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On the Basic Convex Polytopes and *n*-Balls in Complex Dimensions

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Abstract: This paper extends the findings of the prior research concerning *n*-balls, regular *n*-simplices, and *n*-orthoplices in real dimensions using recurrence relations that removed the indefiniteness present in known formulas. The main result of this paper is the proof that these recurrence relations are continuous for complex *n*, whereas in the indefinite points their values are given in the sense of a limit of a function. It is shown that the volume of an *n*-simplex is a bivalued function for $n < 0$, and thus the surfaces of *n*-simplices and *n*-orthoplices are also bivalued functions for $n < 1$. Applications of these formulas to these omnidimensional polytopes inscribed in and circumscribed about *n*-balls reveal previously unknown properties of these geometric objects in negative, real dimensions. In particular for $0 < n < 1$ the volumes of the omnidimensional polytopes are larger than volumes of circumscribing *n*-balls, while their volumes and surfaces are smaller than volumes of inscribed *n*-balls.

Keywords: regular basic convex polytopes; negative dimensions; fractal dimensions; complex dimensions; circumscribed and inscribed polytopes

1. Introduction

The dimension *n* is intuitively defined as a number of coordinates of a point within Euclidean space $\mathbb{R}^n$. However, this is not the only possible definition [1, 2] of a dimension of a set. Negative dimensions [2-5] , for example, can be defined by analytic continuations from positive dimensions [6] and refer to densities, rather than to sizes as the natural ones and. Fractional (or fractal), including negative [7], dimensions have been shown to be consistent with experimental results and enable the examination of transport parameters in multiphase fractal media [8, 9]. Complex [2], including complex fractional [10], dimensions can also be considered.

This study extends prior research [11] presenting novel recurrence relations for volumes and surfaces of *n*-balls, regular *n*-simplices, and *n*-orthoplices in real dimensions, to complex, continuous dimensions. The recurrence relations presented in the prior research remove indefiniteness and singularities present in known formulas, revealing the properties of the relevant geometric objects in negative and real dimensions. It was signaled in the prior research that these recurrence relations are continuous on their domains of definitions for $n \in \mathbb{R}$, whereas the starting points for fractional dimensions can be provided, e.g., using spline interpolation between two (or three in the case of *n*-balls) subsequent integer dimensions. It was also shown that for $n < -1$, $n \in \mathbb{Z}$, *n*-simplices are undefined in the negative, integer dimensions, and their volumes and surfaces are imaginary in the negative, fractional ones and divergent with decreasing $n \in \mathbb{R}$. The volumes and surfaces of *n*-cubes inscribed in *n*-balls in negative dimensions were also examined in the prior research and shown to be complex, and associated with integral powers of the imaginary unit in negative, integer dimensions. The conclusion of the prior research was that out of three regular, convex polytopes present in all natural dimensions, only n-orthoplices and n-cubes (and n-balls) are defined in all negative, integer dimensions. The reader is referred to this previous publication [11] for full details.

This study shows that the recurrence relations of the prior research [11] are indeed continuous, whereas their values at the singular points can be given in the sense of a limit of a function. The properties of the three omnidimensional, regular, convex polytopes, present for all $n \in \mathbb{N}_0$ [12], inscribed in and circumscribed about n-balls are presented. It is also shown, in the exemplary case of n-balls, that volumes and surfaces of these geometric objects are, in general, complex numbers if $n \in \mathbb{C}$.

The paper is structured as follows. Section 2 summarizes known non-recurrence formulas for omnidimensional, regular, convex polytopes in natural dimensions that are employed in the further sections of the paper. Section 3 summarizes recurrence relations for these geometric objects presented in the prior research [11]. In Section 4 it is shown that these recurrence relations can be naturally extended to complex, continuous dimensions. Section 5 examines the properties of the omnidimensional, regular, convex polytopes inscribed in and circumscribed about n-balls for $n \in \mathbb{R}$. Section 6 presents complex volumes and surfaces of n-balls in complex dimensions. Section 7 hints possible applications and concludes the findings of this paper.

2. Known non-Recurrence Formulas

The known volume of an n-ball (B) is

$$V_n(R)_B = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n, \tag{1}$$

where $\Gamma(\mathbb{C} \to \mathbb{C})$ is the Euler's gamma function and R denotes the n-ball radius. This implies that volumes of n-balls are complex in complex dimensions (cf. Section 6) and becomes

$$V_{2k}(R)_B = \frac{\pi^k R^{2k}}{k!} \tag{2}$$

for even $n = 2k$, $k \in \mathbb{N}_0$, and

$$V_{2k-1}(R)_B = \frac{2^{2k} \pi^{k-1} k!}{(2k)!} R^{2k-1} \tag{3}$$

for odd $n = 2k - 1$, $k \in \mathbb{N}$.

Known [13] $(n-1)$-dimensional surface of an n-ball is

$$S_n(R)_B = \frac{n}{R} V_n(R)_B. \tag{4}$$

Known [14, 15] volume of a regular n-simplex (S) having the edge length A is

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$$V_n(A)_S = \frac{\sqrt{n+1}}{n!\sqrt{2^n}} A^n, \tag{5}$$

Aby n-simplex has $n + 1$ $(n - 1)$-facets [13]. Therefore, its surface is

$$S_n(A)_S = (n+1)V_{n-1}(A)_S. \tag{6}$$

Known [13] volume of n-orthoplex (O) is

$$V_n(A)_O = \frac{\sqrt{2^n}}{n!} A^n. \tag{7}$$

Any n-orthoplex has 2^n facets [13], which are regular $(n - 1)$-simplices. Therefore, its surface is

$$S_n(A)_O = 2^n V_{n-1}(A)_S. \tag{8}$$

Since the factorial is defined only for non-negative integers, formulas (2), (3), (5), and (7) are indefinite in negative dimensions.

3. Recurrence Relations in Integer Dimensions

The volume of an n-ball can be expressed [13] in terms of the volume of an $(n - 2)$-ball of the same radius as a recurrence relation

$$V_n(R)_B = \frac{2\pi R^2}{n} V_{n-2}(R)_B, \tag{9}$$

where $V_0(R)_B := 1$ and $V_1(R)_B := 2R$. It was shown in the prior research [11] that the relation (9) can be extended into negative dimensions as

$$V_n(R)_B = \frac{n+2}{2\pi R^2} V_{n+2}(R)_B, \tag{10}$$

solving (9) for V_{n-2} and assigning new $n \in \mathbb{Z}$ as the previous $n - 2$.

A radius recurrence relation

$$f_n \doteq \frac{2}{n} f_{n-2}, \tag{11}$$

defined in the prior research [11] for $n \in \mathbb{N}$, where $f_0 := 1$ and $f_1 := 2$, allows for expressing the volume n-ball as

$$V_n(R)_B \doteq f_n \pi^{\lfloor n/2 \rfloor} R^n, \tag{12}$$

where "$\lfloor x \rfloor$" denotes the floor function giving the greatest integer less than or equal to its argument x. It was further shown in the prior research [11] that the relation (11) can be analogously as formula (9) extended into negative dimensions as

$$f_n = \frac{n+2}{2} f_{n+2}, \tag{13}$$

which removes the indefiniteness of the factorial present in Formulas (2)-(3) and allows to define $f_{-1} := 1$, $f_0 := 1$ to initiate (11) or (13).

It was shown in the prior research [11] that formula (5) can be written as a recurrence relation

$$V_n(A)_S \doteq AV_{n-1}(A)_S \sqrt{\frac{n+1}{2n^3}}, \tag{14}$$

with $V_0(A)_S := 1$, to remove the indefiniteness of the factorial for $n < 1$. Formula (14) can be solved for V_{n-1}. Assigning new $n \in \mathbb{Z}$ as the previous $n - 1$, yields [11]

$$V_n(A)_S = \frac{V_{n+1}(A)_S}{A} \sqrt{\frac{2(n+1)^3}{n+2}}, \tag{15}$$

which also removes the singularity for $n = 0$ present in known formula (5) defining the volumes of regular n-simplices in natural dimensions.

Finally it was shown in the prior research [11] that formula (7) can be written as a recurrence relation

$$V_n(A)_O \doteq AV_{n-1}(A)_O \frac{\sqrt{2}}{n}, \tag{16}$$

with $V_0(A)_O := 1$, to remove the indefiniteness of the factorial for $n < 1$. Solving (16) for V_{n-1} and assigning new $n \in \mathbb{Z}$ as the previous $n - 1$, yields [11]

$$V_n(A)_O = V_{n+1}(A)_O \frac{n+1}{A\sqrt{2}}, \tag{17}$$

which also removes singularity for $n = 0$ present in known formula (7) defining the volumes of n-orthoplices the regular n-simplices in natural dimensions.

4. Continuous Relations in Complex Dimensions

The recurrence relations presented in the preceding section can be naturally extended to complex, continuous dimensions.

Theorem 1.

Recurrence relations (9), (10), (12) (n-balls) are continuous for $n \in \mathbb{C}$, wherein for $n = -2k - 2$, $k \in \mathbb{N}_0$ their values are given in the sense of a limit of a function.

Proof 1.

Comparing (1) with (10) and setting $m = n + 2$ and $k = m/2$, yields

$$\begin{gathered}V_n(R)_B=\frac{\pi^{n/2}}{\Gamma(n/2+1)}R^n=\frac{n+2}{2\pi R^2}V_{n+2}(R)_B\\V_{n+2}(R)_B=\frac{\pi^{n/2}2\pi^{2/2}}{(n+2)\Gamma(n/2+1)}R^{n+2}\quad V_m(R)_B=\frac{\pi^{m/2}2}{m\Gamma(m/2)}R^m=\frac{\pi^k}{k\Gamma(k)}R^{2k},\\V_n(R)_B=\frac{\pi^{n/2}}{\Gamma(n/2+1)}R^n\quad V_n(D)_B=\frac{\pi^{n/2}}{2^n\Gamma(n/2+1)}D^n\end{gathered}\tag{18}$$

which recovers (1), as $n\Gamma(n/2)/2 = \Gamma(n/2 + 1)$ for $n > 0$, $n \in \mathbb{C}$. On the other hand, (10) corresponds to (12)

$$\begin{gathered}V_n(R)_B=\frac{n+2}{2\pi R^2}V_{n+2}(R)_B=\frac{n+2}{2}f_{n+2}\pi^{\lfloor n/2\rfloor}R^n\\V_{n+2}(R)_B=\pi^1f_{n+2}\pi^{\lfloor n/2\rfloor}R^{n+2}\quad V_m(R)_B=f_m\pi^{1+\lfloor(m-2)/2\rfloor}R^m=f_m\pi^{\lfloor m/2\rfloor}R^m\end{gathered},\tag{19}$$

for $n \in \mathbb{C}$, which completes the proof. □

Also

$$\lim_{n\to-2k-2,\,k\in\mathbb{N}_0}\pi^{n/2}D^n2^{-n}\frac{1}{\Gamma(n/2+1)}=a\cdot0=0,\tag{20}$$

where a $\neq 0$, $a \in \mathbb{C}$.

Using (4) and (18) the surface of an n-balls is given by

$$S_n(D)_B=\frac{2^{1-n}n\pi^{n/2}}{\Gamma(n/2+1)}D^{n-1}.\tag{21}$$

Theorem 2.

Recurrence relations (14), (15) (regular n-simplices) are continuous for $n \in \mathbb{C}$, wherein for $n = -k - 1$, $k \in \mathbb{N}_0$ their values are given in the sense of a limit of a function.

Proof 2.

Expressing the factorial in (5) by the gamma function, comparing (5) with (15), and setting $m = n + 1$, yields

$$V_n(A)_S = \frac{\sqrt{n+1}}{n!\sqrt{2^n}} A^n = \frac{\sqrt{n+1}}{\Gamma(n+1)2^{n/2}} A^n = \frac{V_{n+1}(A)_S}{A}\sqrt{\frac{2(n+1)^3}{n+2}}$$
$$V_{n+1}(A)_S = \frac{\sqrt{n+1}\sqrt{n+2}}{\Gamma(n+1)2^{(n+1)/2}\sqrt{(n+1)^3}} A^{n+1} \quad , \tag{22}$$
$$V_m(A)_S = \frac{m\sqrt{m}\sqrt{m+1}}{\Gamma(m+1)2^{m/2}\sqrt{m^3}} A^m \quad V_n(A)_S = \frac{\sqrt{n+1}}{\Gamma(n+1)2^{n/2}} A^n \begin{cases} 1 & n \geq 0 \\ \pm 1 & n < 0 \end{cases}$$

which recovers (5), as $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, and completes the proof. □

Also

$$\lim_{n \to -k-1,\, k \in \mathbb{N}_0} 2^{-n/2} A^n \sqrt{n+1} \frac{1}{\Gamma(n+1)} = a \cdot 0 = 0, \tag{23}$$

where $a \in \mathbb{C}$.

For $n < -1$ n-simplex volume formula (22) is imaginary and for $n < 0$ it is a bivalued function, as $n\sqrt{n}/\sqrt{n^3} = 1$ only for $n \in \mathbb{R}$, $n > 0$. Thus, its general form is

$$V_n(A)_S = \frac{\sqrt{n+1}}{\Gamma(n+1)2^{n/2}} A^m \frac{n\sqrt{n}}{\sqrt{n^3}}. \tag{24}$$

Using (6) and (22) the surface of a regular n-simplex is given by

$$S_n(A)_S = \frac{n(n+1)\sqrt{n}}{\Gamma(n+1)2^{(n-1)/2}} A^{n-1} \begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}. \tag{25}$$

For $n < 0$ n-simplex surface formula (25) is imaginary and for $n < 1$ it is a bivalued function, as $(n-1)\sqrt{(n-1)}/\sqrt{(n-1)^3} = 1$ only for $n \in \mathbb{R}$, $n > 1$. Thus, its general form is

$$S_n(A)_S = \frac{n(n+1)\sqrt{n}}{\Gamma(n+1)2^{(n-1)/2}} A^{n-1} \frac{(n-1)\sqrt{n-1}}{\sqrt{(n-1)^3}}. \tag{26}$$

Theorem 3.

Recurrence relations (16), (17) (n-orthoplices) are continuous for $n \in \mathbb{C}$, wherein for $n = -k-1$, $k \in \mathbb{N}_0$ their values are given in the sense of a limit of a function.

Proof 3.

Expressing the factorial in (7) by the gamma function, comparing (7) with (17), and setting $m = n + 1$, yields

$$\begin{aligned} &V_n(A)_O = \frac{\sqrt{2^n}}{n!}A^n = \frac{\sqrt{2^n}}{\Gamma(n+1)}A^n = V_{n+1}(A)_O \frac{n+1}{A\sqrt{2}} \\ &V_{n+1}(A)_O = \frac{2^{(n+1)/2}}{(n+1)\Gamma(n+1)}A^{n+1} \\ &V_m(A)_O = \frac{\sqrt{2^m}}{m\Gamma(m)}A^m \quad V_n(A)_O = \frac{2^{n/2}}{\Gamma(n+1)}A^n \end{aligned}, \tag{27}$$

which recovers (7), as $n\Gamma(n) = \Gamma(n + 1)$ for $n \in \mathbb{C}\backslash\{n \in \mathbb{Z}, n \leq -1\}$ and $\Gamma(n + 1) = n!$ for $n \in \mathbb{N}_0$, and completes the proof. □

Using (8) and (27) the surface of an n-orthoplex is given by

$$S_n(A)_O = \frac{n2^{(n+1)/2}\sqrt{n}}{\Gamma(n+1)}A^{n-1}\begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}. \tag{28}$$

For $n < 0$, $n \notin \mathbb{Z}$ n-orthoplex surface formula (28) is imaginary and for $n < 1$ it is a bivalued function, as $(n - 1)\sqrt{(n - 1)}/\sqrt{(n - 1)^3} = 1$ only for $n \in \mathbb{R}$, $n > 1$. Thus, its general form is

$$S_n(A)_O = \frac{n2^{(n+1)/2}\sqrt{n}}{\Gamma(n+1)}A^{n-1}\frac{(n-1)\sqrt{n-1}}{\sqrt{(n-1)^3}}. \tag{29}$$

Continuous recurrence relations (18)-(29) are shown in Figures 1-4.

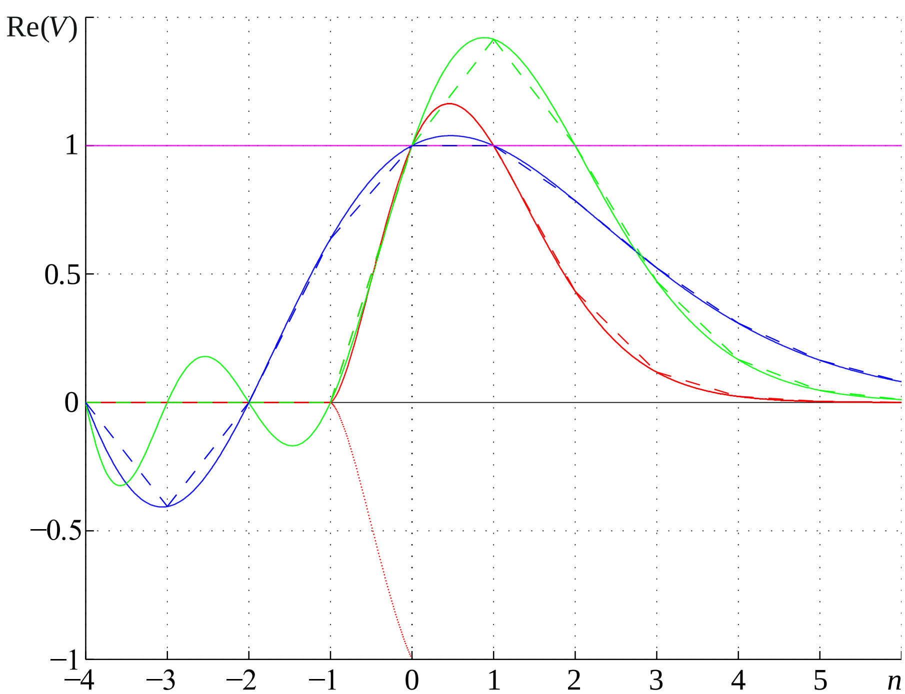

Figure 1. Graphs of the real part of volumes (V) of unit edge length regular n-simplices (red), n-orthoplices (green), n-cubes (pink), and unit diameter n-balls (blue), along with the integer recurrence relations (dashed lines) and the branch for n-simplices (dotted line) for $n = [-4, 6]$.

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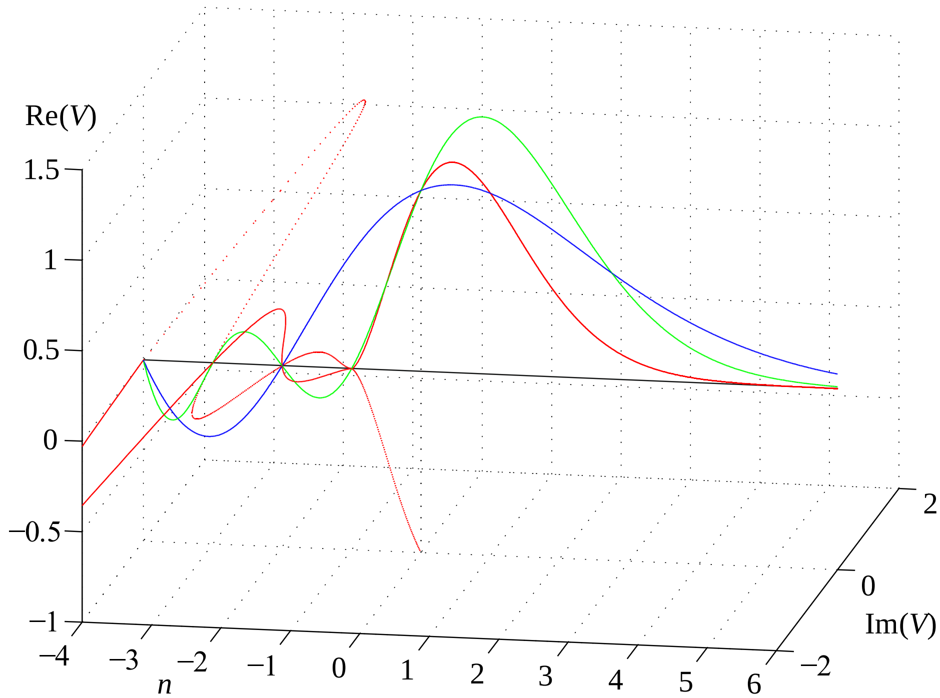

Figure 2. Graphs of volumes (V) of unit edge length regular n-simplices (red), n-orthoplices (green), and unit diameter n-balls (blue), along with the branch for n-simplices (dotted line) for $n = [-4, 6]$.

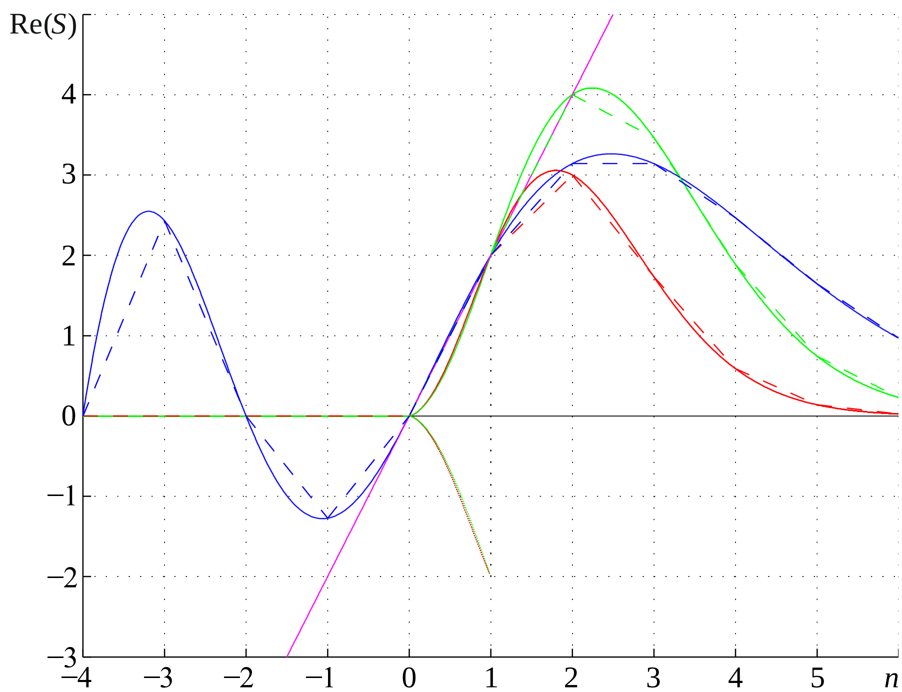

Figure 3. Graphs of the real part of surfaces (S) of unit edge length regular n-simplices (red), n-orthoplices (green), n-cubes (pink), and unit diameter n-balls (blue), along with the integer recurrence relations (dashed lines) and the branches for n-simplices and n-orthoplices (dotted lines) for $n = [-4, 6]$.

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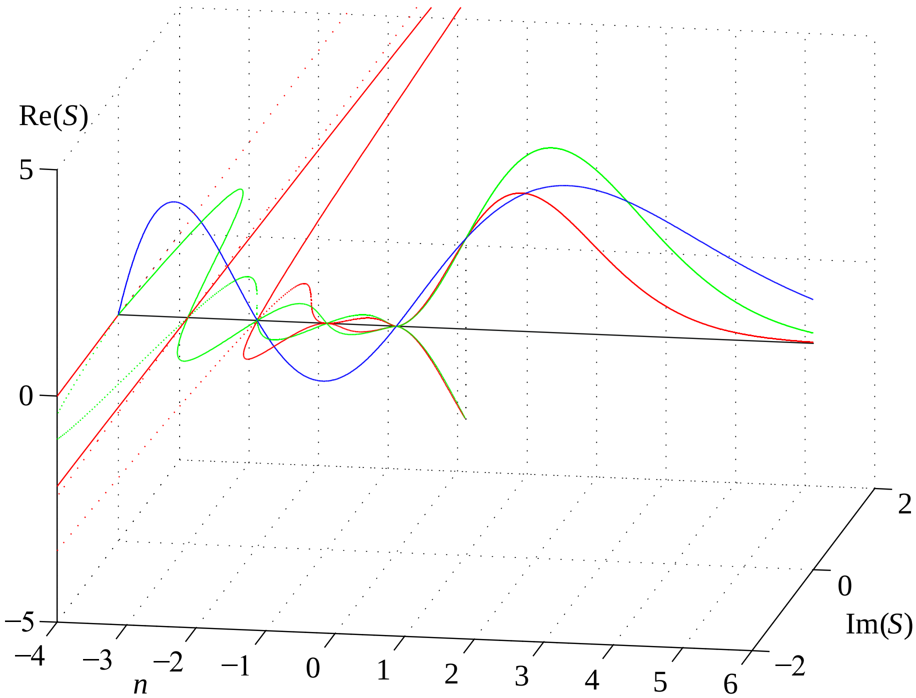

Figure 4. Graphs of surfaces (S) of unit edge length regular n-simplices (red), n-orthoplices (green), and unit diameter n-balls (blue), along with the branches for n-simplices and n-orthoplices (dotted lines) for $n = [-4, 6]$.

5. Basic Regular Polytopes Inscribed in and Circumscribed About *n*-Balls

Anyone of the three regular polytopes can be inscribed in and circumscribed about an n-ball, and this is considered in this section on the basis of the continuous relations presented in the previous one. The principal branches of their volumes and surfaces are summarized in the Table 1. n-balls are defined in terms of their diameters, which concept is closer to the concept of the edge length of a polytope.

Table 1. Volumes and surfaces of regular n-simplices, n-orthoplices, and n-cubes inscribed in and circumscribed about an n-balls.

	inscribed in n-ball (IB)		circumscribed about n-ball (CB)	
	(V) volume/D^n	(S) surface/D^{n-1}	(V) volume/D^n	(S) surface/D^{n-1}
(S)	$\frac{n^{-n/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^n}$ (2)	$\frac{n^{(4-n)/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^{n-1}}$ (2)	$\frac{n^{n/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^n}$ (2)	$\frac{n^{(2+n)/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^{n-1}}$ (2)
(O)	$\frac{1}{\Gamma(n+1)}$ (1)	$\frac{2n^{3/2}}{\Gamma(n+1)}$ (2)	$\frac{n^{n/2}}{\Gamma(n+1)}$ (1)	$\frac{2n^{n/2+1}}{\Gamma(n+1)}$ (2)
(C)	$n^{-n/2}$ (1)	$2n^{(3-n)/2}$ (1)	1 (1)	$2n$ (1)

(1) one branch, (2) two branches.

5.1 Regular n-Simplices Inscribed in and Circumscribed About n-Balls

The diameter D_{BCS} an n-ball circumscribed about a regular n-simplex (BCS) is known [15] to be

$$D_{BCS} = \frac{\sqrt{2n}}{\sqrt{n+1}} A, \tag{30}$$

where A is the edge length. Hence, the edge length A_{SIB} of a regular n-simplex inscribed (SIB) inside an n-ball (B) with diameter D is

$$A_{SIB} = \frac{\sqrt{n+1}}{\sqrt{2n}} D, \tag{31}$$

so that its volume (22) becomes

$$V_n\left(A_{SIB}\right)_S = \frac{n^{-n/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^n} D^n \begin{cases} 1 & n \geq 0 \\ \pm 1 & n < 0 \end{cases}, \tag{32}$$

For $n < -1$, $n \notin \mathbb{Z}$ the inscribed n-simplex volume (32) is imaginary (as $n^{-n/2}$ introduces the imaginary unit for $n < 0$) and divergent with decreasing n, for $n < 0$ it is a bivalued function of n, and is complex for $-1 < n < 0$, whereas in this range both branches are right-handed. It is also zero for $n = -k$, $k \in \mathbb{N}$.

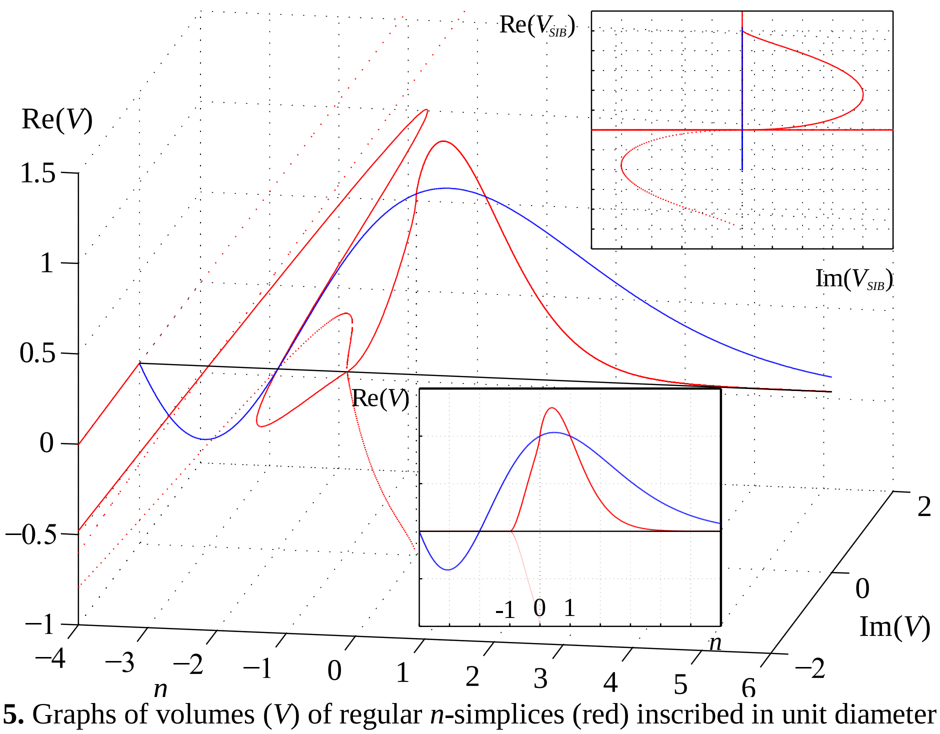

Figure 5. Graphs of volumes (V) of regular n-simplices (red) inscribed in unit diameter n-balls and volumes of unit diameter n-balls (blue) for $n = [-4, 6]$ (inset for $n = [-1, 0]$).

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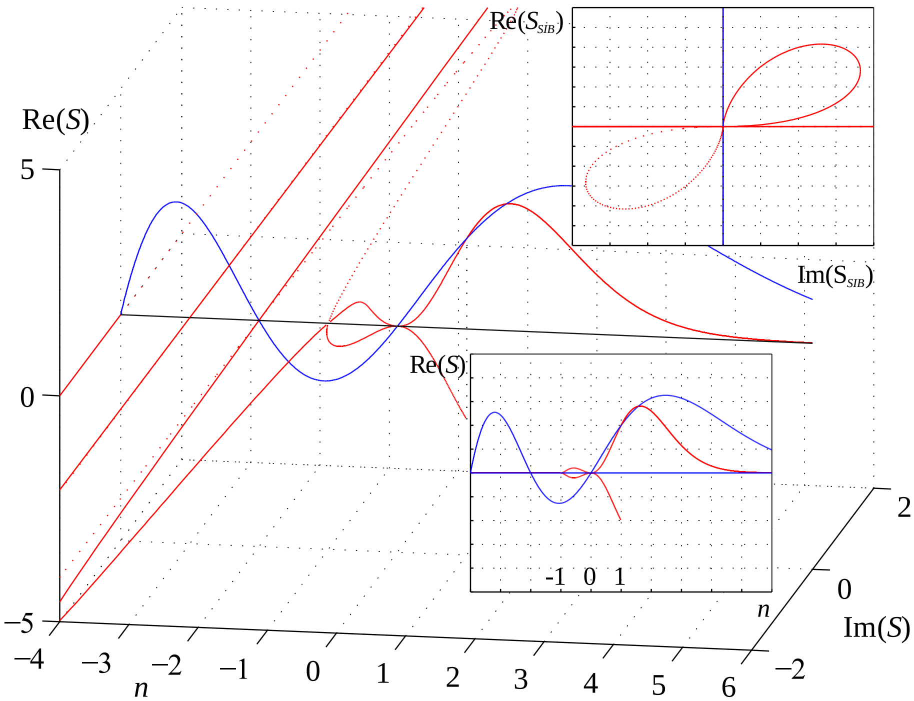

Figure 6. Graphs of surfaces (S) of regular n-simplices (red) inscribed in unit diameter n-balls and surfaces of unit diameter n-balls (blue) for n = [−4, 6] (inset for n = [−1, 0]).

Similarly, the surface (25) of a regular inscribed n-simplex with edge length A given by (31) becomes

$$S_n\left(A_{SIB}\right)_S = \frac{n^{(4-n)/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^{n-1}} D^{n-1} \begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}, \tag{33}$$

as shown in in Figure 6. For $n < -1$, $n \notin \mathbb{Z}$ the inscribed n-simplex surface (33) is imaginary and divergent with decreasing n, for $n < 1$ it is a bivalued function of n, and is complex for $-1 < n < 0$, whereas in this range both branches are right-handed. It is also zero for $n = -k$, $k \in \mathbb{N}_0$.

The diameter D_{BIS} of an n-ball inscribed inside a regular n-simplex (BIS) is known [15] to be

$$D_{BIS} = \frac{\sqrt{2}}{\sqrt{n}\sqrt{n+1}} A, \tag{34}$$

where A is the edge length. Hence, the edge length A_{SCB} of a regular n-simplex circumscribed (SCB) about an n-ball (B) with diameter D is

$$A_{SCB} = \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{2}} D, \tag{35}$$

so that its volume (22) becomes

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$$V_n\left(A_{SCB}\right)_S = \frac{n^{n/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^n} D^n \begin{cases} 1 & n \geq 0 \\ \pm 1 & n < 0 \end{cases}, \tag{36}$$

as shown in in Figure 7. For $n < 0$ the circumscribed n-simplex volume (36) is a complex, bivalued function of n, whereas both branches are left-handed and convergent to zero with decreasing n. For $0 < n < 1$ it is smaller than the volume of the inscribed n-ball (cf. Table 4). Also it is real for $n = -(k + 1)/2$, $k \in \mathbb{N}$ (cf. Appendix 1).

Similarly, the surface (25) of a regular circumscribed n-simplex with edge length A_{SCB} (35) becomes

$$S_n\left(A_{SCB}\right)_S = \frac{n^{(2+n)/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^{n-1}} D^{n-1} \begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}, \tag{37}$$

as shown in in Figure 8. For $n < 1$ the circumscribed n-simplex surface formula (37) branches, for $n < 0$ is complex and is initially divergent to achieve real minimum at $n = -3.5$ and then becomes convergent to zero with decreasing n. Both branches are left-handed towards negative infinity or the branch point, and for $0 < n < 1$ it is smaller than the surface of the inscribed n-ball. Also it is real for $n = -(k + 1)/2$, $k \in \mathbb{N}$ (cf. Appendix 2) and zero for $n = -k$, $k \in \mathbb{N}_0$.

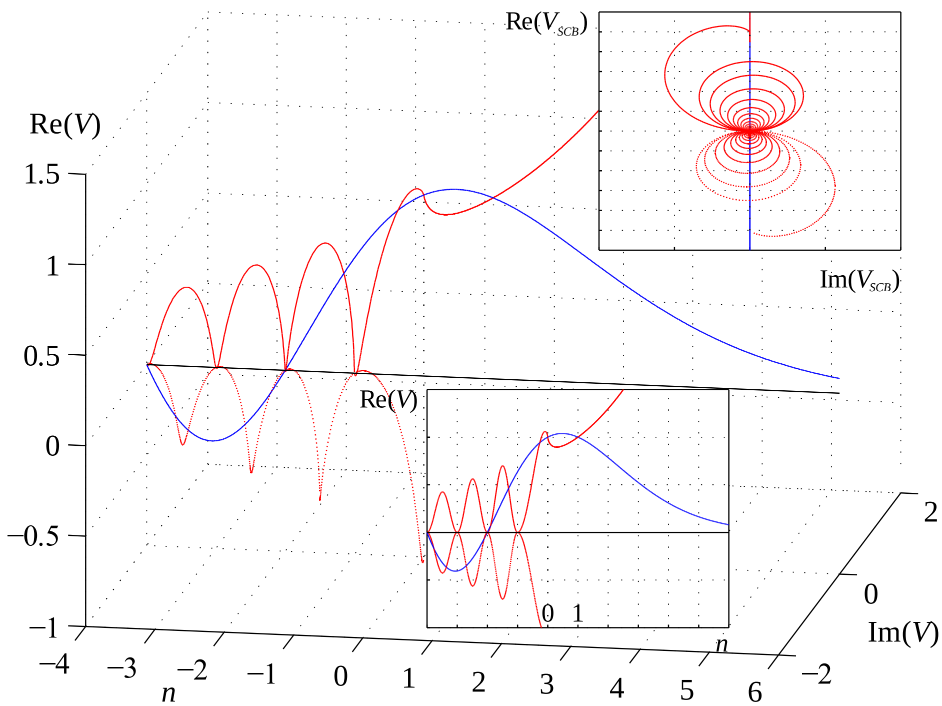


Figure 7. Graphs of volumes (V) of regular n-simplices (red) circumscribed about unit diameter n-balls and volumes of unit diameter n-balls (blue) for $n = [-4, 6]$ (insets for $n = [-30, 0]$).

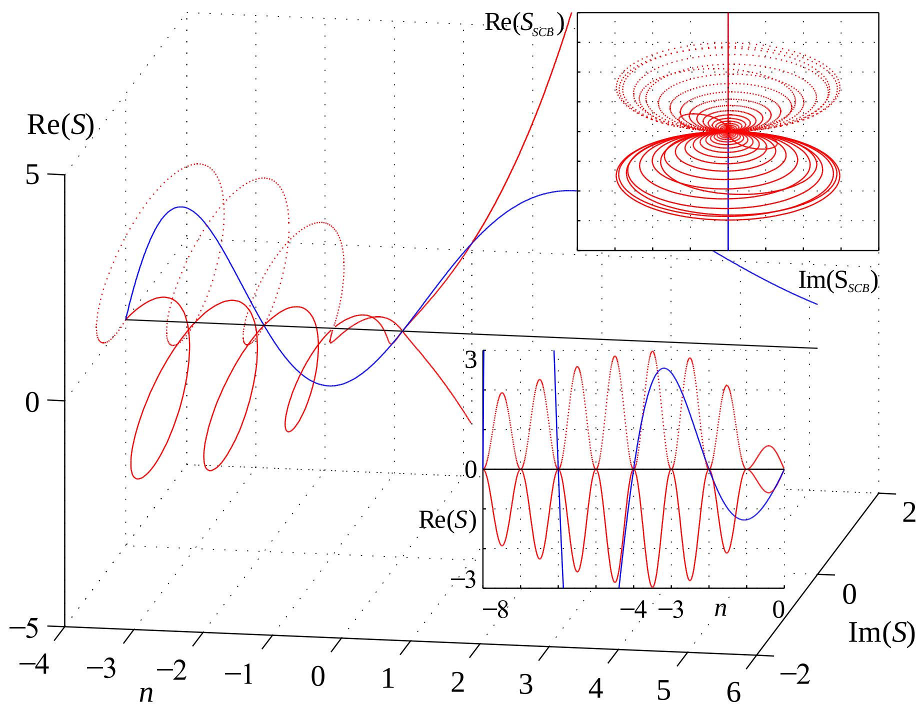


Figure 8. Graphs of surfaces (S) of regular n-simplices (red) circumscribed about unit diameter n-balls and surfaces of unit diameter n-balls (blue) for n = [−4, 6] (inset for n = [−30, 0] and [−8, 0]).

5.2 *n*-Orthoplices Inscribed in and Circumscribed About *n*-Balls

The diameter D_{BCO} of an n-ball circumscribed about an n-orthoplex (BCO) is known [16] to be

$$D_{BCO} = \sqrt{2}A, \tag{38}$$

where A is the edge length. Hence, the edge length A_{OIB} of an n-orthoplex inscribed inside an n-ball (OIB) with diameter D is

$$A_{OIB} = \frac{1}{\sqrt{2}} D, \tag{39}$$

so that its volume (27) becomes

$$V_n\left(A_{OIB}\right)_O = \frac{1}{\Gamma\left(n+1\right)} D^n, \tag{40}$$

as shown in in Figure 9. The inscribed n-orthoplex volume formula (40) is real for $n \in \mathbb{R}$, where for $n = -k$, $k \in \mathbb{N}$, its zero values are given in the sense of a limit of a function (cf. (23)), and for $0 < n < 1$ it is larger than the volume of the circumscribing n-ball (cf. Table 4).

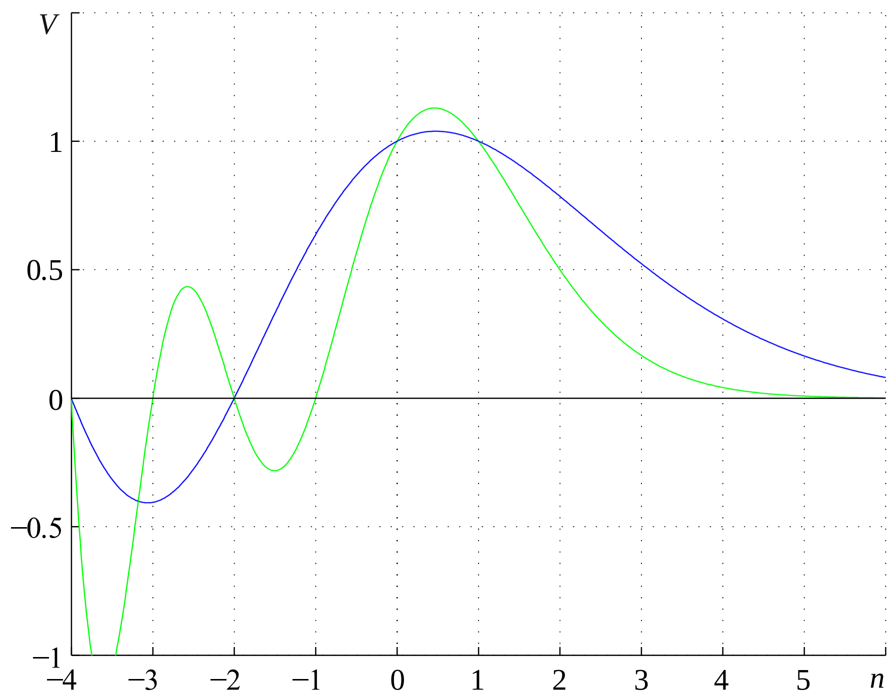


Figure 9. Graphs of volumes (V) of n-orthoplices (green) inscribed in unit diameter n-balls and volumes of unit diameter n-balls (blue) for $n = [-4, 6]$.

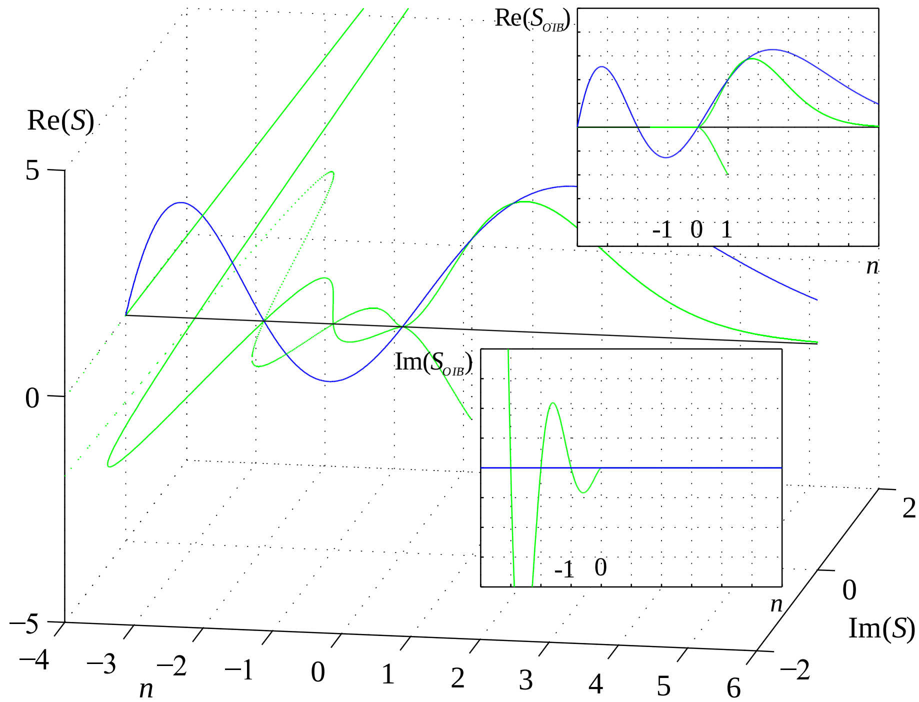


Figure 10. Graphs of surfaces (S) of n-orthoplices (green) inscribed in unit diameter n-balls and surfaces of unit diameter n-balls (blue) for $n = [-4, 6]$.

Similarly, the surface (28) of the inscribed n-orthoplex with edge length A given by (39) becomes

$$S_n\left(A_{OIB}\right)_O=\frac{2n^{3/2}}{\Gamma\left(n+1\right)}D^{n-1}\begin{cases}1 & n\geq 1\\ \pm 1 & n<1\end{cases}, \tag{41}$$

as shown in in Figure 10. For $n < 1$ inscribed n-orthoplex surface (41) branches, for $n < 0$, $n \notin \mathbb{Z}$ it is imaginary and oscillatory divergent with decreasing n, and for $n \leq -1$, $n \in \mathbb{Z}$, its zero values are given in the sense of a limit of a function (cf. (23)).

The diameter D_{BIO} of an n-ball inscribed inside an n-orthoplex (BIO) is known [16] to be

$$D_{BIO}=\sqrt{\frac{2}{n}}A, \tag{42}$$

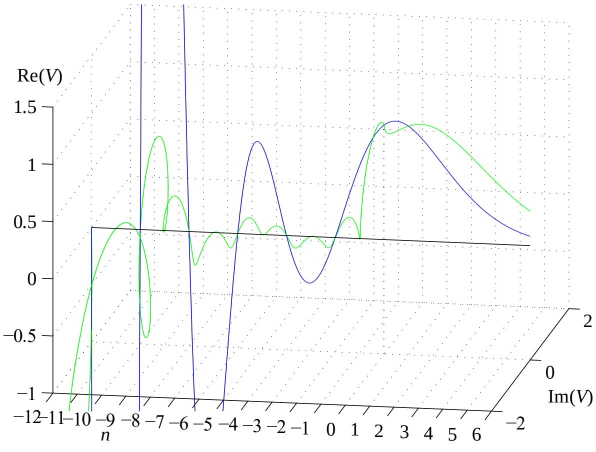
where A is the edge length. Hence, the edge length A_{OCB} of an n-orthoplex circumscribed about an n-ball (OCB) with diameter D is

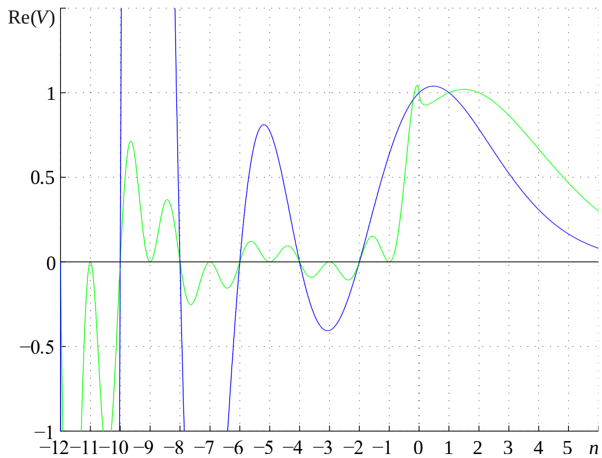
$$A_{OCB}=\sqrt{\frac{n}{2}}D, \tag{43}$$

so that its volume (27) becomes

$$V_n\left(A_{OCB}\right)_O=\frac{n^{n/2}}{\Gamma\left(n+1\right)}D^n, \tag{44}$$

as shown in Figure 11. Circumscribed n-orthoplex volume (44) is a singlevalued function, is complex for $n < 0$, initially convergent, achieves real minimum for $-4 < n < -3$ and becomes left-handedly oscillatory divergent with decreasing n, crossing the quadrants of the complex plane in the order $\{\mathrm{Re}(V_{OCB}) > 0,\ \mathrm{Im}(V_{OCB}) < 0\}$, $\{\mathrm{Re}(V_{OCB}) > 0,\ \mathrm{Im}(V_{OCB}) > 0\}$, $\{\mathrm{Re}(V_{OCB}) < 0,\ \mathrm{Im}(V_{OCB}) > 0\}$, and $\{\mathrm{Re}(V_{OCB}) < 0,\ \mathrm{Im}(V_{OCB}) < 0\}$. For $n = -k$, $k \in \mathbb{N}$, its zero values are given in the sense of a limit of a function (cf. (23)). For $0 < n < 1$ it is smaller than the volume of the inscribed n-ball (cf. Table 4).





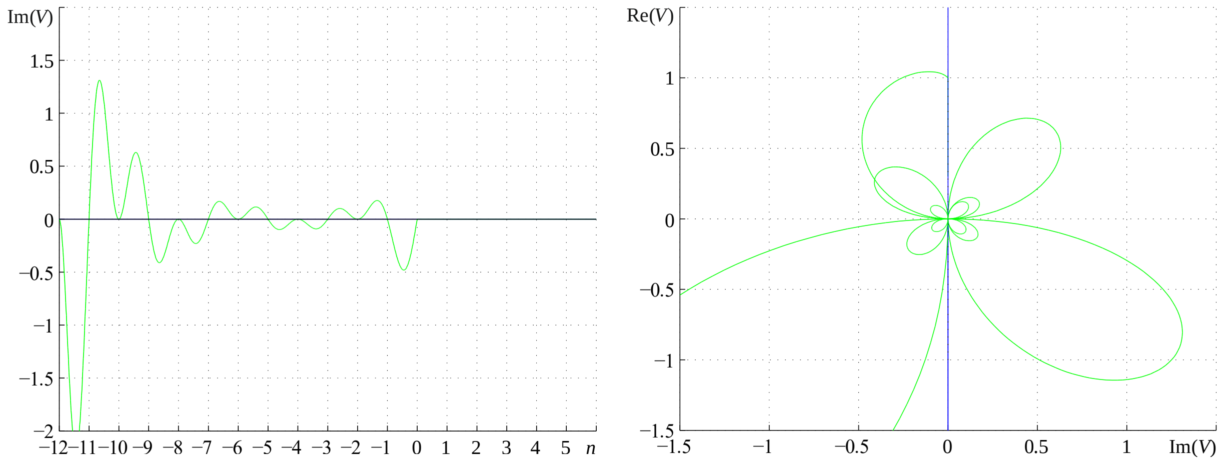
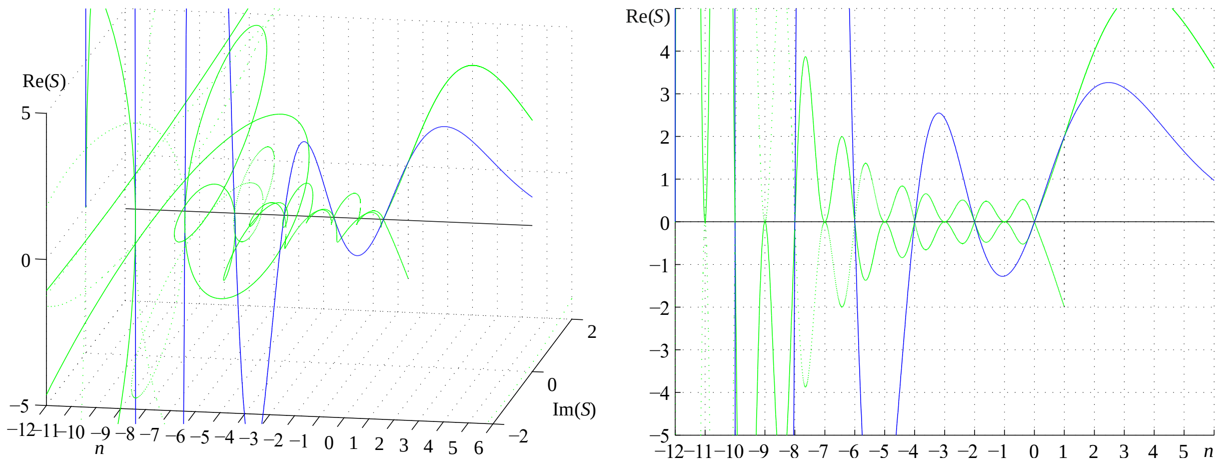


Figure 11. Graphs of volumes (V) of n-orthoplices (green) circumscribed about unit diameter n-balls and volumes of unit diameter n-balls (blue) for n = [−12, 6].

Similarly, the surface (28) of the circumscribed n-orthoplex with edge length A given by (43) becomes

$$S_n\left(A_{OCB}\right)_O = \frac{2n^{n/2+1}}{\Gamma\left(n+1\right)} D^{n-1} \begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}, \tag{45}$$

as shown in Figure 12. Circumscribed n-orthoplex surface (45) is a bivalued function for $n < 1$, is complex for $n < 0$ and initially convergent to achieve real minimum for $-2 < n < -1$ and then becomes oscillatory, left-handedly divergent with decreasing n. The principal branch crosses the quadrants of the complex plane in the order $\{\mathrm{Re}(V_{OCB}) < 0,\ \mathrm{Im}(V_{OCB}) > 0\}$, $\{\mathrm{Re}(V_{OCB}) < 0,\ \mathrm{Im}(V_{OCB}) < 0\}$, $\{\mathrm{Re}(V_{OCB}) > 0,\ \mathrm{Im}(V_{OCB}) < 0\}$, and $\{\mathrm{Re}(V_{OCB}) > 0,\ \mathrm{Im}(V_{OCB}) > 0\}$. For $n \leq -1$, $n \in \mathbb{Z}$, its zero values are given in the sense of a limit of a function (cf. (23)). For $0 < n < 1$ it is smaller than the surface of the inscribed n-ball (cf. Table 5).



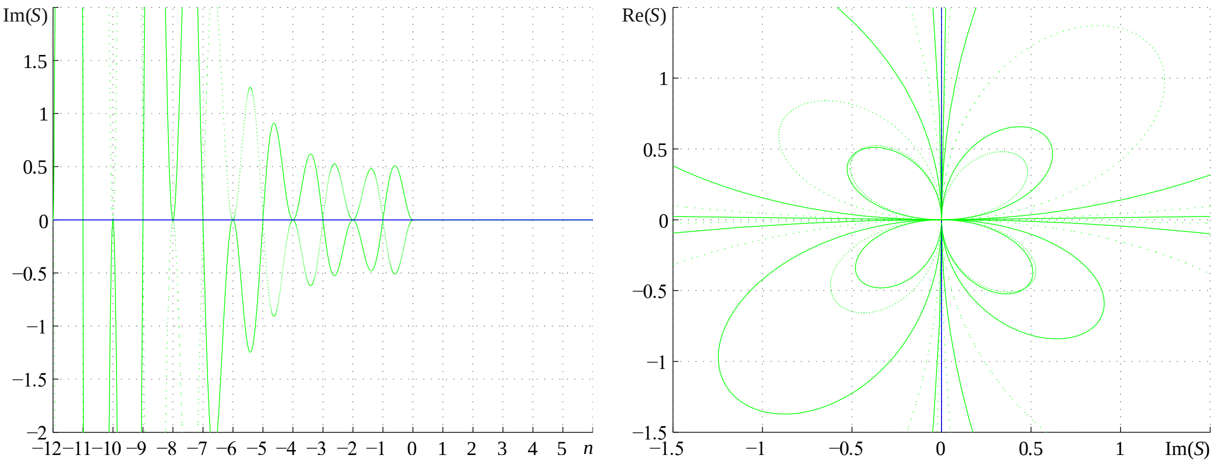


Figure 12. Graphs of surfaces (S) of n-orthoplices (green) circumscribed about unit diameter n-balls and surfaces of unit diameter n-balls (blue) for $n = [-12, 6]$.

5.3 *n*-Cubes Inscribed in and Circumscribed About *n*-Balls

The edge length A_{CCB} of an n-cube circumscribed about an n-ball (CCB) corresponds to the diameter D of this n-ball. Thus, the volume of this cube is simply

$$V_n\left(A_{CCB}\right)_C = D^n, \tag{46}$$

and the surface is

$$S_n\left(A_{CCB}\right)_C = 2nD^{n-1}. \tag{47}$$

The edge length A_{CIB} of an n-cube inscribed inside an n-ball (CIB) of diameter D is $A_{CIB} = D/\sqrt{n}$, which is singular for $n = 0$ and complex for $n < 0$, rendering [11] the following volume and the surface of an n-cube inscribed in an n-ball

$$V_n\left(A_{CIB}\right)_C = n^{-n/2}D^n, \tag{48}$$

$$S_n\left(A_{CIB}\right)_C = 2n^{(3-n)/2}D^{n-1}. \tag{49}$$

The reflection relation can be obtained setting $m = -n$ in (48), yielding [11] the volume and the surface

$$V_m\left(A_{CIB}\right)_C = i^m m^{m/2}D^{-m}, \tag{50}$$

$$S_m\left(A_{CIB}\right)_C = -2i^{m+1}m^{(3+m)/2}D^{-m-1}, \tag{51}$$

which are complex for $m \in \mathbb{R}$. Volumes (48) and (50) correspond to each other [11] for $n \leq 0$, $n \in \mathbb{R}$ and for $n = 2k$, $k \in \mathbb{Z}$, as shown in Figure 13 (left column). Surfaces (49) and (51) correspond to each other [11] for $n \in \mathbb{R}$, $n \leq 0$, and for $n = 2k - 1$, $k \in \mathbb{Z}$, as shown in Figure 13 (right column).

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For $n \geq 0$ (by convention $0^0 := 1$) the inscribed n-cube volume (48) is real, complex if $n < 0$, becoming real if n is negative and even and imaginary if n is negative and odd, and divergent with decreasing n. For $0 < n < 1$ it is larger than the volume of the circumscribing n-ball. For $n \geq 0$ the inscribed n-cube surface (49) is real, complex if $n < 0$, becoming real if n is negative and odd and imaginary if n is negative and even, and divergent with decreasing n.

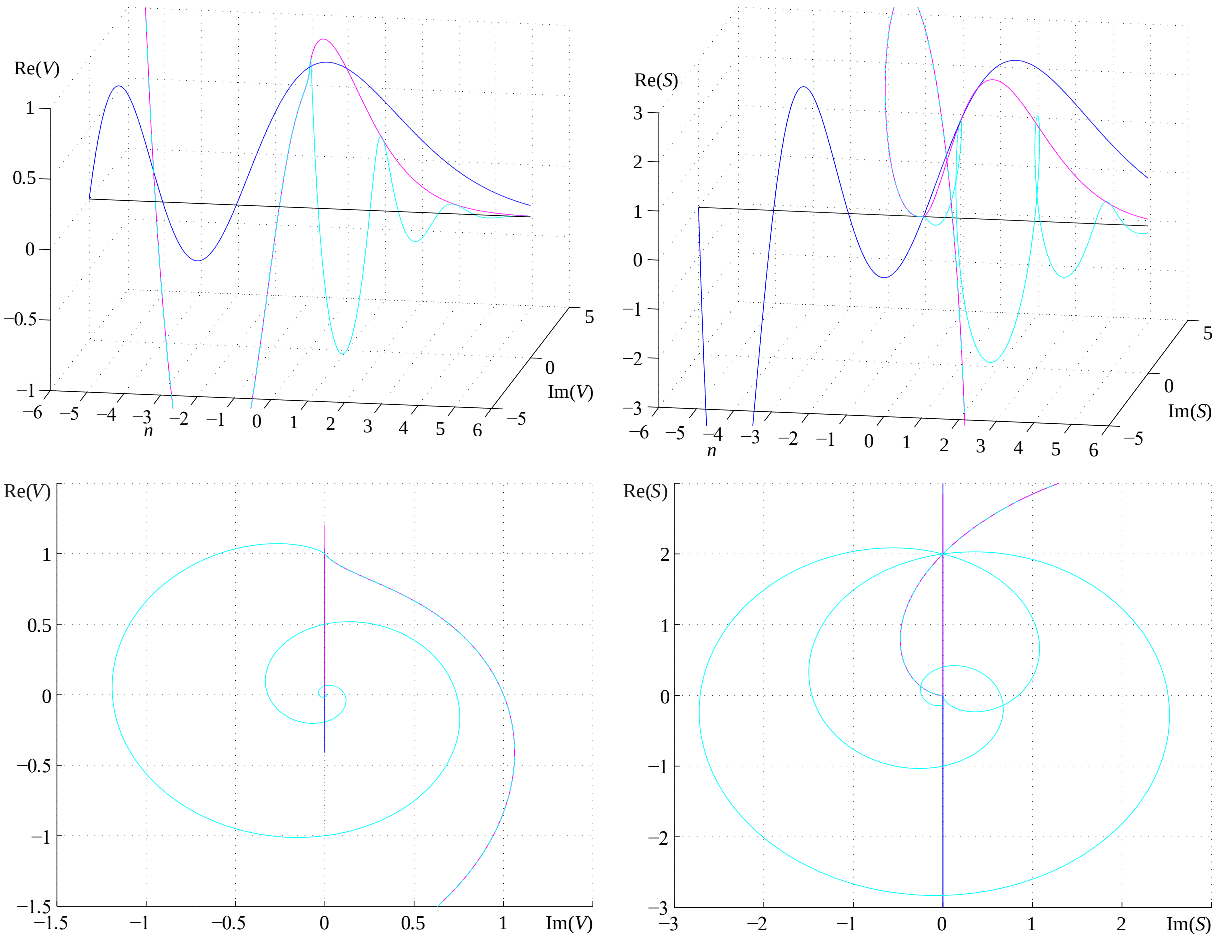

Figure 13. Graphs of volumes of n-cubes (pink) inscribed in unit diameter n-balls with the reflection relation (cyan) and volumes of unit diameter n-balls (blue) for $n = [-6, 6]$.

Furthermore, the following holds [11] for (48) and (50) with $m = n$

$$V_n\left(A_{CIB}\right)_C V_m\left(A_{CIB}\right)_C \overset{m=n}{=} D^n n^{-n/2} i^n D^{-n} n^{n/2} = i^n, \tag{52}$$

and the following relations

$$\frac{V_n\left(A_{SCB}\right)_S}{V_n\left(A_{SIB}\right)_S} = n^n, \tag{53}$$

$$\frac{S_n\left(A_{SCB}\right)_S}{S_n\left(A_{SIB}\right)_S} = n^{n-1}, \tag{54}$$

$$\frac{V_n\left(A_{OCB}\right)_O}{V_n\left(A_{OIB}\right)_O} = \frac{V_n\left(A_{CCB}\right)_C}{V_n\left(A_{CIB}\right)_C} = n^{n/2}, \tag{55}$$

$$\frac{S_n\left(A_{OCB}\right)_O}{S_n\left(A_{OIB}\right)_O} = \frac{S_n\left(A_{CCB}\right)_C}{S_n\left(A_{CIB}\right)_C} = n^{(n-1)/2}, \tag{56}$$

relating formulas (36), (32); (37), (33); (44), (40); (46), (48); (45), (41); and (47), (49) with each other, can be easily obtained.

6. The volume of an *n*-Ball in Complex Dimensions

The gamma function is defined for all complex numbers except the non-positive integers. Thus [17], for $n = a + ib$, where if $n \in \mathbb{Z}$, $a \geq -1$

$$\pi^{n/2} = \pi^{(a+ib)/2} = \pi^{a/2}\left[\cos\left(\frac{b}{2}\ln(\pi)\right) + i\sin\left(\frac{b}{2}\ln(\pi)\right)\right], \tag{57}$$

$$R^n = R^{a+ib} = R^a\left[\cos\left(b\ln(R)\right) + i\sin\left(b\ln(R)\right)\right], \tag{58}$$

the volume (1) and surface (4) become

$$V_n(R)_B = \pi^{\frac{a}{2}} R^a \frac{\left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right] + i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}}{\Gamma\left(\frac{a+ib}{2}+1\right)}, \tag{59}$$

$$S_n(R)_B = (a+ib)\pi^{\frac{a}{2}} R^{a-1} \frac{\left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right] + i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}}{\Gamma\left(\frac{a+ib}{2}+1\right)}, \tag{60}$$

where we have used $\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a + b)$ and $\cos(a)\sin(b) + \sin(a)\cos(b) = \sin(a + b)$, as shown in Figure 14 for unit radius *n*-balls.

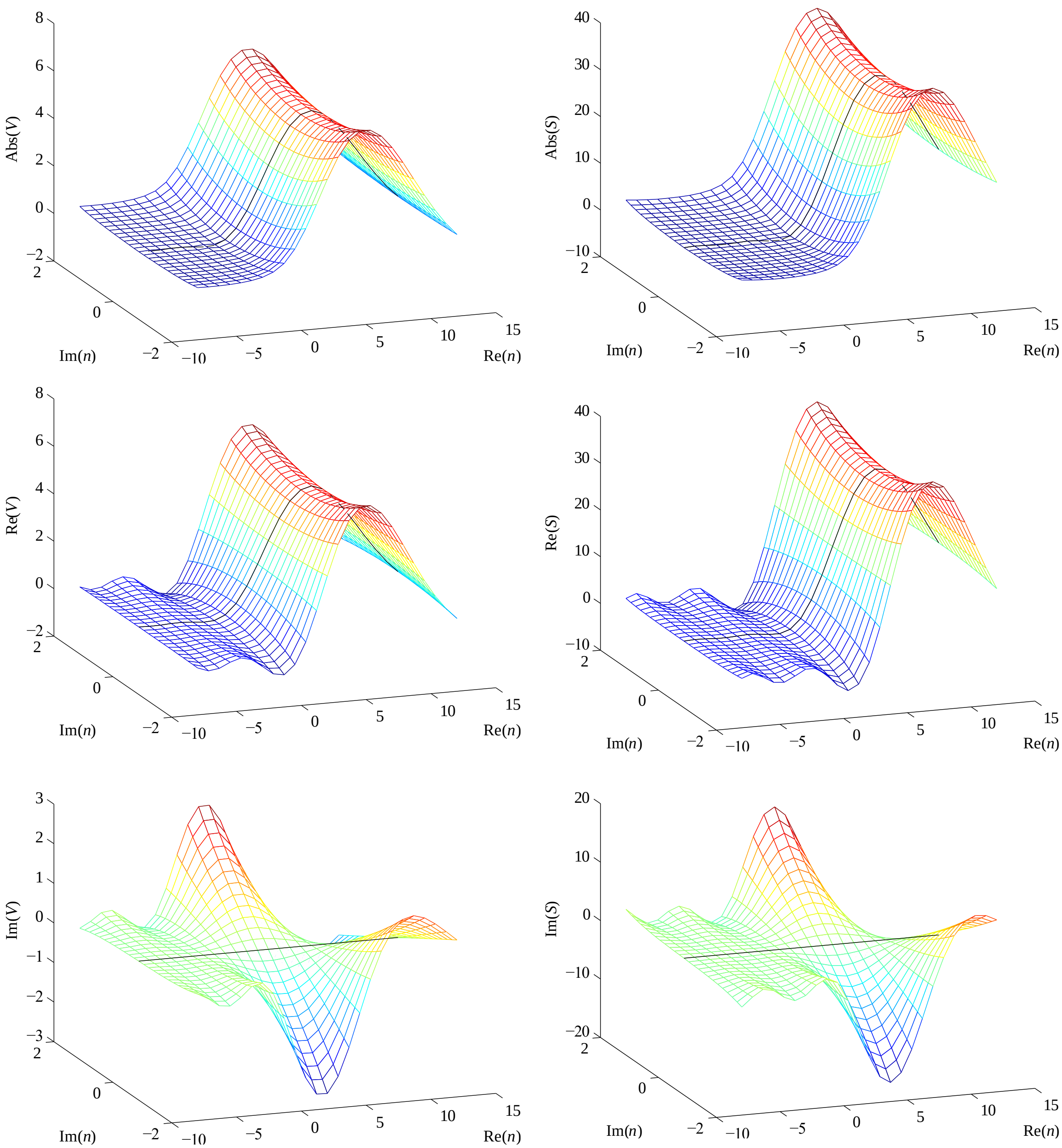

Figure 14. Graphs of complex volumes (V) and surfaces (S) of unit radius n-balls in complex dimensions $n = a + ib$ for a = [-10, 15], b = [-2, 2].

In particular for $n = 3 + ib$, $b \in \mathbb{R}$ (spacetime dimensionality) equation (59) becomes

$$V_n\left(R\right)_B = \pi^{\frac{3}{2}} R^3 \frac{\left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right] + i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}}{\Gamma\left(\frac{3+ib}{2}+1\right)}, \tag{61}$$

which reduces to familiar $V_3(R)_B = 4\pi R^3/3$ for $n = 3 + 0i$, i.e. at the present moment. Note that the imaginary part of the volume (59), in a way, establishes the arrow of time.

7. Conclusions

The recurrence relations (9), (10), (12), and (14)-(17) defining volumes and surfaces of regular n-simplices, n-orthoplices and n-balls can be expressed by the gamma function (18), (21); (22), (25); (27), (28) and thus are continuous for $n \in \mathbb{C}$. For $n = -2k - 2$, $k \in \mathbb{N}_0$ in the case of n-balls, and for $n = -k - 1$, $k \in \mathbb{N}_0$ in the case of n-simplices and n-orthoplices their values are given in the sense of a limit of a function. The starting points for fractional dimensions are given due to the continuity of the gamma function.

In the negative dimensions the volume of an n-simplex is a bivalued function. Thus, the surfaces of n-simplices and n-orthoplices are also bivalued functions for $n < 1$. Moreover, as the gamma function is a function of a complex argument and value, these volumes and surfaces inherit this gamma function property.

Applications of these formulas to the omnidimensional polytopes inscribed in and circumscribed about an n-balls revealed the properties of these geometric objects in negative, real dimensions, as summarized in Tables 2-5. In particular for $0 < n < 1$ the volumes of the omnidimensional polytopes are larger than volumes of circumscribing n-balls, while their volumes and surfaces are smaller than volumes of inscribed n-balls.

The results of this study could perhaps be applied in linguistic statistics, where the dimension in the distribution for frequency dictionaries is chosen to be negative [3], in fog computing, where n-simplex is related to a full mesh pattern, n-orthoplex is linked to a quasi-full mesh structure, and n-cube is referred to as a certain type of partial mesh layout [18], and in molecular physics and crystallography.

Table 2. Volumes of omnidimensional polytopes inscribed in and circumscribed about unit diameter n-balls ($n \in \mathbb{R}$ unless stated otherwise; "no complex" means that the relation is real or imaginary; for $n \to -\infty$ all relations are oscillatory divergent).

<table>
<tr><th>n</th><th>bival.</th><th>complex</th><th>real</th><th>imaginary</th><th>zero</th><th>div./conv.</th></tr>
<tr><td>V_B (18)</td><td>no</td><td rowspan="2">no</td><td>$n \in \mathbb{R}$</td><td>no</td><td>$n = -2k$, $k \in \mathbb{N}_0$</td><td rowspan="3">0, $n \to \infty$
$-\infty$, $n \to -\infty$</td></tr>
<tr><td>V_S (22)</td><td rowspan="3">$n < 0$</td><td>$n = -k$, $k \in \mathbb{N}$
$n \geq -1$, $n \in \mathbb{R}$</td><td rowspan="2">$n < -1$, $n \notin \mathbb{Z}$</td><td rowspan="6">$n = -k$, $k \in \mathbb{N}$</td></tr>
<tr><td>V_{SIB} (32)</td><td>$-1 < n < 0$
(RH)</td><td>$n = -k$, $k \in \mathbb{N}$
$n \geq 0$, $n \in \mathbb{R}$</td></tr>
<tr><td>V_{SCB} (36)</td><td>$n < 0$
(LH)</td><td>$n = -(k + 1)/2$, $k \in \mathbb{N}$
$n \geq 0$, $n \in \mathbb{R}$</td><td rowspan="5">no</td><td>∞, $n \to \infty$
0, $n \to -\infty$</td></tr>
<tr><td>V_O (27)</td><td rowspan="6">no</td><td rowspan="2">no</td><td rowspan="2">$n \in \mathbb{R}$</td><td rowspan="3">0, $n \to \infty$
$-\infty$, $n \to -\infty$</td></tr>
<tr><td>V_{OIB} (40)</td></tr>
<tr><td>V_{OCB} (44)</td><td>$n < 0$
(LH)</td><td>$n = -k$, $k \in \mathbb{N}$
$n \geq 0$, $n \in \mathbb{R}$</td></tr>
<tr><td>V_C</td><td>no</td><td>$n \in \mathbb{R}$</td><td rowspan="3">no</td><td>const</td></tr>
<tr><td>V_{CIB} (48)</td><td>$n < 0$</td><td>$n = -2k$, $k \in \mathbb{N}$
$n \geq 0$, $n \in \mathbb{R}$</td><td>$n = -2k - 1$,
$k \in \mathbb{N}_0$</td><td>0, $n \to \infty$
$-\infty$, $n \to -\infty$</td></tr>
<tr><td>V_{CCB} (46)</td><td>no</td><td>$n \in \mathbb{R}$</td><td>no</td><td>const</td></tr>
</table>

Table 3. Surfaces of omnidimensional polytopes inscribed in and circumscribed about unit diameter n-balls ($n \in \mathbb{R}$ unless stated otherwise; "no complex" means that the relation is real or imaginary; for $n \to -\infty$ all relations are oscillatory divergent, with the exception of S_C and S_{CCB}).

<table>
<tr><th>n</th><th>bival.</th><th>complex</th><th>real</th><th>imaginary</th><th>zero</th><th>div./conv.</th></tr>
<tr><td>S_B (21)</td><td>no</td><td rowspan="2">no</td><td>$n \in \mathbb{R}$</td><td>no</td><td>$n = -2k,\ k \in \mathbb{N}_0$</td><td rowspan="3">$0,\ n \to \infty$
$-\infty,\ n \to -\infty$</td></tr>
<tr><td>S_S (25)</td><td rowspan="6">$n < 1$</td><td rowspan="2">$n = -k,\ k \in \mathbb{N}$
$n \geq 0,\ n \in \mathbb{R}$</td><td>$n < 0,\ n \notin \mathbb{Z}$</td><td rowspan="6">$n = -k,\ k \in \mathbb{N}_0$</td></tr>
<tr><td>S_{SIB} (33)</td><td>$-1 < n < 0$
(RH)</td><td>$n < -1,\ n \notin \mathbb{Z}$</td></tr>
<tr><td>S_{SCB} (37)</td><td>$n < 0$
(LH)</td><td>$n = -(k+1)/2,\ k \in \mathbb{N}$
$n \geq 0,\ n \in \mathbb{R}$</td><td>no</td><td>$\infty,\ n \to \infty$
$0,\ n \to -\infty$</td></tr>
<tr><td>S_O (28)</td><td rowspan="2">no</td><td rowspan="3">$n = -k,\ k \in \mathbb{N}$
$n \geq 0,\ n \in \mathbb{R}$</td><td rowspan="2">$n < 0,\ n \notin \mathbb{Z}$</td><td rowspan="3">$0,\ n \to \infty$
$-\infty,\ n \to -\infty$</td></tr>
<tr><td>S_{OIB} (41)</td></tr>
<tr><td>S_{OCB} (45)</td><td>$n < 0$
(LH)</td><td>no</td></tr>
<tr><td>S_C</td><td rowspan="3">no</td><td>no</td><td>$n \in \mathbb{R}$</td><td>no</td><td>$n = 0$</td><td>$\infty,\ n \to \infty$
$-\infty,\ n \to -\infty$</td></tr>
<tr><td>S_{CIB} (49)</td><td>$n < 0$</td><td>$n = -4k-1,\ k \in \mathbb{N}_0$
$n \geq 0,\ n \in \mathbb{R}$</td><td>$n = -4k-3$
$k \in \mathbb{N}_0$</td><td>no</td><td>$0,\ n \to \infty$
$-\infty,\ n \to -\infty$</td></tr>
<tr><td>S_{CCB} (47)</td><td>no</td><td>$n \in \mathbb{R}$</td><td>no</td><td>$n = 0$</td><td>$\infty,\ n \to \infty$
$-\infty,\ n \to -\infty$</td></tr>
</table>

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Data Availability Statement: https://github.com/szluk/balls_simplices_orthoplices (accessed on August 18, 2022).

Conflicts of Interest: The author declares no conflict of interest.

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Appendix 1

The volume (36) of a regular n-simplex circumscribed about n-ball is real for $D \in \mathbb{R}$ and for

$$n = \frac{-(2k+1)}{2}, k \in \mathbb{N} \quad n = \left\{-\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \ldots\right\}. \tag{62}$$

We start by assuming that the principal branch of (36) is real and amounts some $a \in \mathbb{R}$ for these n. Then

$$V_n\left(A_{SCB}\right)_S 2^n = \frac{\left((-1)\frac{1+2k}{2}\right)^{\frac{-(1+2k)}{4}}\left(\frac{1-2k}{2}\right)^{\frac{1-2k}{4}}}{\Gamma\left(\frac{1-2k}{2}\right)} = a2^n = b \in \mathbb{R}$$
$$\left((-1)\frac{1+2k}{2}\right)^{\frac{-(1+2k)}{4}}\left(\frac{1-2k}{2}\right)^{\frac{1-2k}{4}} = b\Gamma\left(\frac{1-2k}{2}\right) = c \in \mathbb{R} \tag{63}$$

for some b, $c \in \mathbb{R}$, since for

$$k = \{1,2,3,\ldots\} \in \mathbb{N} \quad \frac{1-2k}{2} = \left\{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\right\} \quad \Gamma\left(\frac{1-2k}{2}\right) \in \mathbb{R}. \tag{64}$$

Thus

$$\frac{(1-2k)^{1-2k}}{(1+2k)^{1+2k}} = d^4 (-1)^{1+2k} = e \in \mathbb{R}, \tag{65}$$

for some d, $e \in \mathbb{R}$, since $(-1)^3 = (-1)^5 = \ldots. = -1 \in \mathbb{R}$, $2^n \in \mathbb{R}$ for $n \in \mathbb{R}$, and $(1 - 2k)^{1-2k}/(1 + 2k)^{1+2k} \in \mathbb{R}$, which completes the proof. □

Also

$$\begin{aligned} V_{-1/2}\left(A_{SCB}\right)_S &= \frac{(-1/2)^{-1/4} (1/2)^{1/4} 2^{1/2}}{\Gamma(1/2)} = \frac{(1/2)^{1/4} 2^{1/2}}{(-1)^{1/4} (1/2)^{1/4} \sqrt{\pi}} = \\ &= \frac{2^{1/2}}{(-1)^{1/4}\sqrt{\pi}} = \frac{1-i}{\sqrt{\pi}} \approx 0.564 - i0.564 \end{aligned}. \tag{66}$$

<u>Appendix 2</u>

The proof of the surface (37) of a regular n-simplex circumscribed about n-ball being real for $D \in \mathbb{R}$ and for n given by (62) is analogous as the previous one. We start the proof, assuming that the principal branch of (37) is real and amounts some $a \in \mathbb{R}$ for these n. Then

$$\begin{aligned} &S_n\left(A_{SCB}\right)_S 2^{n-1}\Gamma(n+1) = \left((-1)\frac{1+2k}{2}\right)^{\frac{3-2k}{4}} \left(\frac{1-2k}{2}\right)^{\frac{1-2k}{4}} = a2^{n-1}\Gamma(n+1) = b \in \mathbb{R} \\ &(1+2k)^{3-2k}(1-2k)^{1-2k} = b^4(-1)^{2k-3} = c \in \mathbb{R} \end{aligned}, \tag{67}$$

for some b, $c \in \mathbb{R}$, since in the considered case the gamma function (64) is real, $2^n \in \mathbb{R}$ for $n \in \mathbb{R}$, $(-1)^{2k-3} = -1$, and $(1 + 2k)^{3-2k}/(1 - 2k)^{1-2k} \in \mathbb{R}$, which completes the proof. □

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Table 4. Particular values of volumes of omnidimensional polytopes inscribed in and circumscribed about unit diameter *n*-balls (principal branch only).

n	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$
V_B	0.331	$2/\pi \approx 0.637$	0.867	1	1.039	1	0.908
V_{SIB}	$\frac{-i3^{3/4}}{\sqrt{\pi}} \approx -i1.286$	0	$\frac{1-i}{\sqrt{2\pi}} \approx 0.399(1-i)$	1	$\frac{3^{3/4}}{\sqrt{\pi}} \approx 1.286$	1	$\frac{5^{5/4}}{3^{7/4}\sqrt{\pi}} \approx 0.617$
V_{SCB}	$\frac{2^{3/2}}{3^{3/4}\sqrt{\pi}} \approx 0.7$	0	$\frac{1-i}{\sqrt{\pi}} \approx 0.564(1-i)$	1	$\frac{3^{3/4}}{\sqrt{2\pi}} \approx 0.909$	1	$\frac{5^{5/4}}{3^{1/4}2^{3/2}\sqrt{\pi}} \approx 1.133$
V_{OIB}	$\frac{-1}{2\sqrt{\pi}} \approx -0.282$	0	$\frac{1}{\sqrt{\pi}} \approx 0.564$	1	$\frac{2}{\sqrt{\pi}} \approx 1.128$	1	$\frac{4}{3\sqrt{\pi}} \approx 0.752$
V_{OCB}	$\frac{1+i}{2^{3/4}3^{3/4}\sqrt{\pi}} \approx 0.147(1+i)$	0	$\frac{1-i}{2^{1/4}\sqrt{\pi}} \approx 0.474(1-i)$	1	$\frac{2^{3/4}}{\sqrt{\pi}} \approx 0.949$	1	$\frac{2^{5/4}}{2^{1/4}\sqrt{\pi}} \approx 1.02$
V_{CIB}	$\frac{(i-1)3^{3/4}}{2^{-5/4}} \approx 0.958(i-1)$	i	$\frac{(1+i)}{2^{3/4}} \approx 0.595(i+1)$	1	$2^{1/4} \approx 1.189$	1	$\frac{2^{3/4}}{3^{3/4}} \approx 0.738$
V_{CCB}	1	1	1	1	1	1	1

Table 5. Particular values of surfaces of omnidimensional polytopes inscribed in and circumscribed about unit diameter *n*-balls (principal branch only).

n	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$
S_B	-0.992	$-4/\pi \approx -1.273$	-0.867	0	1.039	2	2.723
S_{SIB}	$\frac{-i3^{11/4}}{2\sqrt{\pi}} \approx -5.787i$	0	$\frac{1+i}{\sqrt{\pi}\,2^{3/2}} \approx 0.199(1+i)$	0	$\frac{3^{3/4}}{2\sqrt{\pi}} \approx 0.643$	2	$\frac{3^{1/4}5^{5/4}}{2\sqrt{\pi}} \approx 2.776$
S_{SCB}	$\frac{-2^{3/2}}{3^{1/4}\sqrt{\pi}} \approx -2.1$	0	$\frac{i-1}{\sqrt{\pi}} \approx 0.564(i-1)$	0	$\frac{3^{3/4}}{\sqrt{2\pi}} \approx 0.909$	2	$\frac{3^{3/4}5^{5/4}}{2^{3/2}\sqrt{\pi}} \approx 3.4$
S_{OIB}	$\frac{3^{3/2}i}{2^{3/2}\sqrt{\pi}} \approx 1.036i$	0	$\frac{-i}{\sqrt{2\pi}} \approx -0.399i$	0	$\frac{\sqrt{2}}{\sqrt{\pi}} \approx 0.798$	2	$\frac{2^{3/2}\sqrt{3}}{\sqrt{\pi}} \approx 2.764$
S_{OCB}	$\frac{-(1+i)3^{1/4}}{2^{1/4}\sqrt{\pi}} \approx -0.442(1+i)$	0	$\frac{(-1+i)}{2^{1/4}\sqrt{\pi}} \approx -0.474(-1+i)$	0	$\frac{2^{3/4}}{\sqrt{\pi}} \approx 0.949$	2	$\frac{2^{5/4}3^{3/4}}{\sqrt{\pi}} \approx 3.059$
S_{CIB}	$\frac{(1+i)3^{9/4}}{2^{7/4}} \approx 3.521(1+i)$	2	$\frac{1-i}{2^{5/4}} \approx 0.42(1-i)$	0	$2^{-1/4} \approx 0.841$	2	$3^{3/4}2^{1/4} \approx 2.711$
S_{CCB}	-3	-2	-1	0	1	2	3