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On the Basic Convex Polytopes and *n*-Balls in Complex Dimensions

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Abstract: This paper extends the findings of the prior research concerning n-balls, regular n-simplices, and n-orthoplices in real dimensions using recurrence relations that removed the indefiniteness present in known formulas. The main result of this paper is a proof that these recurrence relations are continuous for complex n, wherein the volume of an n-simplex is a multivalued function for $n < 0$, and thus the surfaces of n-simplices and n-orthoplices are also multivalued functions for $n < 1$. Applications of these formulas to n-simplices, n-orthoplices, and n-cubes inscribed in and circumscribed about n-balls reveal previously unknown properties of these geometric objects in negative, real dimensions. In particular, it is shown that the volume and surface of a regular n-simplex inscribed in an n-ball are complex for $-1 < n < 0$, imaginary for $n < -1$, and divergent with decreasing n; the volume and surface of a regular n-simplex circumscribed about an n-ball is complex for $n < 0$ and left-handedly respectively convergent to zero or divergent towards infinity; and the volume and surface of an n-orthoplex circumscribed about an n-ball is complex for $n < 0$ and oscillatory divergent towards infinity with decreasing n.

Keywords: regular basic convex polytopes; negative dimensions; fractal dimensions; complex dimensions; circumscribed and inscribed polytopes

1. Introduction

The notion of dimension n of a set has various definitions [1, 2]. Natural dimensions define a minimum number of independent parameters (coordinates) needed to specify a point within Euclidean space $\mathbb{R}^n$, where $n = -1$ is the dimension of the empty set, the void, having zero volume and indefinite surface. Negatively dimensional spaces can be defined by analytic continuations from positive dimensions [3]. Negative dimensions [2, 4-6] refer to densities, rather than to sizes as the natural ones. Fractional (or fractal) dimensions extend the notion of dimension to real, including negative [7], numbers. Negative dimensions are considered in probabilistic fractal measures [8]. Fractal dimension and lacunarity [9, 10] allow for an investigation of the fractal nature of prime sequences [11]. Fractal dimensions have been shown to be consistent with experimental results and enable the examination of transport parameters in multiphase fractal media, including permeability, thermal dispersion, and conductivities (both thermal and electrical) [12]. The probability models for pore distribution and for permeability of porous media can also be expressed as a function of fractal dimensions [13]. The fractal dimension of the function is shown to be a linear function of the order of fractional integro-differentiation [14]. Recently, there has been a surge of interest in applications of topology, and of persistent homology in particular. Several authors have proposed estimators of fractal dimension defined in terms of minimum spanning trees and higher dimensional persistent homology [15]. Complex [2] and complex fractal [16] dimensions can also be considered. Furthermore, geometric concepts (such as lengths, volumes, and surfaces) can be related to negative, fractional, and complex numbers. Complex geodesic paths emerge in the presence of black hole singularities [17] and when studying entropic dynamics on curved statistical manifolds [18]. Fractional derivatives of complex functions could be able to describe different physical phenomena [19].

In $\mathbb{R}^2$, there is a countably infinite number of regular, convex polygons; in $\mathbb{R}^3$, there are five regular, convex Platonic solids; in $\mathbb{R}^4$, there are six regular, convex polytopes. For $n > 4$, there are only three: self-dual n-simplex and n-cube dual to n-orthoplex [20]. Furthermore, $\mathbb{R}^n$ is also equipped with a perfectly regular, convex n-ball. The properties of these three regular, convex polytopes in natural dimensions are well known [21-23]. Fractal dimensions of hyperfractals based on these polytopes in natural dimensions were disclosed in [24]. This study extends prior research [25] on recurrence relations that removed the indefiniteness present in known formulas concerning n-balls, n-simplices, and n-orthoplices in real dimensions, to complex, continuous dimensions.

The paper is structured as follows. Section 2 presents known non-recurrence formulas for volumes and surfaces of n-balls, regular n-simplices, and n-orthoplices in natural dimensions. Section 3 presents recurrence relations for these geometric objects in integer dimensions, while in Section 4 they are extended to complex, continuous dimensions. Section 5 refers to regular n-simplices, n-orthoplices, and n-cubes inscribed in and circumscribed about n-balls in real dimensions. Section 6 presents complex volumes of n-balls in complex dimensions. Section 7 concludes the findings of this paper and hints their possible applications.

2. Known non-Recurrence Formulas

The volume of an n-ball (B) is known to be

$$V_n(R)_B = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n, \tag{1}$$

where $\Gamma(\mathbb{C} \to \mathbb{C})$ is the Euler's gamma function and R is the n-ball radius. This implies complex n-ball volumes (cf. Section 6) and becomes

$$V_{2k}(R)_B = \frac{\pi^k R^{2k}}{k!} \tag{2}$$

if n is even ($n = 2k$, $k \in \mathbb{N}_0$) and

$$V_{2k-1}(R)_B = \frac{2^{2k}\pi^{k-1}k!}{(2k)!} R^{2k-1} \tag{3}$$

if n is odd ($n = 2k - 1$, $k \in \mathbb{N}$).

It is also known [22] that the $(n - 1)$-dimensional surface of an n-ball can be expressed as

$$S_n(R)_B = \frac{n}{R} V_n(R)_B. \tag{4}$$

The volume of a regular n-simplex (S) is known [21, 26] to be

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$$V_n\left(A\right)_S = \frac{\sqrt{n+1}}{n!\sqrt{2^n}} A^n, \tag{5}$$

where A is the edge length. A regular n-simplex has $n + 1$ $(n - 1)$-facets [22], so its surface is

$$S_n\left(A\right)_S = \left(n+1\right) V_{n-1}\left(A\right)_S. \tag{6}$$

The volume of n-orthoplex (O) is known [22] to be

$$V_n\left(A\right)_O = \frac{\sqrt{2^n}}{n!} A^n. \tag{7}$$

As n-orthoplex has 2^n facets [22], being regular $(n - 1)$-simplices, its surface is

$$S_n\left(A\right)_O = 2^n V_{n-1}\left(A\right)_S. \tag{8}$$

Formulas (2), (3), (5), and (7) are indefinite in negative dimensions since the factorial is defined only for non-negative integers.

3. Recurrence Relations in Integer Dimensions

It is known [22] to expresses the volume of an n-ball in terms of the volume of an $(n - 2)$-ball of the same radius as a recurrence relation

$$V_n\left(R\right)_B = \frac{2\pi R^2}{n} V_{n-2}\left(R\right)_B, \tag{9}$$

where $V_0(R)_B := 1$ and $V_1(R)_B := 2R$. The relation (9) can be extended [25] into negative dimensions as

$$V_n\left(R\right)_B = \frac{n+2}{2\pi R^2} V_{n+2}\left(R\right)_B, \tag{10}$$

solving (9) for V_{n-2} and assigning new $n \in \mathbb{Z}$ as the previous $n - 2$.

A radius recurrence relation [25]

$$f_n \doteq \quad , \tag{11}$$

for $n \in \mathbb{N}$, where $f_0 := 1$ and $f_1 := 2$, allows for expressing the volume n-ball as

$$V_n\left(R\right)_B \doteq \quad R^n, \tag{12}$$

where "$\lfloor x \rfloor$" denotes the floor function giving the greatest integer less than or equal to its argument x. The sequence (11) allows for presenting an n-ball's volume recurrence relation (12) as a product of the rational factor f_n, the irrational factor π^$\lfloor n/2 \rfloor$, and the metric (radius) factor R^n. The relation (11) can be analogously as (9) extended [25] into negative dimensions as

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$$f_n = \frac{n+2}{2} f_{n+2}, \tag{13}$$

which removes the indefiniteness of the factorial present in Formulas (2)-(3). It is also sufficient to define $f_{-1} := 1$, $f_0 := 1$ (for the empty set and point dimension) to initiate (11) or (13).

In the case of regular n-simplices, (5) can be written [25], with $V_0(A)_S := 1$, as a recurrence relation

$$V_n(A)_S \doteq \qquad 1)_S \sqrt{\frac{n+1}{2n^3}}, \tag{14}$$

which removes the indefiniteness of the factorial for $n < 1$ present in (5). Solving (14) for V_{n-1} and assigning new $n \in \mathbb{Z}$ as the previous $n - 1$, yields [25]

$$V_n(A)_S = \frac{V_{n+1}(A)_S}{A} \sqrt{\frac{2(n+1)^3}{n+2}}, \tag{15}$$

which also removes the singularity for $n = 0$.

In the case of n-orthoplices, Equation (7) can be written [25], with $V_0(A)_O := 1$, as a recurrence relation

$$V_n(A)_O \doteq \qquad 1)_O \frac{\sqrt{2}}{n}, \tag{16}$$

which removes the indefiniteness of the factorial for $n < 1$ present in (7). Solving (16) for V_{n-1} and assigning new $n \in \mathbb{Z}$ as the previous $n - 1$, yields [25]

$$V_n(A)_O = V_{n+1}(A)_O \frac{n+1}{A\sqrt{2}}, \tag{17}$$

which also removes singularity for $n = 0$ and is zero for integer $n \leq -1$.

4. Continuous Relations in Complex Dimensions

Theorem 1.

Recurrence relations (9), (10), (12) (n-balls) are continuous for $n \in \mathbb{C}$, wherein for $n = -2k - 2$, $k \in \mathbb{N}_0$ their values are given in the sense of a limit of a function.

Proof 1.

Comparing (1) with (10) and setting $m = n + 2$ and $k = m/2$, yields

$$V_n(R)_B = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n = \frac{n+2}{2\pi R^2} V_{n+2}(R)_B$$
$$V_{n+2}(R)_B = \frac{\pi^{n/2} 2\pi^{2/2}}{(n+2)\Gamma(n/2+1)} R^{n+2} \quad V_m(R)_B = \frac{\pi^{m/2} 2}{m\Gamma(m/2)} R^m = \frac{\pi^k}{k\Gamma(k)} R^{2k}\,, \tag{18}$$
$$V_n(R)_B = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n \quad V_n(D)_B = \frac{\pi^{n/2}}{2^n\Gamma(n/2+1)} D^n$$

which recovers (1), as $n\Gamma(n/2)/2 = \Gamma(n/2 + 1)$ for $n > 0$, $n \in \mathbb{C}$. On the other hand, (10) corresponds to (12)

$$V_n(R)_B = \frac{n+2}{2\pi R^2} V_{n+2}(R)_B = \frac{n+2}{2} f_{n+2} \pi^{\lfloor n/2 \rfloor} R^n$$
$$V_{n+2}(R)_B = \pi^1 f_{n+2} \pi^{\lfloor n/2 \rfloor} R^{n+2} \quad V_m(R)_B = f_m \pi^{1+\lfloor (m-2)/2 \rfloor} R^m = f_m \pi^{\lfloor m/2 \rfloor} R^m \,, \tag{19}$$

for $n \in \mathbb{C}$, which completes the proof. □

Also

$$\lim_{n\to -2k-2,\, k\in\mathbb{N}} \pi^{n/2} D^n 2^{-n} \frac{1}{\Gamma(n/2+1)} = a \cdot 0 = 0\,, \tag{20}$$

where $a \neq 0$, $a \in \mathbb{C}$.

Using (4) and (18) the surface of an n-balls is given by

$$S_n(D)_B = \frac{2^{1-n} n\pi^{n/2}}{\Gamma(n/2+1)} D^{n-1}. \tag{21}$$

Theorem 2.

Recurrence relations (14), (15) (regular n-simplices) are continuous for $n \in \mathbb{C}$, wherein for $n = -k - 1$, $k \in \mathbb{N}_0$ their values are given in the sense of a limit of a function.

Proof 2.

Expressing the factorial in (5) by the gamma function, comparing (5) with (15), and setting $m = n + 1$, yields

$$V_n(A)_S = \frac{\sqrt{n+1}}{n!\sqrt{2^n}} A^n = \frac{\sqrt{n+1}}{\Gamma(n+1)2^{n/2}} A^n = \frac{V_{n+1}(A)_S}{A}\sqrt{\frac{2(n+1)^3}{n+2}}$$
$$V_{n+1}(A)_S = \frac{\sqrt{n+1}\sqrt{n+2}}{\Gamma(n+1)2^{(n+1)/2}\sqrt{(n+1)^3}} A^{n+1} , \quad (22)$$
$$V_m(A)_S = \frac{m\sqrt{m}\sqrt{m+1}}{\Gamma(m+1)2^{m/2}\sqrt{m^3}} A^m \quad V_n(A)_S = \frac{\sqrt{n+1}}{\Gamma(n+1)2^{n/2}} A^n \begin{cases} 1 & n>0 \\ \pm 1 & n<0 \end{cases}$$

which recovers (5), as $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, and completes the proof. □

Also

$$\lim_{n\to -k-1,\, k\in\mathbb{N}} 2^{-n/2} A^n \sqrt{n+1}\frac{1}{\Gamma(n+1)} = a\cdot 0 = 0, \quad (23)$$

where $a \in \mathbb{C}$.

For $n < -1$ n-simplex volume formula (22) is imaginary and for $n < 0$ it is a multivalued function, as $n\sqrt{n}/\sqrt{n^3} = 1$ only for $n \in \mathbb{R}$, $n > 0$. Thus, its general form is

$$V_n(A)_S = \frac{\sqrt{n+1}}{\Gamma(n+1)2^{n/2}} A^m \frac{n\sqrt{n}}{\sqrt{n^3}} . \quad (24)$$

Using (6) and (22) the surface of a regular n-simplex is given by

$$S_n(A)_S = \frac{n(n+1)\sqrt{n}}{\Gamma(n+1)2^{(n-1)/2}} A^{n-1} \begin{cases} 1 & n>1 \\ \pm 1 & n<1 \end{cases} . \quad (25)$$

For $n < 0$ n-simplex surface formula (25) is imaginary and for $n < 1$ it is a multivalued function, as $(n-1)\sqrt{(n-1)}/\sqrt{(n-1)^3} = 1$ only for $n \in \mathbb{R}$, $n > 1$. Thus, its general form is

$$S_n(A)_S = \frac{n(n+1)\sqrt{n}}{\Gamma(n+1)2^{(n-1)/2}} A^{n-1} \frac{(n-1)\sqrt{n-1}}{\sqrt{(n-1)^3}} . \quad (26)$$

Theorem 3.

Recurrence relations (16), (17) (n-orthoplices) are continuous for $n \in \mathbb{C}$, wherein for $n = -k-1$, $k \in \mathbb{N}_0$ their values are given in the sense of a limit of a function.

Proof 3.

Expressing the factorial in (7) by the gamma function, comparing (7) with (17), and setting $m = n + 1$, yields

$$V_n(A)_O = \frac{\sqrt{2^n}}{n!}A^n = \frac{\sqrt{2^n}}{\Gamma(n+1)}A^n = V_{n+1}(A)_O \frac{n+1}{A\sqrt{2}}$$
$$V_{n+1}(A)_O = \frac{2^{(n+1)/2}}{(n+1)\Gamma(n+1)}A^{n+1} \qquad , \qquad (27)$$
$$V_m(A)_O = \frac{\sqrt{2^m}}{m\Gamma(m)}A^m \quad V_n(A)_O = \frac{2^{n/2}}{\Gamma(n+1)}A^n$$

which recovers (7), as $n\Gamma(n) = \Gamma(n+1)$ for $n \in \mathbb{C}\backslash\{n \in \mathbb{Z}, n \le -1\}$ and $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$, and completes the proof. □

Using (8) and (27) the surface of an n-orthoplex is given by

$$S_n(A)_O = \frac{n2^{(n+1)/2}\sqrt{n}}{\Gamma(n+1)}A^{n-1}\begin{cases} 1 & n > 1 \\ \pm 1 & n < 1 \end{cases}. \qquad (28)$$

For $n < 0$, $n \notin \mathbb{Z}$ n-orthoplex surface formula (28) is imaginary and for $n < 1$ it is a multivalued function, as $(n-1)\sqrt{(n-1)}/\sqrt{(n-1)^3} = 1$ only for $n \in \mathbb{R}$, $n > 1$. Thus, its general form is

$$S_n(A)_O = \frac{n2^{(n+1)/2}\sqrt{n}}{\Gamma(n+1)}A^{n-1}\frac{(n-1)\sqrt{n-1}}{\sqrt{(n-1)^3}}. \qquad (29)$$

Continuous recurrence relations (18)-(29) are shown in Figures 1-4.

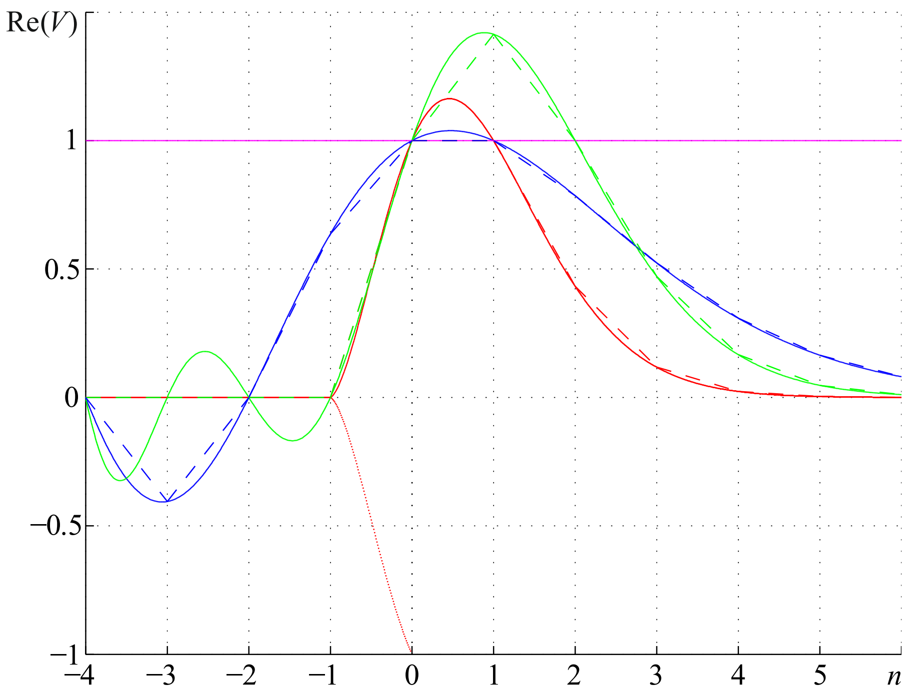

Figure 1. Graphs of the real part of volumes (V) of unit edge length regular n-simplices (red), n-orthoplices (green), n-cubes (pink), and unit diameter n-balls (blue), along with the integer recurrence relations (dashed lines) and the branch for n-simplices (dotted line) for $n = [-4, 6]$.

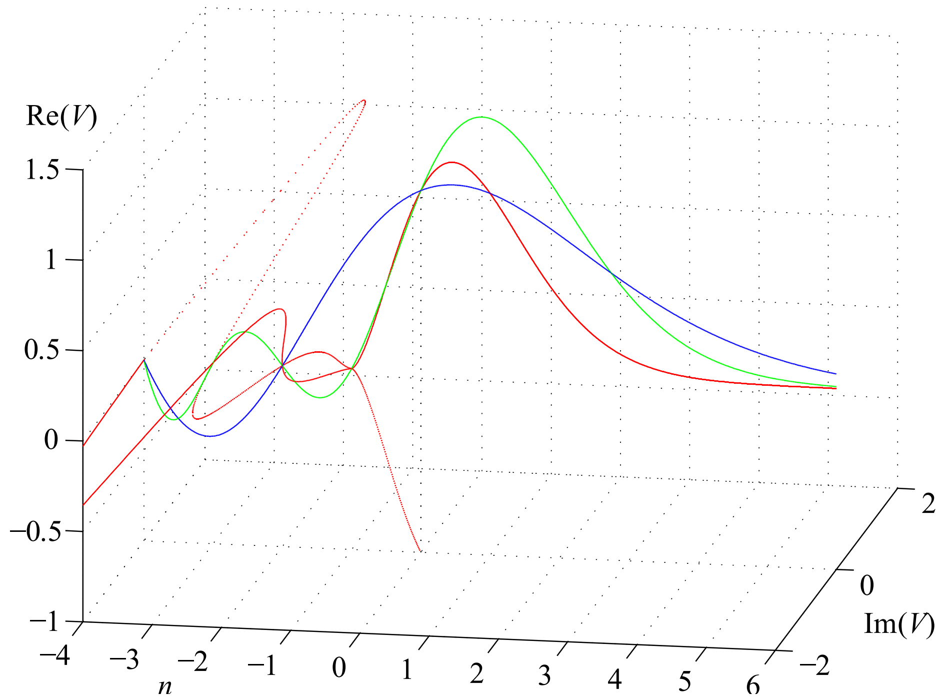

Figure 2. Graphs of volumes (V) of unit edge length regular n-simplices (red), n-orthoplices (green), and unit diameter n-balls (blue), along with the branch for n-simplices (dotted line) for $n = [-4, 6]$.

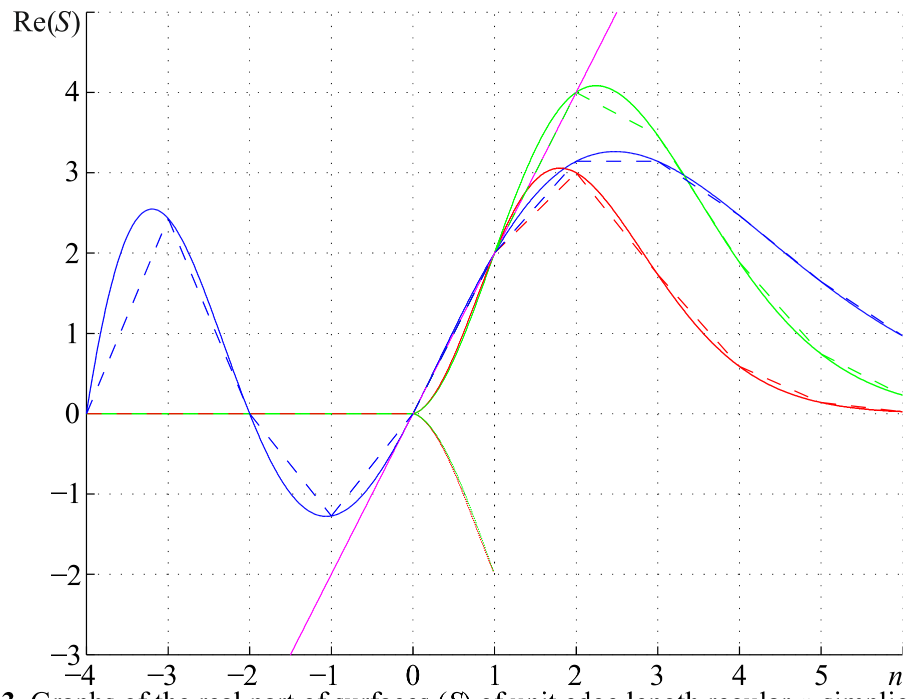

Figure 3. Graphs of the real part of surfaces (S) of unit edge length regular n-simplices (red), n-orthoplices (green), n-cubes (pink), and unit diameter n-balls (blue), along with the integer recurrence relations (dashed lines) and the branches for n-simplices and n-orthoplices (dotted lines) for $n = [-4, 6]$.

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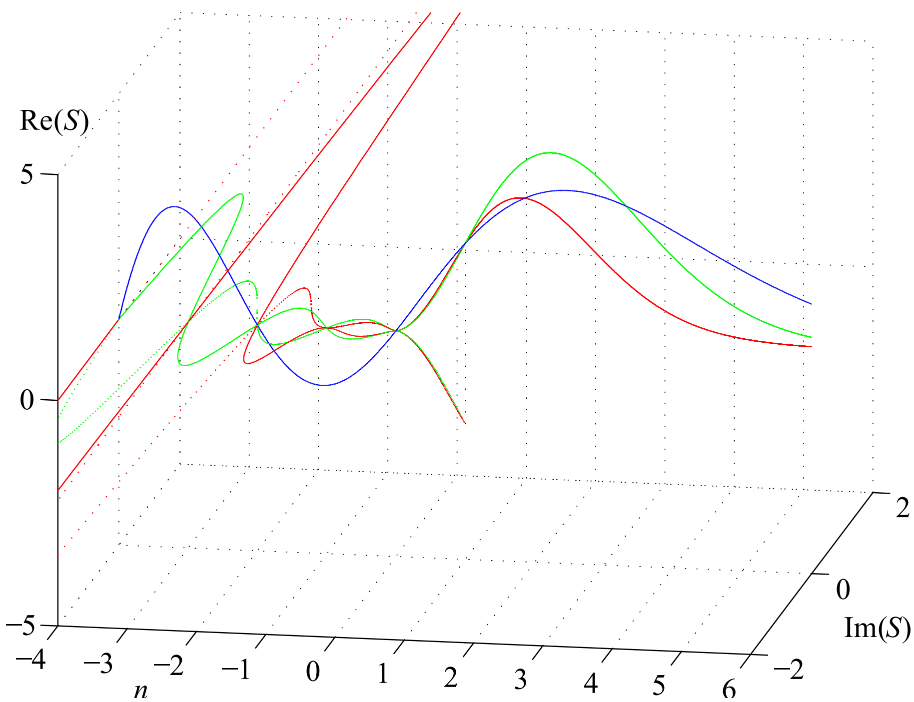

Figure 4. Graphs of surfaces (S) of unit edge length regular n-simplices (red), n-orthoplices (green), and unit diameter n-balls (blue), along with the branches for n-simplices and n-orthoplices (dotted lines) for $n = [-4, 6]$.

5. Basic Regular Polytopes Inscribed in and Circumscribed About *n*-Balls

Anyone of the three regular polytopes can be inscribed in and circumscribed about an n-ball, and this is considered in this section on the basis of the continuous relations presented in the previous one. The principal branches of their volumes and surfaces are summarized in the Table 1. n-balls are defined in terms of their diameters, which concept is closer to the concept of the edge length of a polytope.

Table 1. Volumes and surfaces of regular n-simplices, n-orthoplices, and n-cubes inscribed in and circumscribed about an n-balls.

	inscribed in n-ball (IB)		circumscribed about n-ball (CB)	
	volume/D^n	surface/D^{n-1}	volume/D^n	surface/D^{n-1}
(S)	$\frac{n^{-n/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^n}$ (2)	$\frac{n^{(4-n)/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^{n-1}}$ (2)	$\frac{n^{n/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^n}$ (2)	$\frac{n^{(2+n)/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^{n-1}}$ (2)
(O)	$\frac{1}{\Gamma(n+1)}$ (1)	$\frac{2n^{3/2}}{\Gamma(n+1)}$ (2)	$\frac{n^{n/2}}{\Gamma(n+1)}$ (1)	$\frac{2n^{n/2+1}}{\Gamma(n+1)}$ (2)
(C)	$n^{-n/2}$ (1)	$2n^{(3-n)/2}$ (1)	1 (1)	$2n$ (1)

(1) one branch, (2) two branches.

5.1 Regular n-Simplices Inscribed in and Circumscribed About n-Balls

The diameter D_{BCS} an n-ball circumscribed about a regular n-simplex (BCS) is known [26] to be

$$D_{BCS} = \frac{\sqrt{2n}}{\sqrt{n+1}} A , \tag{30}$$

where A is the edge length. Hence, the edge length A_{SIB} of a regular n-simplex inscribed (SIB) inside an n-ball (B) with diameter D is

$$A_{SIB} = \frac{\sqrt{n+1}}{\sqrt{2n}} D , \tag{31}$$

so that its volume (22) becomes

$$V_n\left(A_{SIB}\right)_S = \frac{n^{-n/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^n} D^n \begin{cases} 1 & n > 0 \\ \pm 1 & n < 0 \end{cases}, \tag{32}$$

as shown in in Figure 5. For $n < -1$ the inscribed n-simplex volume is imaginary and divergent with decreasing n, for $n < 0$ it branches, and is complex for $-1 < n < 0$, where in this case for both branches it is right-handed towards negative infinity or the branch point.

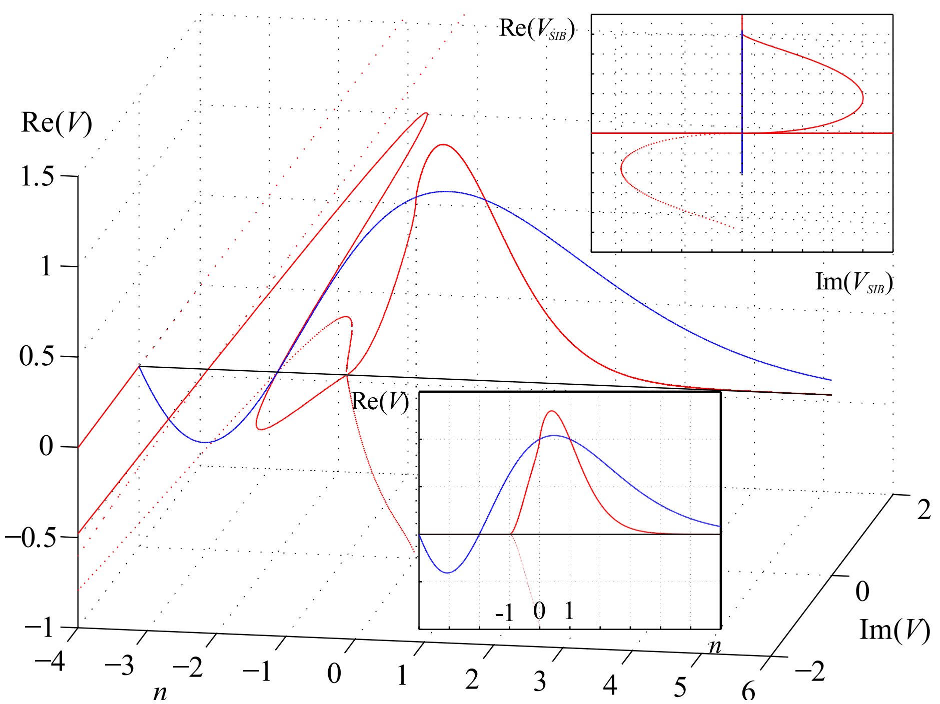

Figure 5. Graphs of volumes (V) of regular n-simplices (red) inscribed in unit diameter n-balls and volumes of unit diameter n-balls (blue) for n = [−4, 6] (inset for n = [−1, 0]).

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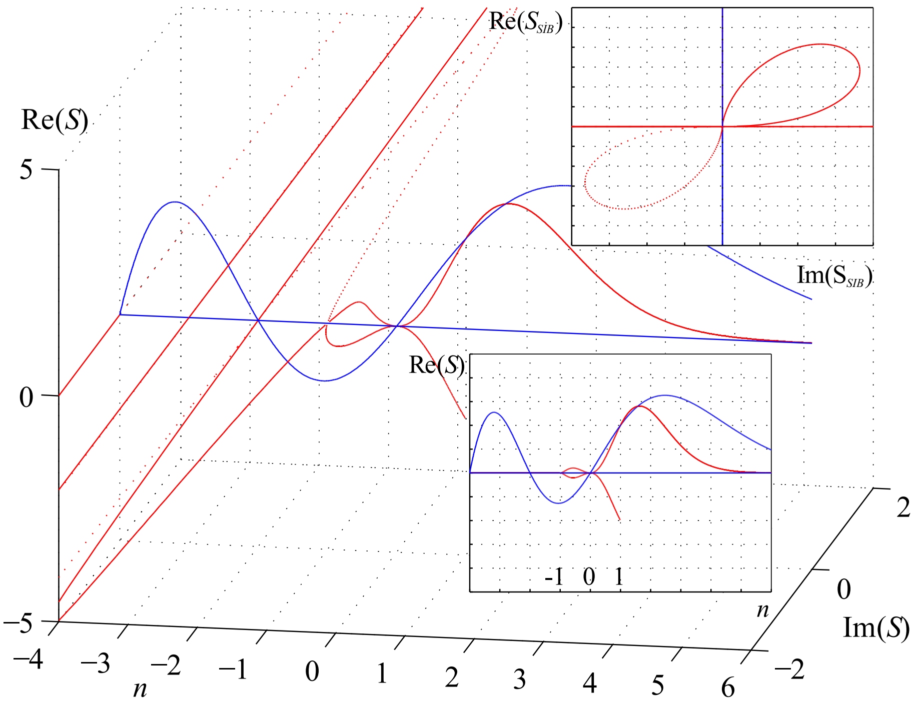

Figure 6. Graphs of surfaces (S) of regular n-simplices (red) inscribed in unit diameter n-balls and surfaces of unit diameter n-balls (blue) for $n = [-4, 6]$ (inset for $n = [-1, 0]$).

Similarly, the surface (25) of a regular inscribed n-simplex with edge length A given by (31) becomes

$$S_n\left(A_{SIB}\right)_S = \frac{n^{(4-n)/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^{n-1}} D^{n-1} \begin{cases} 1 & n>1 \\ \pm 1 & n<1 \end{cases}, \tag{33}$$

as shown in in Figure 6. For $n < -1$ the inscribed n-simplex surface is imaginary and divergent with decreasing n, for $n < 1$ it branches, and is complex for $-1 < n < 0$, where in this case for both branches it is right-handed towards negative infinity or the branch point.

The diameter D_{BIS} of an n-ball inscribed inside a regular n-simplex (BIS) is known [26] to be

$$D_{BIS} = \frac{\sqrt{2}}{\sqrt{n}\sqrt{n+1}} A, \tag{34}$$

where A is the edge length. Hence, the edge length A_{SCB} of a regular n-simplex circumscribed (SCB) about an n-ball (B) with diameter D is

$$A_{SCB} = \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{2}} D, \tag{35}$$

so that its volume (22) becomes

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$$V_n\left(A_{SCB}\right)_S = \frac{n^{n/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^n} D^n \begin{cases} 1 & n > 0 \\ \pm 1 & n < 0 \end{cases}, \tag{36}$$

as shown in in Figure 7. For $n < 0$ the circumscribed n-simplex volume branches, is complex and convergent to zero with decreasing n, where for both branches it is left-handed towards negative infinity or the branch point, and for $0 < n < 1$ it is smaller [sic] than the volume of the inscribed n-ball. Also it is real for $n = -k/2$, $k \in \mathbb{N}_0$, as shown in the drawing, and as follows from numerical calculations. An analytic solution requires further research.

Similarly, the surface (25) of a regular circumscribed n-simplex with edge length A_{SCB} (35) becomes

$$S_n\left(A_{SCB}\right)_S = \frac{n^{(2+n)/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^{n-1}} D^{n-1} \begin{cases} 1 & n > 1 \\ \pm 1 & n < 1 \end{cases}, \tag{37}$$

as shown in in Figure 8. For $n < 1$ the circumscribed n-simplex surface formula (37) branches, for $n < 0$ is complex and divergent with decreasing n, where for both branches it is left-handed towards negative infinity or the branch point, and for $0 < n < 1$ it is smaller than the surface of the inscribed n-ball. Also it is real for $n = -k/2$, $k \in \mathbb{N}_0$, as shown in the drawing, and as follows from numerical calculations. An analytic solution requires further research.

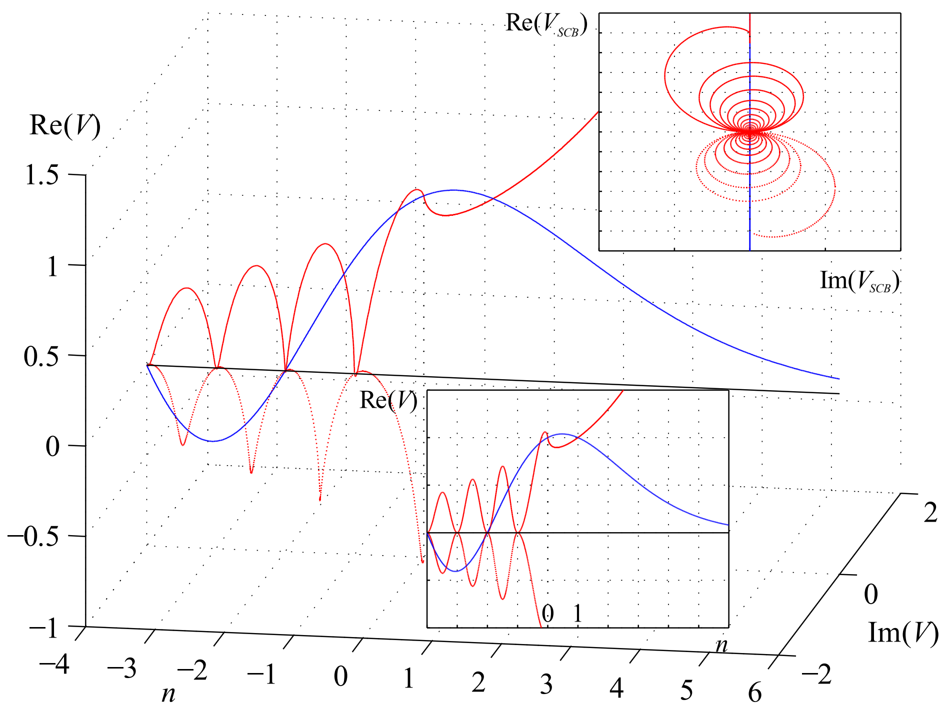

Figure 7. Graphs of volumes (V) of regular n-simplices (red) circumscribed about unit diameter n-balls and volumes of unit diameter n-balls (blue) for n = [−4, 6] (inset for n = [−30, 0]).

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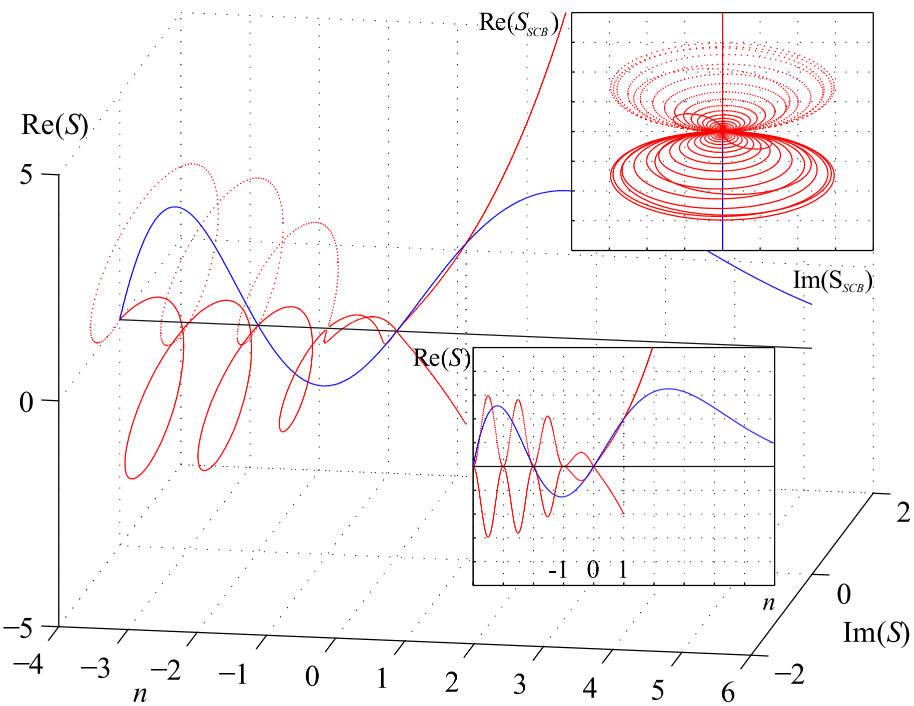

Figure 8. Graphs of surfaces (S) of regular n-simplices (red) circumscribed about unit diameter n-balls and surfaces of unit diameter n-balls (blue) for $n = [-4, 6]$ (inset for $n = [-30, 0]$).

5.2 *n*-Orthoplices Inscribed in and Circumscribed About *n*-Balls

The diameter D_{BCO} of an n-ball circumscribed about an n-orthoplex (BCO) is known [27] to be

$$D_{BCO} = \sqrt{2}A, \tag{38}$$

where A is the edge length. Hence, the edge length A_{OIB} of an n-orthoplex inscribed inside an n-ball (OIB) with diameter D is

$$A_{OIB} = \frac{1}{\sqrt{2}}D, \tag{39}$$

so that its volume (27) becomes

$$V_n\left(A_{OIB}\right)_O = \frac{1}{\Gamma(n+1)}D^n, \tag{40}$$

as shown in in Figure 9. The inscribed n-orthoplex volume formula (40) is real for $n \in \mathbb{R}$, and for $0 < n < 1$ it is intriguingly larger than the volume of the circumscribing n-ball.

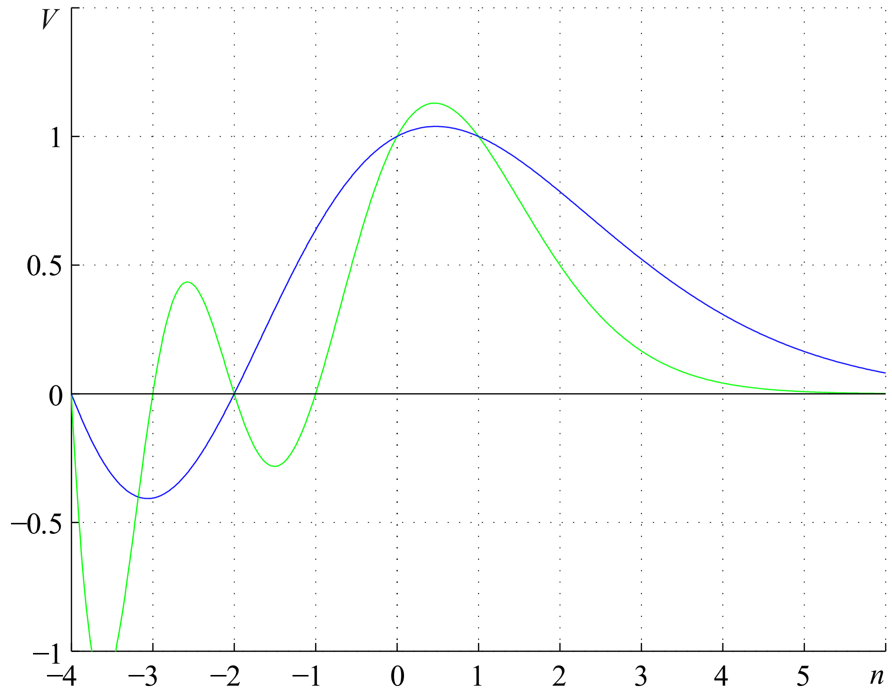

Figure 9. Graphs of volumes (V) of n-orthoplices (green) inscribed in unit diameter n-balls and volumes of unit diameter n-balls (blue) for $n = [-4, 6]$.

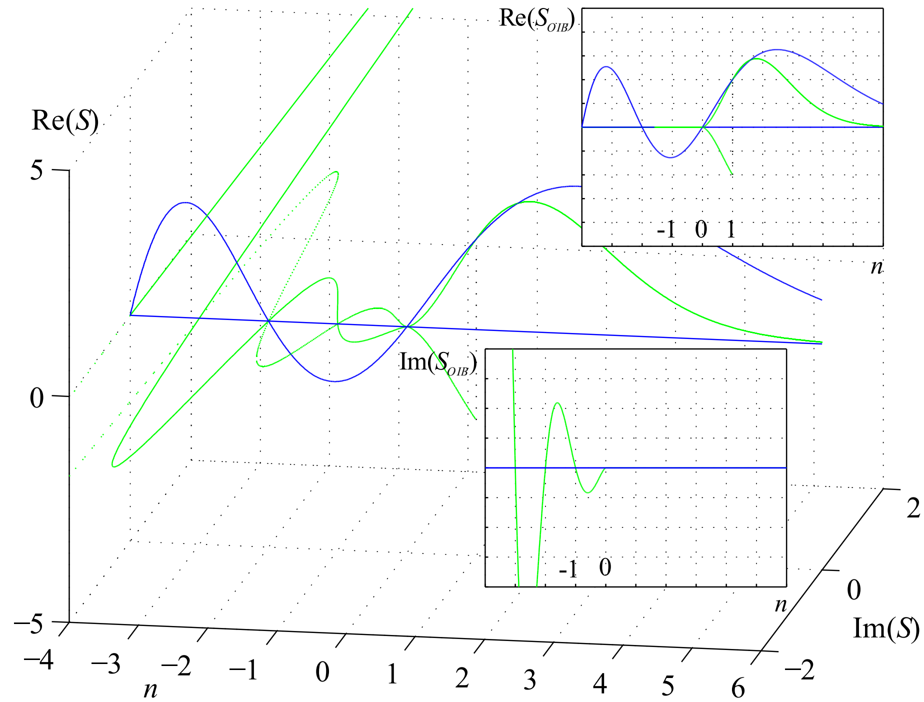

Figure 10. Graphs of surfaces (S) of n-orthoplices (green) inscribed in unit diameter n-balls and surfaces of unit diameter n-balls (blue) for $n = [-4, 6]$.

Similarly, the surface (28) of the inscribed n-orthoplex with edge length A given by (39) becomes

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$$S_n\left(A_{OIB}\right)_O = \frac{2n^{3/2}}{\Gamma\left(n+1\right)} D^{n-1} \begin{cases} 1 & n>1 \\ \pm 1 & n<1 \end{cases}, \tag{41}$$

as shown in in Figure 10. For $n<1$ inscribed n-orthoplex surface branches, and for $n<-1$ is imaginary and divergent with decreasing n.

The diameter D_{BIO} of an n-ball inscribed inside an n-orthoplex (BIO) is known [27] to be

$$D_{BIO} = \sqrt{\frac{2}{n}} A, \tag{42}$$

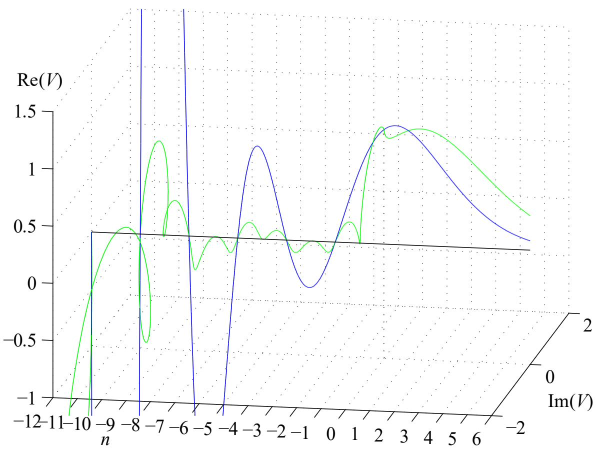
where A is the edge length. Hence, the edge length A_{OCB} of an n-orthoplex circumscribed about an n-ball (OCB) with diameter D is

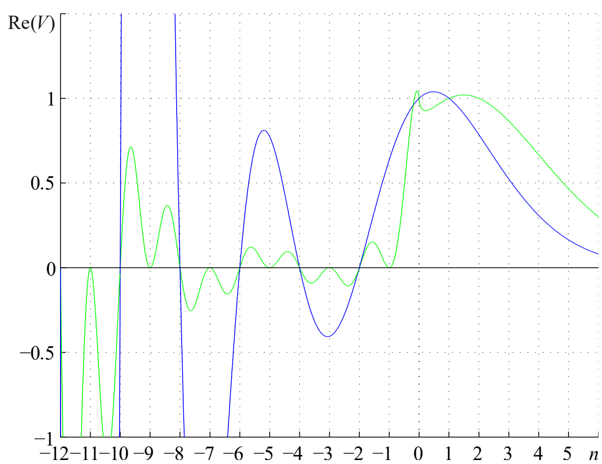
$$A_{OCB} = \sqrt{\frac{n}{2}} D, \tag{43}$$

so that its volume (27) becomes

$$V_n\left(A_{OCB}\right)_O = \frac{n^{n/2}}{\Gamma\left(n+1\right)} D^n, \tag{44}$$

as shown in Figure 11. Circumscribed n-orthoplex volume is a singlevalued function, is complex for $n<0$, oscillatory divergent with decreasing n, where it is left-handed towards negative infinity or the branch point, and crossing the quadrants of the complex plane in the order $\{\mathrm{Re}(V_{OCB})>0, \mathrm{Im}(V_{OCB})<0\}$, $\{\mathrm{Re}(V_{OCB})>0, \mathrm{Im}(V_{OCB})>0\}$, $\{\mathrm{Re}(V_{OCB})<0, \mathrm{Im}(V_{OCB})>0\}$, and $\{\mathrm{Re}(V_{OCB})<0, \mathrm{Im}(V_{OCB})<0\}$. For $0<n<1$ it is smaller than the volume of the inscribed n-ball.





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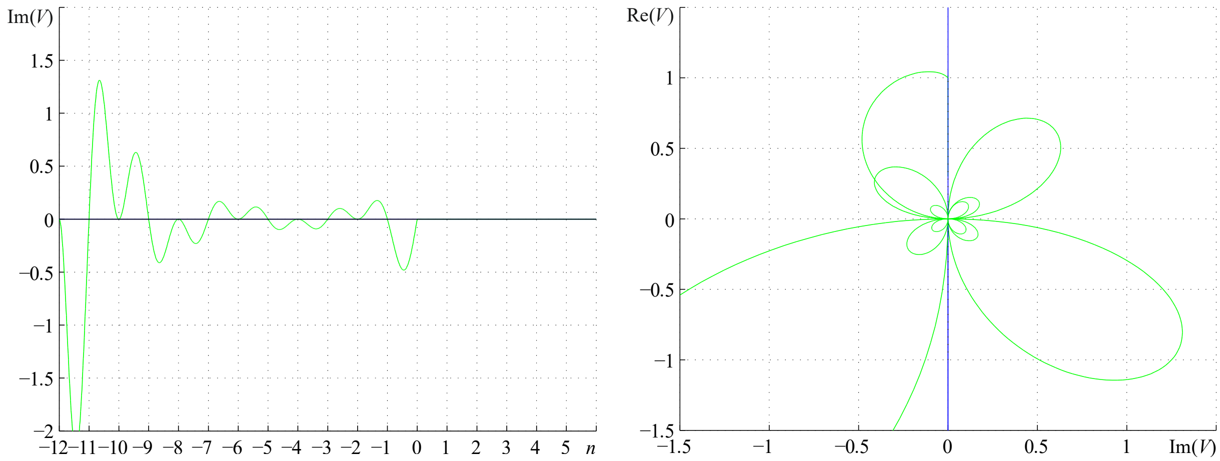
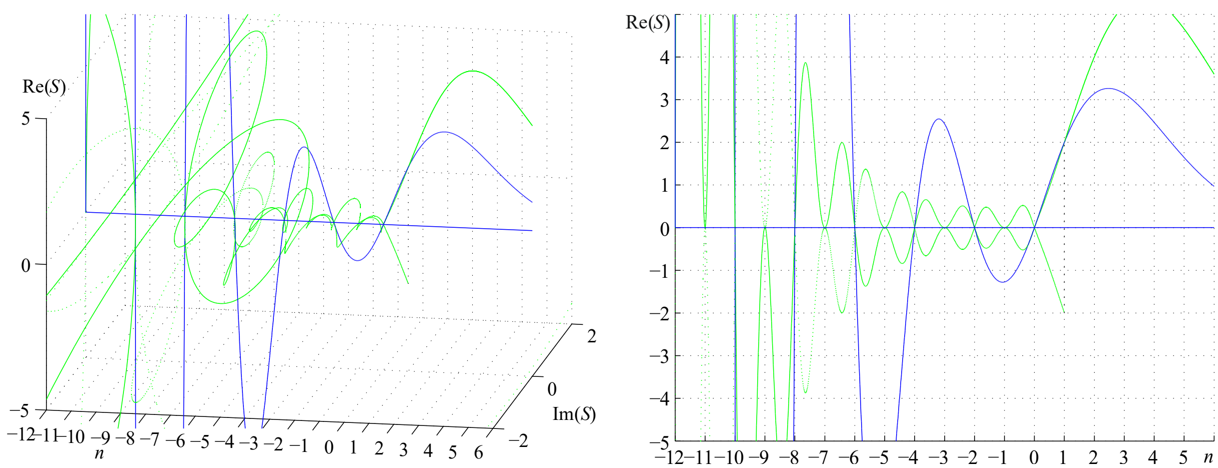


Figure 11. Graphs of volumes (V) of n-orthoplices (green) circumscribed about unit diameter n-balls and volumes of unit diameter n-balls (blue) for n = [−12, 6].

Similarly, the surface (28) of the circumscribed n-orthoplex with edge length A given by (43) becomes

$$S_n\left(A_{OCB}\right)_O = \frac{2n^{n/2+1}}{\Gamma\left(n+1\right)} D^{n-1} \begin{cases} 1 & n>1 \\ \pm 1 & n<1 \end{cases}, \tag{45}$$

as shown in Figure 12. Circumscribed n-orthoplex surface branches for $n < 1$, is complex for $n < 0$ and oscillatory divergent with decreasing n, where for both branches it is left-handed towards negative infinity or the branch point, and the principal branch crosses the quadrants of the complex plane in the order {Re(V_{OCB}) < 0, Im(V_{OCB}) > 0}, {Re(V_{OCB}) < 0, Im(V_{OCB}) < 0}, {Re(V_{OCB}) > 0, Im(V_{OCB}) < 0}, and {Re(V_{OCB}) > 0, Im(V_{OCB}) > 0}. Also it is zero for negative, integer n.



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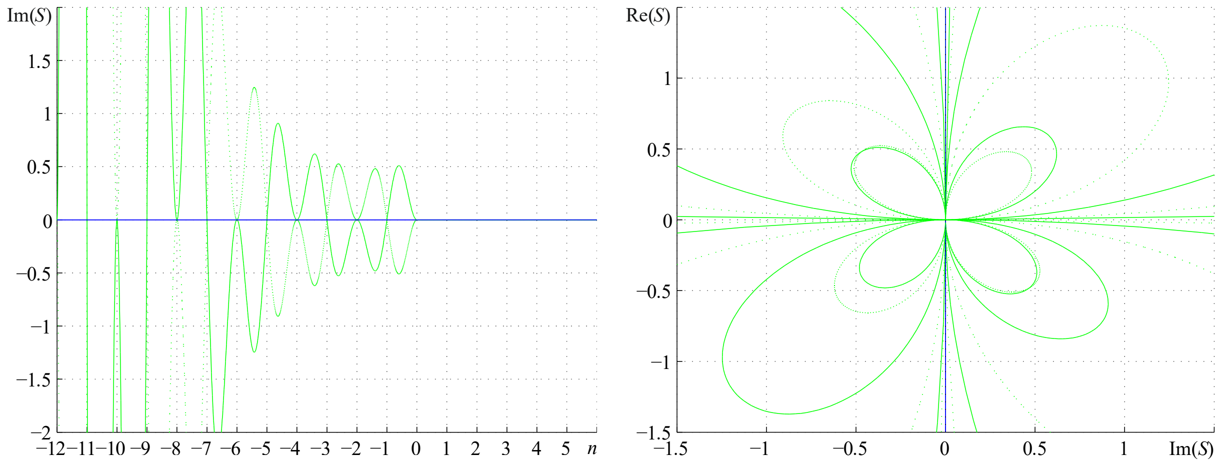

Figure 12. Graphs of surfaces (S) of n-orthoplices (green) circumscribed about unit diameter n-balls and surfaces of unit diameter n-balls (blue) for $n = [-12, 6]$.

5.3 *n*-Cubes Inscribed in and Circumscribed About *n*-Balls

The edge length A_{CCB} of an n-cube circumscribed about an n-ball (CCB) corresponds to the diameter D of this n-ball. Thus, the volume of this cube is simply $V_n(D)_{CCB} = D^n$, and the surface is $S_n(D)_{CCB} = 2nD^{n-1}$.

The edge length A_{CIB} of an n-cube inscribed inside an n-ball (CIB) of diameter D is $A_{CIB} = D/\sqrt{n}$, which is singular for $n = 0$ and complex for $n < 0$, rendering [25] the following volume and the surface of an n-cube inscribed in an n-ball

$$V_n\left(D\right)_{CIB} = n^{-n/2} D^n, \tag{46}$$

$$S_n\left(D\right)_{CIB} = 2n^{(3-n)/2} D^{n-1}. \tag{47}$$

The reflection relation can be obtained setting $m = -n$ in (46), yielding [25] the volume and the surface

$$V_m\left(D\right)_{CIB} = i^m D^{-m} m^{m/2}, \tag{48}$$

$$S_m\left(D\right)_{CIB} = -2i^{m+1} m^{(3+m)/2} D^{-m-1}, \tag{49}$$

which are complex for $m \in \mathbb{R}$. Volumes (46) and (48) correspond to each other [25] for $n \le 0$, $n \in \mathbb{R}$ and for $n = 2k$, $k \in \mathbb{Z}$, as shown in Figure 13 (left column). Surfaces (47) and (49) correspond to each other [25] for $n \in \mathbb{R}$, $n \le 0$, and for $n = 2k - 1$, $k \in \mathbb{Z}$, as shown in Figure 13 (right column).

Furthermore, the following holds [25] for (46) and (48) with $m = n$

$$V_n\left(D\right)_{CIB} V_m\left(D\right)_{CIB} \overset{m=n}{=} D^n n^{-n/2} i^n D^{-n} n^{n/2} = i^n. \tag{50}$$

For $n \ge 0$ (by convention $0^0 := 1$) the inscribed n-cube volume (46) is real, complex if $n < 0$, becoming real if n is negative and even and imaginary if n is negative and odd, and divergent with

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decreasing n. For $0 < n < 1$ it is larger than the volume of the circumscribing n-ball. For $n \geq 0$ the inscribed n-cube surface (47) is real, complex if $n < 0$, becoming real if n is negative and odd and imaginary if n is negative and even, and divergent with decreasing n. For $0 < n < 1$ it is smaller than the surface of the circumscribing n-ball.

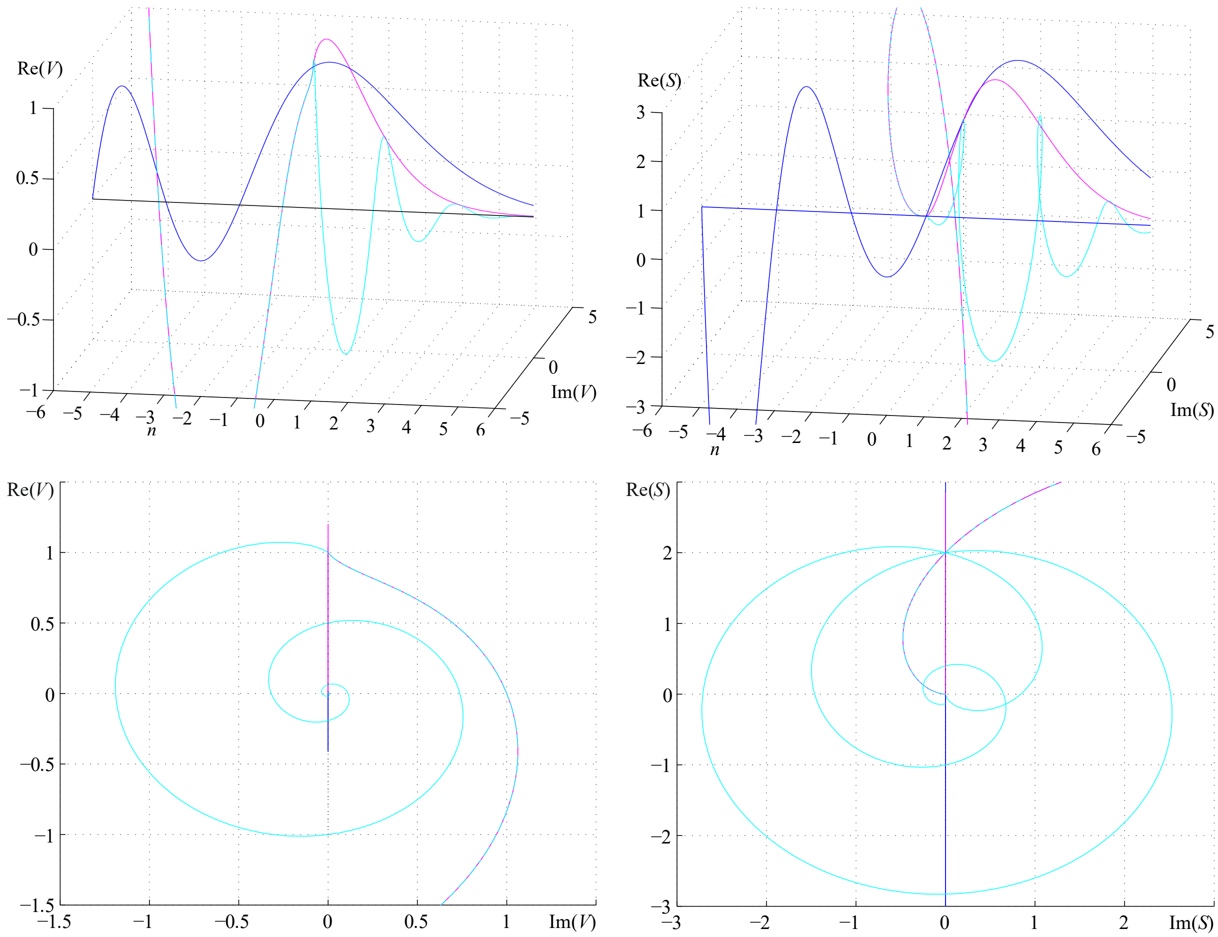

Figure 13. Graphs of volumes of n-cubes (pink) inscribed in unit diameter n-balls with the reflection relation (cyan) and volumes of unit diameter n-balls (blue) for $n = [-6, 6]$.

6. The volume of an *n*-Ball in Complex Dimensions

The gamma function is defined for all complex numbers except the non-positive integers. Thus [28], for $n = a + ib$, where if $n \in \mathbb{Z}$, $a \geq -1$

$$\pi^{n/2} = \pi^{(a+ib)/2} = \pi^{a/2}\left[\cos\left(\frac{b}{2}\ln(\pi)\right) + i\sin\left(\frac{b}{2}\ln(\pi)\right)\right], \tag{51}$$

$$R^n = R^{a+ib} = R^a\left[\cos\left(b\ln(R)\right) + i\sin\left(b\ln(R)\right)\right], \tag{52}$$

the volume (1) and surface (4) become

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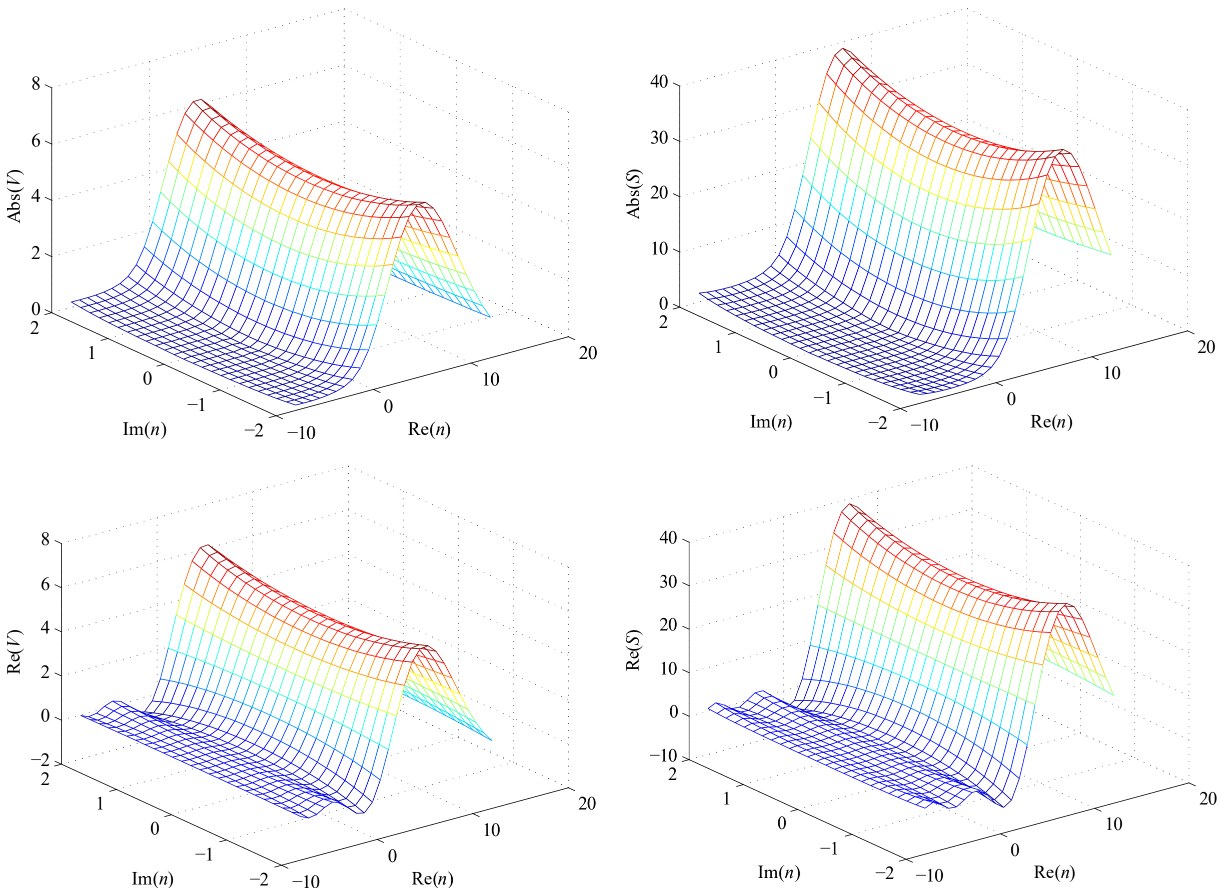
$$V_n\left(R\right)_B = \pi^{\frac{a}{2}} R^a \frac{\left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right] + i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}}{\Gamma\left(\frac{a+ib}{2}+1\right)}, \quad (53)$$

$$S_n\left(R\right)_B = \left(a+ib\right)\pi^{\frac{a}{2}} R^{a-1} \frac{\left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right] + i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}}{\Gamma\left(\frac{a+ib}{2}+1\right)}, \quad (54)$$

where we have used cos(a)cos(b) − sin(a)sin(b) = cos(a + b) and cos(a)sin(b) + sin(a)cos(b) = sin(a + b), as shown in Figure 14 for unit radius n-balls.



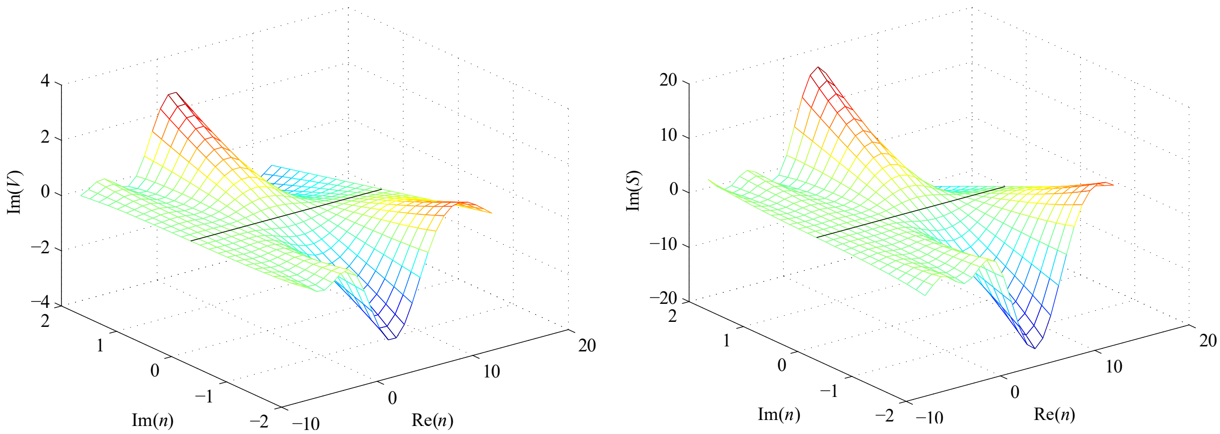


Figure 14. Graphs of complex volumes (V) and surfaces (S) of unit radius n-balls in complex dimensions $n = a + ib$ for a = [-10, 15], b = [-2, 2].

In particular for $n = 3 + ib$, $b \in \mathbb{R}$ (spacetime dimensionality) equation (53) becomes

$$V_n\left(R\right)_B = \pi^{\frac{3}{2}} R^3 \frac{\left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right] + i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}}{\Gamma\left(\frac{3+ib}{2}+1\right)}, \tag{55}$$

which reduces to familiar $V_3(R)_B = 4\pi R^3/3$ for $n = 3 + 0i$, i.e. at the present moment. Note that the imaginary part of the volume (53), in a way, establishes the arrow of time.

7. Conclusion

It was shown that the recurrence relations (9), (10), (12), and (14)-(17) are continuous for $n \in \mathbb{C}$ and can be expressed by the gamma function (18), (22), (27), wherein for $n = -2k - 2$, $k \in \mathbb{N}_0$ in the case of n-balls, and for $n = -k - 1$, $k \in \mathbb{N}_0$ in the case of n-simplices and n-orthoplices their values are given in the sense of a limit of a function. It was further shown that the volume of an n-simplex is a multivalued function for $n < 0$, and thus the surfaces of n-simplices and n-orthoplices are also multivalued functions for $n < 1$.

Applications of these formulas to basic regular polytopes inscribed in and circumscribed about an n-balls reveals the properties of these geometric objects in negative, real dimensions. In particular it is shown that the volume and surface of a regular n-simplex inscribed in an n-ball is complex for $-1 < n < 0$, imaginary for $n < -1$, and divergent with decreasing n; the volume and surface of a regular n-simplex circumscribed about an n-ball is complex for $n < 0$ and left-handedly respectively convergent to zero or divergent towards infinity; and the volume and surface of an n-orthoplex circumscribed about an n-ball is complex for $n < 0$ and oscillatory divergent towards infinity with decreasing n.

The results of this study could perhaps be applied in linguistic statistics, where the dimension in the distribution for frequency dictionaries is chosen to be negative [4], in fog computing, where n-simplex is related to a full mesh pattern, n-orthoplex is linked to a quasi-full mesh structure, and n-

cube is referred to as a certain type of partial mesh layout [29], and in molecular physics and crystallography.

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Data Availability Statement: https://github.com/szluk/balls_simplices_orthoplices (accessed on August 18, 2022).

Conflicts of Interest: The author declares no conflict of interest.

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