \rangle|^2 = \sum_{j=1}^n \sum_{k=1}^n |\langle \tau_j, \tau_k \rangle|^2 \geq \frac{n^2}{d}.$$

Further,

$$\text{(Higher order Welch bounds)} \quad \max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle|^{2m} \geq \frac{1}{n-1} \left[\frac{n}{\binom{d+m-1}{m}} - 1 \right], \quad \forall m \in \mathbb{N}.$$

In particular,

$$\text{(First order Welch bound)} \quad \max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle|^2 \geq \frac{n-d}{d(n-1)}.$$

Theorem 1.1 is a fundamental result in many areas such as in the study of root-mean-square (RMS) absolute cross relation of unit vectors [71], frame potential [9, 14, 18], correlations [70], codebooks [27], numerical search algorithms [85, 86], quantum measurements [73], coding and communications [76, 80], code division multiple access (CDMA) systems [49, 50], wireless systems [68], compressed/compressive sensing [1, 6, 29, 32, 72, 78, 79, 81], ‘game of Sloanes’ [45], equiangular tight frames [77], equiangular lines [7, 8, 15, 17, 20, 23, 26, 30, 31, 34, 35, 37–40, 44, 46, 47, 57, 60, 62, 63, 87], digital fingerprinting [59] etc.

Theorem 1.1 has been upgraded/different proofs were given in [19, 24, 25, 28, 42, 69, 76, 82, 83]. In 2021 M. Krishna derived continuous version of Theorem 1.1 [51]. In 2022 M. Krishna obtained Theorem 1.1

for Hilbert C^* -modules [53], Banach spaces [52], non-Archimedean Hilbert spaces [55], p -adic Hilbert spaces [56] non-Archimedean Banach spaces and p -adic Banach spaces [54].

In this paper we derive continuous non-Archimedean (resp. p -adic) Banach space version of non-Archimedean (resp. p -adic) functional Welch bounds in Theorems 2.4 and (resp. Theorems 3.3). We formulate continuous non-Archimedean Zauner conjectures (Conjectures 2.8 and 2.9) and continuous p -adic Zauner conjectures (Conjectures 3.7 and 3.8).

2. CONTINUOUS NON-ARCHIMEDEAN WELCH BOUNDS

In this section we derive continuous non-Archimedean Banach space version of results derived in [54]. Let \mathbb{K} be a non-Archimedean (complete) valued field satisfying

$$(1) \quad \left| \sum_{j=1}^n \lambda_j^2 \right| = \max_{1 \leq j \leq n} |\lambda_j|^2, \quad \forall \lambda_j \in \mathbb{K}, 1 \leq j \leq n, \forall n \in \mathbb{N}.$$

For examples of such fields, we refer [64]. Throughout this section, we assume that our non-Archimedean field \mathbb{K} satisfies Equation (1). Letter \mathcal{X} stands for a d -dimensional non-Archimedean Banach space over \mathbb{K} . Identity operator on \mathcal{X} is denoted by $I_{\mathcal{X}}$. The dual of \mathcal{X} is denoted by \mathcal{X}^* . In the entire paper, G denotes a locally compact group and μ_G denotes a left Haar measure on G . We assume throughout the paper that the diagonal $\Delta := \{(g, g) : g \in G\}$ is measurable in $G \times G$. All our integrals are weak non-Archimedean Riemann integrals (see [67]).

Theorem 2.1. (First Order Continuous Non-Archimedean Functional Welch Bound) *Let \mathcal{X} be a d -dimensional non-Archimedean Banach space over \mathbb{K} . Let $\{\tau_g\}_{g \in G}$ be a collection in \mathcal{X} and $\{f_g\}_{g \in G}$ be a collection in \mathcal{X}^* satisfying following conditions.*

(i) *For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map*

$$G \ni g \mapsto f_g(x)\phi(\tau_g) \in \mathbb{K}$$

is measurable and integrable.

(ii) *The map*

$$S_{f,\tau} : \mathcal{X} \ni x \mapsto \int_G f_g(x)\tau_g d\mu_G(g) \in \mathcal{X}$$

is a well-defined bounded linear operator.

(iii) *The operator $S_{f,\tau}$ is diagonalizable.*

(iv)

$$\int_{G \times G} |f_g(\tau_h)f_h(\tau_g)| d(\mu_G \times \mu_G)(g, h) < \infty.$$

Then

$$\max \left\{ \left| \int_{\Delta} f_g(\tau_g)^2 d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g, h \in G, g \neq h} |f_g(\tau_h)f_h(\tau_g)| \right\} \geq \frac{1}{|d|} \left| \int_G f_g(\tau_g) d\mu_G(g) \right|^2.$$

In particular, if $f_g(\tau_g) = 1$ for all $g \in G$, then

(*First order continuous non-Archimedean functional Welch bound*)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h)f_h(\tau_g)| \right\} \geq \frac{|\mu(G)|^2}{|d|}.$$

Proof. We first note that

$$\begin{aligned} \text{Tra}(S_{f,\tau}) &= \int_G f_g(\tau_g) d\mu_G(g), \\ \text{Tra}(S_{f,\tau}^2) &= \int_G \int_G f_g(\tau_h)f_h(\tau_g) d\mu_G(g) d\mu_G(h). \end{aligned}$$

Let $\lambda_1, \dots, \lambda_d$ be the diagonal entries in the diagonalization of $S_{f,\tau}$. Then using the diagonalizability of $S_{f,\tau}$ and the non-Archimedean Cauchy-Schwarz inequality (Theorem 2.4.2 [64]), we get

$$\begin{aligned} \left| \int_G f_g(\tau_g) d\mu_G(g) \right|^2 &= |\text{Tra}(S_{f,\tau})|^2 = \left| \sum_{k=1}^d \lambda_k \right|^2 \leq |d| \left| \sum_{k=1}^d \lambda_k^2 \right| = |d| |\text{Tra}(S_{f,\tau}^2)| \\ &= |d| \left| \int_G \int_G f_g(\tau_h)f_h(\tau_g) d\mu_G(g) d\mu_G(h) \right| = |d| \left| \int_{G \times G} f_g(\tau_h)f_h(\tau_g) d(\mu_G \times \mu_G)(g, h) \right| \\ &= |d| \left| \int_{\Delta} f_g(\tau_g)^2 d(\mu_G \times \mu_G)(g, g) + \int_{(G \times G) \setminus \Delta} f_g(\tau_h)f_h(\tau_g) d(\mu_G \times \mu_G)(g, h) \right| \\ &\leq |d| \max \left\{ \left| \int_{\Delta} f_g(\tau_g)^2 d(\mu_G \times \mu_G)(g, g) \right|, \left| \int_{(G \times G) \setminus \Delta} f_g(\tau_h)f_h(\tau_g) d(\mu_G \times \mu_G)(g, h) \right| \right\} \\ &\leq |d| \max \left\{ \left| \int_{\Delta} f_g(\tau_g)^2 d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h)f_h(\tau_g)| \right\}. \end{aligned}$$

Whenever $f_g(\tau_g) = 1$ for all $g \in G$,

$$|\mu(G)|^2 \leq |d| \max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h)f_h(\tau_g)| \right\}.$$

□

Corollary 2.2. (*First Order Continuous Non-Archimedean Welch Bound*) Let \mathcal{X} be a d -dimensional non-Archimedean Hilbert space over \mathbb{K} . Let $\{\tau_g\}_{g \in G}$ be a collection in \mathcal{X} satisfying following conditions.

(i) For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map

$$G \ni g \mapsto \langle x, \tau_g \rangle \phi(\tau_g) \in \mathcal{X}$$

is measurable and integrable.

(ii) The map

$$S_\tau : \mathcal{X} \ni x \mapsto \int_G \langle x, \tau_g \rangle \tau_g d\mu_G(g) \in \mathcal{X}$$

is a well-defined bounded linear operator.

- (iii) The operator S_τ is diagonalizable.
- (iv)

$$\int_{G \times G} |\langle \tau_g, \tau_h \rangle|^2 d(\mu_G \times \mu_G)(g, h) < \infty.$$

Then

$$\max \left\{ \left| \int_{\Delta} \langle \tau_g, \tau_g \rangle^2 d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g, h \in G, g \neq h} |\langle \tau_g, \tau_h \rangle|^2 \right\} \geq \frac{1}{|d|} \left| \int_G \langle \tau_g, \tau_g \rangle d\mu_G(g) \right|^2.$$

In particular, if $\langle \tau_g, \tau_g \rangle = 1$ for all $g \in G$, then

(First order continuous non-Archimedean Welch bound)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g, h \in G, g \neq h} |\langle \tau_g, \tau_h \rangle|^2 \right\} \geq \frac{|\mu(G)|^2}{|d|}.$$

Next we obtain higher order continuous non-Archimedean functional Welch bounds. We need the following vector space result.

Theorem 2.3. [13, 21] If \mathcal{V} is a vector space of dimension d and $Sym^m(\mathcal{V})$ denotes the vector space of symmetric m -tensors, then

$$\dim(Sym^m(\mathcal{V})) = \binom{d+m-1}{m}, \quad \forall m \in \mathbb{N}.$$

Theorem 2.4. (Higher Order Continuous Non-Archimedean Functional Welch Bounds) Let \mathcal{X} be a d -dimensional non-Archimedean Banach space over \mathbb{K} . Let $m \in \mathbb{N}$. Let $\{\tau_g\}_{g \in G}$ be a collection in \mathcal{X} and $\{f_g\}_{g \in G}$ be a collection in \mathcal{X}^* satisfying following conditions.

- (i) For every $x \in Sym^m(\mathcal{X})$ and for every $\phi \in Sym^m(\mathcal{X})^*$, the map

$$G \ni g \mapsto f_g^{\otimes m}(x) \phi(\tau_g^{\otimes m}) \in Sym^m(\mathcal{X})$$

is measurable and integrable.

- (ii) The map

$$S_{f, \tau} : Sym^m(\mathcal{X}) \ni x \mapsto \int_G f_g^{\otimes m}(x) \tau_g^{\otimes m} d\mu_G(g) \in Sym^m(\mathcal{X})$$

is a well-defined bounded linear operator.

- (iii) The operator $S_{f, \tau}$ is diagonalizable.
- (iv)

$$\int_{G \times G} |f_g(\tau_h) f_h(\tau_g)|^m d(\mu_G \times \mu_G)(g, h) < \infty.$$

Then

$$\max \left\{ \left| \int_{\Delta} f_g(\tau_g)^{2m} d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g, h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)|^m \right\} \geq \frac{1}{\left| \binom{d+m-1}{m} \right|} \left| \int_G f_g(\tau_g)^m d\mu_G(g) \right|^2.$$

In particular, if $f_g(\tau_g) = 1$ for all $g \in G$, then

(Higher order continuous non-Archimedean functional Welch bounds)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)|^m \right\} \geq \frac{|\mu(G)|^2}{\binom{d+m-1}{m}}.$$

Proof. Let $\lambda_1, \dots, \lambda_{\dim(\text{Sym}^m(\mathcal{X}))}$ be the diagonal entries in the diagonalization of $S_{f,\tau}$. We note that

$$b \dim(\text{Sym}^m(\mathcal{X})) = \text{Tra}(bI_{\text{Sym}^m(\mathcal{X})}) = \text{Tra}(S_{f,\tau}) = \int_G f_g^{\otimes m}(\tau_g^{\otimes m}) d\mu_G(g),$$

$$b^2 \dim(\text{Sym}^m(\mathcal{X})) = \text{Tra}(b^2 I_{\text{Sym}^m(\mathcal{X})}) = \text{Tra}(S_{f,\tau}^2) = \int_G \int_G f_g^{\otimes m}(\tau_h^{\otimes m}) f_h^{\otimes m}(\tau_g^{\otimes m}) d\mu_G(g) d\mu_G(h).$$

Then

$$\begin{aligned} & \left| \int_G f_g(\tau_g)^m d\mu_G(g) \right|^2 = \left| \int_G f_g^{\otimes m}(\tau_g^{\otimes m}) d\mu_G(g) \right|^2 = |\text{Tra}(S_{f,\tau})|^2 = \left| \sum_{k=1}^{\dim(\text{Sym}^m(\mathcal{X}))} \lambda_k \right|^2 \\ & \leq |\dim(\text{Sym}^m(\mathcal{X}))| \left| \sum_{k=1}^{\dim(\text{Sym}^m(\mathcal{X}))} \lambda_k^2 \right| = |\dim(\text{Sym}^m(\mathcal{X}))| |\text{Tra}(S_{f,\tau}^2)| \\ & = \binom{d+m-1}{m} |\text{Tra}(S_{f,\tau}^2)| = \binom{d+m-1}{m} \left| \int_G \int_G f_g^{\otimes m}(\tau_h^{\otimes m}) f_h^{\otimes m}(\tau_g^{\otimes m}) d\mu_G(g) d\mu_G(h) \right| \\ & = \binom{d+m-1}{m} \left| \int_G \int_G f_g(\tau_h)^m f_h(\tau_g)^m d\mu_G(g) d\mu_G(h) \right| \\ & = \binom{d+m-1}{m} \left| \int_{G \times G} f_g(\tau_h)^m f_h(\tau_g)^m d(\mu_G \times \mu_G)(g, h) \right| \\ & = \binom{d+m-1}{m} \left| \int_{\Delta} f_g(\tau_g)^{2m} d(\mu_G \times \mu_G)(g, g) + \int_{(G \times G) \setminus \Delta} f_g(\tau_h)^m f_h(\tau_g)^m d(\mu_G \times \mu_G)(g, h) \right| \\ & \leq |d| \max \left\{ \left| \int_{\Delta} f_g(\tau_g)^{2m} d(\mu_G \times \mu_G)(g, g) \right|, \left| \int_{(G \times G) \setminus \Delta} f_g(\tau_h)^m f_h(\tau_g)^m d(\mu_G \times \mu_G)(g, h) \right| \right\} \\ & \leq \binom{d+m-1}{m} \max \left\{ \left| \int_{\Delta} f_g(\tau_g)^{2m} d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h)^m f_h(\tau_g)^m| \right\} \\ & = \binom{d+m-1}{m} \max \left\{ \left| \int_{\Delta} f_g(\tau_g)^{2m} d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)|^m \right\}. \end{aligned}$$

Whenever $f_g(\tau_g) = 1$ for all $g \in G$,

$$|\mu(G)|^2 \leq \binom{d+m-1}{m} \max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)|^m \right\}.$$

□

Corollary 2.5. (Higher Order Continuous Non-Archimedean Welch Bounds) Let \mathcal{X} be a d -dimensional non-Archimedean Hilbert space over \mathbb{K} . Let $m \in \mathbb{N}$. Let $\{\tau_g\}_{g \in G}$ be a collection in \mathcal{X} satisfying following conditions.

(i) For every $x \in \text{Sym}^m(\mathcal{X})$ and for every $\phi \in \text{Sym}^m(\mathcal{X})^*$, the map

$$G \ni g \mapsto \langle x, \tau_g^{\otimes m} \rangle \phi(\tau_g^{\otimes m}) \in \text{Sym}^m(\mathcal{X})$$

is measurable and integrable.

(ii) The map

$$S_\tau : \text{Sym}^m(\mathcal{X}) \ni x \mapsto \int_G \langle x, \tau_g^{\otimes m} \rangle \tau_g^{\otimes m} d\mu_G(g) \in \text{Sym}^m(\mathcal{X})$$

is a well-defined bounded linear operator.

(iii) The operator S_τ is diagonalizable.

(iv)

$$\int_{G \times G} |\langle \tau_g, \tau_h \rangle|^{2m} d(\mu_G \times \mu_G)(g, h) < \infty.$$

Then

$$\max \left\{ \left| \int_{\Delta} \langle \tau_g, \tau_g \rangle^{2m} d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g, h \in G, g \neq h} |\langle \tau_g, \tau_h \rangle|^{2m} \right\} \geq \frac{1}{\left| \binom{d+m-1}{m} \right|} \left| \int_G \langle \tau_g, \tau_g \rangle^m d\mu_G(g) \right|^2.$$

In particular, if $\langle \tau_g, \tau_g \rangle = 1$ for all $g \in G$, then

(Higher order continuous non-Archimedean Welch bounds)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g, h \in G, g \neq h} |\langle \tau_g, \tau_h \rangle|^{2m} \right\} \geq \frac{|\mu(G)|^2}{\left| \binom{d+m-1}{m} \right|}.$$

Motivated from Theorem 2.1 and Corollary 2.2 we formulate the following questions.

Question 2.6. Let \mathbb{K} be a non-Archimedean field satisfying Equation (1) and \mathcal{X} be a d -dimensional non-Archimedean Banach space over \mathbb{K} . For which locally compact group G with measurable diagonal Δ , there exist a collection $\{\tau_g\}_{g \in G}$ in \mathcal{X} and a collection $\{f_g\}_{g \in G}$ in \mathcal{X}^* satisfying the following.

(i) For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map

$$G \ni g \mapsto f_g(x) \phi(\tau_g) \in \mathcal{X}$$

is measurable and integrable.

(ii) The map

$$S_{f, \tau} : \mathcal{X} \ni x \mapsto \int_G f_g(x) \tau_g d\mu_G(g) \in \mathcal{X}$$

is a well-defined bounded linear operator.

(iii) The operator $S_{f, \tau}$ is diagonalizable.

(iv)

$$\int_{G \times G} |f_g(\tau_h) f_h(\tau_g)| d(\mu_G \times \mu_G)(g, h) < \infty.$$

(v) $f_g(\tau_g) = 1$ for all $g \in G$.

(vi)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g, h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)| \right\} = \frac{|\mu(G)|^2}{|d|}.$$

(vii) $\|f_g\| = 1, \|\tau_g\| = 1$ for all $g \in G$.

Question 2.7. Let \mathbb{K} be a non-Archimedean field satisfying Equation (1) and \mathcal{X} be a d -dimensional non-Archimedean Hilbert space over \mathbb{K} . For which locally compact group G with measurable diagonal Δ , there exists a collection $\{\tau_g\}_{g \in G}$ in \mathcal{X} satisfying the following.

(i) For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map

$$G \ni g \mapsto \langle x, \tau_g \rangle \phi(\tau_g) \in \mathcal{X}$$

is measurable and integrable.

(ii) The map

$$S_\tau : \mathcal{X} \ni x \mapsto \int_G \langle x, \tau_g \rangle \tau_g d\mu_G(g) \in \mathcal{X}$$

is a well-defined bounded linear operator.

(iii) The operator S_τ is diagonalizable.

(iv)

$$\int_{G \times G} |\langle \tau_g, \tau_h \rangle|^2 d(\mu_G \times \mu_G)(g, h) < \infty.$$

(v) $\langle \tau_g, \tau_g \rangle = 1$ for all $g \in G$.

(vi)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g, h \in G, g \neq h} |\langle \tau_g, \tau_h \rangle|^2 \right\} = \frac{|\mu(G)|^2}{|d|}.$$

A particular case of Questions 2.6 and 2.7 are following continuous non-Archimedean versions of Zauner conjecture (see [2–5, 10–12, 33, 36, 43, 48, 51, 58, 66, 74, 88] for Zauner conjecture in Hilbert spaces, [53] for Zauner conjecture in Hilbert C^* -modules, [52] for Zauner conjecture in Banach spaces, [55] for Zauner conjecture in non-Archimedean Hilbert spaces, [56] for Zauner conjecture in p -adic Hilbert spaces and [54] for Zauner conjecture for non-Archimedean Banach spaces and p -adic Banach spaces).

Conjecture 2.8. (Continuous Non-Archimedean Functional Zauner Conjecture) Let \mathbb{K} non-Archimedean field satisfying Equation (1). Given locally compact group G and for each $d \in \mathbb{N}$, there exist a collection $\{\tau_g\}_{g \in G}$ in \mathbb{K}^d (w.r.t. any non-Archimedean norm) and a collection $\{f_g\}_{g \in G}$ in $(\mathbb{K}^d)^*$ satisfying the following.

(i) For every $x \in \mathbb{K}^d$ and for every $\phi \in (\mathbb{K}^d)^*$, the map

$$G \ni g \mapsto f_g(x) \phi(\tau_g) \in \mathbb{K}^d$$

is measurable and integrable.

(ii) *The map*

$$S_{f,\tau} : \mathbb{K}^d \ni x \mapsto \int_G f_g(x) \tau_g d\mu_G(g) \in \mathbb{K}^d$$

is a well-defined bounded linear operator.

(iii) *The operator $S_{f,\tau}$ is diagonalizable.*

(iv)

$$\int_{G \times G} |f_g(\tau_h) f_h(\tau_g)| d(\mu_G \times \mu_G)(g, h) < \infty.$$

(v) $f_g(\tau_g) = 1$ *for all* $g \in G$.

(vi)

$$|(\mu_G \times \mu_G)(\Delta)| = |f_g(\tau_h) f_h(\tau_g)| = \frac{|\mu(G)|^2}{|d|}, \quad \forall g, h \in G, g \neq h.$$

(vii) $\|f_g\| = 1, \|\tau_g\| = 1$ *for all* $g \in G$.

Conjecture 2.9. (*Continuous Non-Archimedean Zauner Conjecture*) *Let \mathbb{K} non-Archimedean field satisfying Equation (1). Given locally compact group G and for each $d \in \mathbb{N}$, there exists a collection $\{\tau_g\}_{g \in G}$ in \mathbb{K}^d satisfying the following.*

(i) *For every $x \in \mathbb{K}^d$ and for every $\phi \in (\mathbb{K}^d)^*$, the map*

$$G \ni g \mapsto \langle x, \tau_g \rangle \phi(\tau_g) \in \mathbb{K}^d$$

is measurable and integrable.

(ii) *The map*

$$S_\tau : \mathbb{K}^d \ni x \mapsto \int_G \langle x, \tau_g \rangle \tau_g d\mu_G(g) \in \mathbb{K}^d$$

is a well-defined bounded linear operator.

(iii) *The operator S_τ is diagonalizable.*

(iv)

$$\int_{G \times G} |\langle \tau_g, \tau_h \rangle|^2 d(\mu_G \times \mu_G)(g, h) < \infty.$$

(v) $\langle \tau_g, \tau_g \rangle = 1$ *for all* $g \in G$.

(vi)

$$|(\mu_G \times \mu_G)(\Delta)| = |\langle \tau_g, \tau_h \rangle|^2 = \frac{|\mu(G)|^2}{|d|}, \quad \forall g, h \in G, g \neq h.$$

There are four bounds which are companions of Welch bounds in Hilbert spaces. To recall them we need the notion of Gerzon's bound.

Definition 2.10. [45] *Given $d \in \mathbb{N}$, define Gerzon's bound*

$$\mathcal{Z}(d, \mathbb{K}) := \begin{cases} d^2 & \text{if } \mathbb{K} = \mathbb{C} \\ \frac{d(d+1)}{2} & \text{if } \mathbb{K} = \mathbb{R}. \end{cases}$$

Theorem 2.11. [16, 22, 41, 45, 61, 65, 75, 85] *Define $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and $m := \dim_{\mathbb{R}}(\mathbb{K})/2$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in \mathbb{K}^d , then*

(i) (*Bukh-Cox bound*)

$$\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{\mathcal{Z}(n-d, \mathbb{K})}{n(1+m(n-d-1)\sqrt{m^{-1}+n-d}) - \mathcal{Z}(n-d, \mathbb{K})} \quad \text{if } n > d.$$

(ii) (*Orthoplex/Rankin bound*)

$$\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{1}{\sqrt{d}} \quad \text{if } n > \mathcal{Z}(d, \mathbb{K}).$$

(iii) (*Levenstein bound*)

$$\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \sqrt{\frac{n(m+1) - d(md+1)}{(n-d)(md+1)}} \quad \text{if } n > \mathcal{Z}(d, \mathbb{K}).$$

(iv) (*Exponential bound*)

$$\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq 1 - 2n^{\frac{-1}{d-1}}.$$

Motivated from Theorem 2.11, Theorem 2.1 and Corollary 2.2 we ask the following problems.

Question 2.12. *Whether there is a continuous non-Archimedean functional version of Theorem 2.11? In particular, does there exists a version of*

- (i) *continuous non-Archimedean functional Bukh-Cox bound?*
- (ii) *continuous non-Archimedean functional Orthoplex/Rankin bound?*
- (iii) *continuous non-Archimedean functional Levenstein bound?*
- (iv) *continuous non-Archimedean functional Exponential bound?*

Question 2.13. *Whether there is a continuous non-Archimedean version of Theorem 2.11? In particular, does there exists a version of*

- (i) *continuous non-Archimedean Bukh-Cox bound?*
- (ii) *continuous non-Archimedean Orthoplex/Rankin bound?*
- (iii) *continuous non-Archimedean Levenstein bound?*
- (iv) *continuous non-Archimedean Exponential bound?*

3. CONTINUOUS P-ADIC WELCH BOUNDS

In this section we derive p-adic Banach space version of results done in [56]. Let p be a prime and \mathbb{Q}_p be the field of p-adic numbers. In this section, \mathcal{X} is a d -dimensional p-adic Banach space over \mathbb{Q}_p .

Theorem 3.1. (First Order Continuous p-adic Functional Welch Bound) *Let p be a prime and \mathcal{X} be a d -dimensional p-adic Banach space over \mathbb{Q}_p . Let $\{\tau_g\}_{g \in G}$ be a collection in \mathcal{X} and $\{f_g\}_{g \in G}$ be a collection in \mathcal{X}^* satisfying following conditions.*

- (i) *For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map*

$$G \ni g \mapsto f_g(x)\phi(\tau_g) \in \mathcal{X}$$

is measurable and integrable.

- (ii) *There exists $b \in \mathbb{Q}_p$ such that*

$$\int_G f_g(x)\tau_g d\mu_G(g) = bx, \quad \forall x \in \mathcal{X}.$$

(iii)

$$\int_{G \times G} |f_g(\tau_h) f_h(\tau_g)| d(\mu_G \times \mu_G)(g, h) < \infty.$$

Then

$$\max \left\{ \left| \int_{\Delta} f_g(\tau_g)^2 d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g, h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)| \right\} \geq \frac{1}{|d|} \left| \int_G f_g(\tau_g) d\mu_G(g) \right|^2.$$

In particular, if $f_g(\tau_g) = 1$ for all $g \in G$, then**(First order continuous p -adic functional Welch bound)**

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g, h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)| \right\} \geq \frac{|\mu(G)|^2}{|d|}.$$

Proof. Define $S_{f, \tau} : \mathcal{X} \ni x \mapsto \int_G f_g(x) \tau_g d\mu_G(g) \in \mathcal{X}$. Then

$$bd = \text{Tra}(bI_{\mathcal{X}}) = \text{Tra}(S_{f, \tau}) = \int_G f_g(\tau_g) d\mu_G(g),$$

$$b^2d = \text{Tra}(b^2I_{\mathcal{X}}) = \text{Tra}(S_{f, \tau}^2) = \int_G \int_G f_g(\tau_h) f_h(\tau_g) d\mu_G(g) d\mu_G(h).$$

Therefore

$$\begin{aligned} \left| \int_G f_g(\tau_g) d\mu_G(g) \right|^2 &= |\text{Tra}(S_{f, \tau})|^2 = |bd|^2 = |d| |b^2d| = |d| \left| \int_G \int_G f_g(\tau_h) f_h(\tau_g) d\mu_G(g) d\mu_G(h) \right| \\ &= |d| \left| \int_{G \times G} f_g(\tau_h) f_h(\tau_g) d(\mu_G \times \mu_G)(g, h) \right| \\ &= |d| \left| \int_{\Delta} f_g(\tau_g)^2 d(\mu_G \times \mu_G)(g, g) + \int_{(G \times G) \setminus \Delta} f_g(\tau_h) f_h(\tau_g) d(\mu_G \times \mu_G)(g, h) \right| \\ &\leq |d| \max \left\{ \left| \int_{\Delta} f_g(\tau_g)^2 d(\mu_G \times \mu_G)(g, g) \right|, \left| \int_{(G \times G) \setminus \Delta} f_g(\tau_h) f_h(\tau_g) d(\mu_G \times \mu_G)(g, h) \right| \right\} \\ &\leq |d| \max \left\{ \left| \int_{\Delta} f_g(\tau_g)^2 d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g, h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)| \right\}. \end{aligned}$$

Whenever $f_g(\tau_g) = 1$ for all $g \in G$,

$$|\mu(G)|^2 \leq |d| \max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g, h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)| \right\}.$$

□

Corollary 3.2. (First Order Continuous p -adic Welch Bound) Let p be a prime and \mathcal{X} be a d -dimensional p -adic Hilbert space over \mathbb{Q}_p . Let $\{\tau_g\}_{g \in G}$ be a collection in \mathcal{X} satisfying following conditions.

(i) For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map

$$G \ni g \mapsto \langle x, \tau_g \rangle \phi(\tau_g) \in \mathcal{X}$$

is measurable and integrable.

(ii) There exists $b \in \mathbb{Q}_p$ such that

$$\int_G \langle x, \tau_g \rangle \tau_g d\mu_G(g) = bx, \quad \forall x \in \mathcal{X}.$$

(iii)

$$\int_{G \times G} |\langle \tau_g, \tau_h \rangle|^2 d(\mu_G \times \mu_G)(g, h) < \infty.$$

Then

$$\max \left\{ \left| \int_{\Delta} \langle \tau_g, \tau_g \rangle^2 d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g, h \in G, g \neq h} |\langle \tau_g, \tau_h \rangle|^2 \right\} \geq \frac{1}{|d|} \left| \int_G \langle \tau_g, \tau_g \rangle d\mu_G(g) \right|^2.$$

In particular, if $\langle \tau_g, \tau_g \rangle = 1$ for all $g \in G$, then

(First order continuous p -adic Welch bound)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g, h \in G, g \neq h} |\langle \tau_g, \tau_h \rangle|^2 \right\} \geq \frac{|\mu(G)|^2}{|d|}.$$

Now we derive higher order version of Theorem 3.1.

Theorem 3.3. (Higher Order Continuous p -adic Functional Welch Bounds) Let p be a prime and \mathcal{X} be a d -dimensional p -adic Banach space over \mathbb{Q}_p . Let $m \in \mathbb{N}$. Let $\{\tau_g\}_{g \in G}$ be a collection in \mathcal{X} and $\{f_g\}_{g \in G}$ be a collection in \mathcal{X}^* satisfying following conditions.

(i) For every $x \in \text{Sym}^m(\mathcal{X})$ and for every $\phi \in \text{Sym}^m(\mathcal{X})^*$, the map

$$G \ni g \mapsto f_g^{\otimes m}(x) \phi(\tau_g^{\otimes m}) \in \text{Sym}^m(\mathcal{X})$$

is measurable and integrable.

(ii) There exists $b \in \mathbb{Q}_p$ satisfying

$$\int_G f_g^{\otimes m}(x) \tau_g^{\otimes m} d\mu_G(g) = bx, \quad \forall x \in \text{Sym}^m(\mathcal{X}).$$

(iii)

$$\int_{G \times G} |f_g(\tau_h) f_h(\tau_g)|^m d(\mu_G \times \mu_G)(g, h) < \infty.$$

Then

$$\max \left\{ \left| \int_{\Delta} f_g(\tau_g)^{2m} d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g, h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)|^m \right\} \geq \frac{1}{\left| \binom{d+m-1}{m} \right|} \left| \int_G f_g(\tau_g)^m d\mu_G(g) \right|^2.$$

In particular, if $f_g(\tau_g) = 1$ for all $g \in G$, then

(Higher order p -adic continuous functional Welch bound)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)|^m \right\} \geq \frac{|\mu(G)|^2}{\binom{d+m-1}{m}}.$$

Proof. Define $S_{f,\tau} : \text{Sym}^m(\mathcal{X}) \ni x \mapsto \int_G f_g^{\otimes m}(x) \tau_g^{\otimes m} d\mu_G(g) \in \text{Sym}^m(\mathcal{X})$. Then

$$b \dim(\text{Sym}^m(\mathcal{X})) = \text{Tra}(bI_{\text{Sym}^m(\mathcal{X})}) = \text{Tra}(S_{f,\tau}) = \int_G f_g^{\otimes m}(\tau_g^{\otimes m}) d\mu_G(g),$$

$$b^2 \dim(\text{Sym}^m(\mathcal{X})) = \text{Tra}(b^2 I_{\text{Sym}^m(\mathcal{X})}) = \text{Tra}(S_{f,\tau}^2) = \int_G \int_G f_g^{\otimes m}(\tau_h^{\otimes m}) f_h^{\otimes m}(\tau_g^{\otimes m}) d\mu_G(g) d\mu_G(h).$$

Therefore

$$\begin{aligned} & \left| \int_G f_g(\tau_g)^m d\mu_G(g) \right|^2 = \left| \int_G f_g^{\otimes m}(\tau_g^{\otimes m}) d\mu_G(g) \right|^2 = |\text{Tra}(S_{f,\tau})|^2 = |b \dim(\text{Sym}^m(\mathcal{X}))|^2 \\ & = |\dim(\text{Sym}^m(\mathcal{X}))| |b^2 \dim(\text{Sym}^m(\mathcal{X}))| \\ & = |\dim(\text{Sym}^m(\mathcal{X}))| \left| \int_G \int_G f_g^{\otimes m}(\tau_h^{\otimes m}) f_h^{\otimes m}(\tau_g^{\otimes m}) d\mu_G(g) d\mu_G(h) \right| \\ & = \binom{d+m-1}{m} \left| \int_G \int_G f_g^{\otimes m}(\tau_h^{\otimes m}) f_h^{\otimes m}(\tau_g^{\otimes m}) d\mu_G(g) d\mu_G(h) \right| \\ & = \binom{d+m-1}{m} \left| \int_G \int_G f_g(\tau_h)^m f_h(\tau_g)^m d\mu_G(g) d\mu_G(h) \right| \\ & = \binom{d+m-1}{m} \left| \int_{G \times G} f_g(\tau_h)^m f_h(\tau_g)^m d(\mu_G \times \mu_G)(g, h) \right| \\ & = \binom{d+m-1}{m} \left| \int_{\Delta} f_g(\tau_g)^{2m} d(\mu_G \times \mu_G)(g, g) + \int_{(G \times G) \setminus \Delta} f_g(\tau_h)^m f_h(\tau_g)^m d(\mu_G \times \mu_G)(g, h) \right| \\ & \leq \binom{d+m-1}{m} \max \left\{ \left| \int_{\Delta} f_g(\tau_g)^{2m} d(\mu_G \times \mu_G)(g, g) \right|, \left| \int_{(G \times G) \setminus \Delta} f_g(\tau_h)^m f_h(\tau_g)^m d(\mu_G \times \mu_G)(g, h) \right| \right\} \\ & \leq \binom{d+m-1}{m} \max \left\{ \left| \int_{\Delta} f_g(\tau_g)^{2m} d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h)^m f_h(\tau_g)^m| \right\} \\ & = \binom{d+m-1}{m} \max \left\{ \left| \int_{\Delta} f_g(\tau_g)^{2m} d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)|^m \right\}. \end{aligned}$$

Whenever $f_g(\tau_g) = 1$ for all $g \in G$,

$$|\mu(G)|^2 \leq \binom{d+m-1}{m} \max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)|^m \right\}.$$

□

Corollary 3.4. (Higher Order Continuous p -adic Welch Bounds) Let p be a prime and \mathcal{X} be a d -dimensional p -adic Hilbert space over \mathbb{Q}_p . Let $m \in \mathbb{N}$. Let $\{\tau_g\}_{g \in G}$ be a collection in \mathcal{X} satisfying following conditions.

(i) For every $x \in \text{Sym}^m(\mathcal{X})$ and for every $\phi \in \text{Sym}^m(\mathcal{X})^*$, the map

$$G \ni g \mapsto \langle x, \tau_g^{\otimes m} \rangle \phi(\tau_g^{\otimes m}) \in \text{Sym}^m(\mathcal{X})$$

is measurable and integrable.

(ii) There exists $b \in \mathbb{Q}_p$ satisfying

$$\int_G \langle x, \tau_g^{\otimes m} \rangle \tau_g^{\otimes m} d\mu_G(g) = bx, \quad \forall x \in \text{Sym}^m(\mathcal{X}).$$

(iii)

$$\int_{G \times G} |\langle \tau_g, \tau_h \rangle|^{2m} d(\mu_G \times \mu_G)(g, h) < \infty.$$

Then

$$\max \left\{ \left| \int_{\Delta} \langle \tau_g, \tau_g \rangle^{2m} d(\mu_G \times \mu_G)(g, g) \right|, \sup_{g, h \in G, g \neq h} |\langle \tau_g, \tau_h \rangle|^m \right\} \geq \frac{1}{\left| \binom{d+m-1}{m} \right|} \left| \int_G \langle \tau_g, \tau_g \rangle^m d\mu_G(g) \right|^2.$$

In particular, if $\langle \tau_g, \tau_g \rangle = 1$ for all $g \in G$, then

(Higher order continuous p -adic Welch bound)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g, h \in G, g \neq h} |\langle \tau_g, \tau_h \rangle|^{2m} \right\} \geq \frac{|\mu(G)|^2}{\left| \binom{d+m-1}{m} \right|}.$$

Using Theorem 3.1 and Corollary 3.2 we ask the following questions.

Question 3.5. Let p be a prime and \mathcal{X} be a d -dimensional p -adic Banach space over \mathbb{Q}_p . For which locally compact group G with measurable diagonal Δ , there exist a collection $\{\tau_g\}_{g \in G}$ in \mathcal{X} and a collection $\{f_g\}_{g \in G}$ in \mathcal{X}^* satisfying the following.

(i) For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map

$$G \ni g \mapsto f_g(x) \phi(\tau_g) \in \mathcal{X}$$

is measurable and integrable.

(ii) There exists $b \in \mathbb{Q}_p$ such that

$$\int_G f_g(x) \tau_g d\mu_G(g) = bx, \quad \forall x \in \mathcal{X}.$$

(iii)

$$\int_{G \times G} |f_g(\tau_h) f_h(\tau_g)| d(\mu_G \times \mu_G)(g, h) < \infty.$$

(iv) $f_g(\tau_g) = 1$ for all $g \in G$.

(v)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h \in G, g \neq h} |f_g(\tau_h) f_h(\tau_g)| \right\} = \frac{|\mu(G)|^2}{|d|}.$$

(vi) $\|f_g\| = 1, \|\tau_g\| = 1$ for all $g \in G$.

Question 3.6. Let p be a prime and \mathcal{X} be a d -dimensional p -adic Hilbert space over \mathbb{Q}_p . For which locally compact group G with measurable diagonal Δ , there exists a collection $\{\tau_g\}_{g \in G}$ in \mathcal{X} satisfying the following.

(i) For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map

$$G \ni g \mapsto \langle x, \tau_g \rangle \phi(\tau_g) \in \mathcal{X}$$

is measurable and integrable.

(ii) There exists $b \in \mathbb{Q}_p$ such that

$$\int_G \langle x, \tau_g \rangle \tau_g d\mu_G(g) = bx, \quad \forall x \in \mathcal{X}.$$

(iii)

$$\int_{G \times G} |\langle \tau_g, \tau_h \rangle|^2 d(\mu_G \times \mu_G)(g, h) < \infty.$$

(iv) $\langle \tau_g, \tau_g \rangle = 1$ for all $g \in G$.

(v)

$$\max \left\{ |(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h \in G, g \neq h} |\langle \tau_g, \tau_h \rangle|^2 \right\} = \frac{|\mu(G)|^2}{|d|}.$$

(vi) $\|\tau_g\| = 1$ for all $g \in G$.

A particular case of Questions 3.5 and 3.6 are following continuous p -adic Zauner conjectures.

Conjecture 3.7. (Continuous p -adic Functional Zauner Conjecture) Let p be a prime. Given locally compact group G and for each $d \in \mathbb{N}$, there exist a collection $\{\tau_g\}_{g \in G}$ in \mathbb{Q}_p^d (w.r.t. any non-Archimedean norm) and a collection $\{f_g\}_{g \in G}$ in $(\mathbb{Q}_p^d)^*$ satisfying the following.

(i) For every $x \in \mathbb{Q}_p^d$ and for every $\phi \in (\mathbb{Q}_p^d)^*$, the map

$$G \ni g \mapsto f_g(x) \phi(\tau_g) \in \mathbb{Q}_p^d$$

is measurable and integrable.

(ii) There exists $b \in \mathbb{Q}_p$ such that

$$\int_G f_g(x) \tau_g d\mu_G(g) = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$\int_{G \times G} |f_g(\tau_h) f_h(\tau_g)| d(\mu_G \times \mu_G)(g, h) < \infty.$$

(iv) $f_g(\tau_g) = 1$ for all $g \in G$.

(v)

$$|(\mu_G \times \mu_G)(\Delta)| = |f_g(\tau_h)f_h(\tau_g)| = \frac{|\mu(G)|^2}{|d|}, \quad \forall g, h \in G, g \neq h.$$

(vi) $\|f_g\| = 1, \|\tau_g\| = 1$ for all $g \in G$.

Conjecture 3.8. (Continuous p -adic Zauner Conjecture) Let p be a prime. Given locally compact group G and for each $d \in \mathbb{N}$, there exists a collection $\{\tau_g\}_{g \in G}$ in \mathbb{Q}_p^d satisfying the following.

(i) For every $x \in \mathbb{Q}_p^d$ and for every $\phi \in (\mathbb{Q}_p^d)^*$, the map

$$G \ni g \mapsto \langle x, \tau_g \rangle \phi(\tau_g) \in \mathbb{Q}_p^d$$

is measurable and integrable.

(ii) There exists $b \in \mathbb{Q}_p$ such that

$$\int_G \langle x, \tau_g \rangle \tau_g d\mu_G(g) = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$\int_{G \times G} |\langle \tau_g, \tau_h \rangle|^2 d(\mu_G \times \mu_G)(g, h) < \infty.$$

(iv) $\langle \tau_g, \tau_g \rangle = 1$ for all $g \in G$.

(v)

$$|(\mu_G \times \mu_G)(\Delta)| = |\langle \tau_g, \tau_h \rangle|^2 = \frac{|\mu(G)|^2}{|d|}, \quad \forall g, h \in G, g \neq h.$$

(vi) $\|\tau_g\| = 1$ for all $g \in G$.

Theorem 2.11, Theorem 3.1 and Corollary 3.2 give the following problems.

Question 3.9. Whether there is a continuous p -adic functional version of Theorem 2.11? In particular, does there exists a version of

- (i) continuous p -adic functional Bukh-Cox bound?
- (ii) continuous p -adic functional Orthoplex/Rankin bound?
- (iii) continuous p -adic functional Levenstein bound?
- (iv) continuous p -adic functional Exponential bound?

Question 3.10. Whether there is a continuous p -adic version of Theorem 2.11? In particular, does there exists a version of

- (i) continuous p -adic Bukh-Cox bound?
- (ii) continuous p -adic Orthoplex/Rankin bound?
- (iii) continuous p -adic Levenstein bound?
- (iv) continuous p -adic Exponential bound?

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