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Article

A generalized model for pricing financial derivatives consistent with efficient markets hypothesis – a refinement of the Black-Scholes model

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Abstract: This research article provides criticism and arguments why the canonical framework for derivatives pricing is incomplete and why the delta-hedging approach is not appropriate. An argument is put forward, based on the efficient market hypothesis, why a proper risk-adjusted discount rate should enter into the Black-Scholes model instead of the risk-free rate. The resulting pricing equation for derivatives and in particular the formula for European call options is then shown to depend explicitly on the drift of the underlying asset, which is following a geometric Brownian motion. It is conjectured that with the generalized model, the predicted results by the model could be closer to real data. The adjusted pricing model could partly also explain the mystery of volatility smile. The present model also provides answers to many finance professionals and academics who have been intrigued by the risk-neutral features of the original Black-Scholes pricing framework. The model provides generally different fair values for financial derivatives compared to the Black-Scholes model. In particular, the present model predicts that the original Black-Scholes model tends to undervalue for example European call options.

Keywords: options pricing; financial derivatives; efficient market hypothesis; martingale; Feynman-Kac; Black-Scholes

1. Introduction

Pricing of financial derivatives was revolutionized in 1973, when the famous Black-Scholes framework was introduced (Black and Scholes 1973). The value of, say a European call option is given by a linear parabolic partial differential equation, and an explicit formula is available to compute the value of the option, given parameters. The explicit formulas are obtained by transforming the Black-Scholes partial differential equation (PDE) into a constant coefficient PDE and using Fourier methods for example. The Black-Scholes PDE can also be seen as a Hamilton-Jacobi-Bellman equation for a certain stochastic control problem (Lindgren 2020). The parameters needed are the risk-free rate, volatility, exercise price and time to maturity. In empirical terms, the Black-Scholes model does not predict true market values of options, so as a scientific model, it performs rather poorly. However, it has been thought that it gives a reasonable benchmark for traders and financial markets professionals, as well as for risk managers. One classical narrative against the assumptions of the model comes from (Bergman 1982) and (Musiela and Rutkowski 2005), where it is argued that the hedging portfolio is not self-financing in the first place in the original model. The present approach goes further and claims that the whole premise of the model is too narrow and in particular the argument related to delta-hedging is almost irrelevant to any real speculant, hedger or investment professional.

One of the hardest concepts to understand intuitively within the Black-Scholes model is indeed the fact that in the Black-Scholes formulas, the fair price of the option does not depend on the drift of the underlying asset. This is a result of the framework, even though the geometric Brownian motion for the underlying asset is assumed to have some non-zero drift. The prediction of the model is rather counterintuitive - one would assume that

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placing one's bet on an option with a higher positive drift would affect the value of the respective call option in a positive manner. The Black-Scholes model in effect works as if the market was risk-neutral as a whole, or in other words, as if the drift of the asset was equal to the risk-free rate.

The mathematical reason why the true drift is irrelevant in the Black-Scholes model is nevertheless a direct consequence of the hedging portfolio approach and the reason is the following: in the hedging portfolio, one is shorting the underlying asset an amount which is the delta of the option. In terms of the hedging portfolio, if the asset has a higher positive drift, the call option goes up in terms of value, but the short position goes down in terms of value and the effects cancel each other out exactly. That is the reason why the hedging portfolio should be instantaneously risk free and yield the risk free rate. This is the core of the Black-Scholes reasoning. The value of the option is determined as a kind of residual, in order to keep the hedging portfolio locally risk-free. There are however at least two main problems with this approach.

First, the fair price of a call option given by the Black-Scholes model is thus the fair value merely for the hedging portfolio holder, and this is the main problem with the model. If a randomly chosen market participant buys a call option on say a stock of a blue-chip company, it is unreasonable to assume that he or she holds a short position on the company exactly an amount corresponding to the delta of the call option. If a speculant holds a long call option, his portfolio is probably not completely offset by shorting the respective underlying asset.

On top of this conceptual problem, there is the well-known technical challenge in terms of self-financing portfolios. The key assumption in the Black-Scholes model is the assumed self-financing of the hedging portfolio, in other words, it is assumed that there is no net flows of funding in or out of the hedging portfolio. A self-financing delta-hedging portfolio in this case means that the holdings of the option and underlying are not changed, i.e. the changes in the value of the portfolio come purely from changes in the value of the option and the underlying asset. However, this seems to be false and is quite clearly argumented in (Bergman 1982) and (Bartels 1995). This possibly fundamentally flawed assumption of a self-financing hedging portfolio might mean in itself that the rate of return on the hedged portfolio is not truly riskless. If this is indeed the case, the Black-Scholes pricing model assumptions are fundamentally problematic. Then again, the false assumption of a self-financing portfolio does not destroy the Black-Scholes model as such, as is argued in (Bana 2007). On top of this, naturally the assumption of geometric Brownian motion for the underlying asset is itself perhaps not fully correct, but is less of concern. Whether a geometric Brownian motion is a proper model for asset price dynamics is an important issue, but it does not affect qualitatively the reasoning within the Black-Scholes model.

In literature concerning imperfect hedging portfolios due to market incompleteness, the value of the contingent claim is shown to lie within some hedging bounds, see for example (Hao 2008). On the other hand, costly short-selling has been shown to affect the bid-ask-spreads of options, see (Atmaz and Basak 2019). An equal risk pricing rule in incomplete markets was developed in (Guo and Zhu 2017), and further developed in (Marzban et al. 2022). For equal risk pricing using deep learning, see (Carbonneau and Godin 2021). These approaches do not however consider the fundamental problem of delta-hedging, so that usually holding a long call option, the portfolio is probably not completely offset by shorting the respective underlying asset.

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2. Properly anticipated prices in the options pricing framework - a general framework for pricing derivatives without the hedging portfolio

The present model argues that the canonical framework for options pricing based on Black-Scholes pricing is to be generalized to be consistent with the efficient markets hypothesis as put forward by Paul Samuelson (Samuelson 1965, Samuelson 1973). The traditional approach of delta-hedging is therefore not suitable in general; instead, we need to consider first what the appropriate discount rate of an efficient financial market is. The underlying asset following geometric Brownian motion is not a martingale due to its drift as such, but properly discounted it is. To require the martingale property from the discounted price process is in line with (Samuelson 1973) is a mathematical consequence of the efficient market hypothesis (Fama 1965, Fama 1970). The key assumption thus is that the properly discounted process must be a martingale in an efficient market.

Consider an asset such as a common stock.

Suppose that an asset price X_τ follows geometric Brownian motion:

$$dX_\tau = \mu X_\tau d\tau + \sigma X_\tau dW_\tau, \ (1)$$

with some drift of instantaneous return $\mu > 0$ and volatility $\sigma > 0$. W_τ is a standard Brownian motion, and we consider valuing a generic financial derivative written on the asset.

In line with the formulation of the efficient market hypothesis by Samuelson (Samuelson 1973), we now require that the discounted price process is a martingale. Given that for a geometric Brownian motion, the expectation for the price at future time $T > 0$ is given by:

$$E_t(X_T) = X_t e^{\mu(T-t)}, \ (2)$$

where E_t is an expectation operator with respect to the probability measure generated by the Brownian motion. We see immediately, that we need to introduce a discount factor $e^{-\mu(T-t)}$ for the market in order to obtain:

$$X_t = E_t\left(e^{-\mu(T-t)} X_T\right). \ (3)$$

This requirement of market efficiency can be interpreted as follows: the expected discounted price of an asset at future time must be equal to the current price of the asset. In other words, the risk aversion preferences of the market as a whole are reflected in the discount factor and all relevant information is already reflected in the current price of the asset.

Consider now a financial derivative written on the asset. The financial derivative has a payoff at terminal time T and a respective payoff function $\varphi(X_T)$. In line with above, we are looking the fair value independent of individual risk preferences of market participants, instead we discount the derivative payoff using the market discount function above and evaluate the expectation according to the physical or real probability measure:

$$C(x(t), t) = E_t\left(e^{-\mu(T-t)} \varphi(X_T)\right), \ (4)$$

where $C(x(t), t)$ is the fair value of the financial derivative at time t when the asset has some known price $x(t)$. It is straightforward to use the Feynman-Kac formula (Pavliotis 2014) to write down the partial differential equation describing the evolution of the value of the financial derivative:

$$\frac{\partial C}{\partial t}(x,t) + \mu x \frac{\partial C}{\partial x}(x,t) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(x,t) - \mu C(x,t) = 0. \ (5)$$

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With the initial condition $C(x,T) = \varphi(X_T)$. Notice that the only difference to the canonical Black-Scholes partial differential equation is that the risk-free rate is replaced with the discount rate reflecting the efficiency of the financial market. As an instructive example, consider now pricing a plain vanilla European call option. The payoff is $\varphi(X_T) = max\,(X_T - K, 0)$, where $K > 0$ is the exercise or strike price of the call option maturing at time $T > t$. As everything else is the same as in the Black-Scholes pricing PDE, we can deduce the price for a European call option easily by using the well-known formulas and by just replacing the risk-free rate in the formulas with the drift of the underlying asset:

$$C(X_t, t) = N(d_1)X_t - N(d_2)Ke^{-\mu(T-t)} \quad (6)$$

where N is the cumulative distribution function of the standard normal variable, and

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(Log\left(\frac{X_t}{K}\right) + \left(\mu + \frac{1}{2}\sigma^2\right)(T-t)\right) \quad (7)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \quad (8)$$

It is instructive to note that now the price of the European call depends explicitly on the drift of the asset, as it should intuitively be and the Rho of the option for the call is positive so that an increase in the drift increases the value of the European long call. The only difference with the canonical Black-Scholes model is therefore that the risk-free rate is replaced by the drift of the underlying asset performing geometric Brownian motion.

3. Discussion and conclusions

It is argued that based on the assumption that in an efficient market the risk-adjusted discounted expected price of an asset following geometric Brownian motion should be the current price, it is deduced that the correct risk-adjusted discount rate should be the drift of the asset process. These assumptions will lead to derivatives pricing, where the fair value of the financial derivative can be evaluated as a conditional expectation, discounted at the above rate reflecting market efficiency. The fair value for an option can then be solved by using the Feynman-Kac formula, leading to a modified Black-Scholes PDE, where the drift of the underlying process is explicitly present.

The results thus suggest that the approach based on the hedging portfolio is too limited. The fair price of an option in the Black-Scholes approach is based on the idea of a hedging portfolio. The value of the financial derivative is forced to be such that the hedging portfolio yields exactly the risk-free rate. In the present approach, it is argued that the delta-hedge approach is not sufficient in general, as it implicitly requires that the option holders have that delta-hedged portfolio. For an investor with the delta-hedging portfolio, it is indeed true that the drift of the underlying asset does not make any difference. If the underlying goes up, the short position on the underlying goes down and the call option gains in value; these effects cancel each other out. However, for a general investor, there is no perfectly hedging portfolio and the increase in the drift of the asset has an effect on the value of the call option, for example.

In this article, it is shown mathematically, based on the theory of efficient markets, why the drift should in fact matter when pricing financial derivatives. The present model predicts that for a generic call option holder, the price of a European call option depends explicitly on the drift of the underlying asset. The results might lead to better empirical results when comparing the actual prices of options in the markets and the theoretical prices predicted by the present model. Furthermore, volatility smile (Derman and Miller 2016) should be examined through the lens of this extended model. The higher discount

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rate compared to the original Black-Scholes model implies that the fair value of the financial derivative depends in general on the drift of the underlying asset performing geometric Brownian motion. Therefore, if an option holder is long in a European call option, the value of the call is higher for underlying assets, which have higher instantaneous return or drift. The reason is simply that the fair value of the option is not valued in terms of a hedging portfolio, but instead demanding that the market risk aversion is such that assets with a drift are martingales when discounted properly. The well-known discrepancy between option market data and the values predicted by the Black-Scholes model has been empirically verified many times.

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