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The study shows that the volumes and surfaces of n -balls, n -simplices, and n -orthoplices are holomorphic functions of n , which makes those objects omnidimensional, that is well defined in any complex dimension. Applications of these formulas to the omnidimensional polytopes inscribed in and circumscribed about n -balls reveal previously unknown properties of these geometric objects. In particular, for $0 < n < 1$, the volumes of the omnidimensional polytopes are larger than those of circumscribing n -balls, and both their volumes and surfaces are smaller than those of inscribed n -balls. The surface of an n -simplex circumscribing unit diameter n -ball is spirally convergent to zero with real n approaching negative infinity but first has a local maximum at $n = -3.5$. The surface of an n -orthoplex circumscribing unit diameter n -ball is spirally divergent with real n approaching negative infinity but first has a local minimum at $n = -1.5$, where its real and imaginary parts are equal to each other; similarly, as its volume, where the similar local minimum occurs at $n = -3.5$. Reflection functions for volumes and surfaces of these polytopes inscribed in and circumscribed about n -balls are proposed. symmetries of products and quotients of volumes in complex dimensions n and $-n$ and of surfaces in complex dimensions n and $2 - n$ are shown to be independent of the metric factor and the gamma function. Specific symmetries also hold between volumes and surfaces in dimensions $n = -1/2$ and $n = 1/2$.

Keywords: regular basic convex polytopes; circumscribed and inscribed polytopes; negative dimensions; fractal dimensions; complex dimensions; emergent dimensionality; mathematical physics

I. INTRODUCTION

The notion of dimension n is intuitively defined as a natural number of coordinates of a point within Euclidean space $\mathbb{R}^n$. However, this is not the only possible definition [1, 2] of a dimension of a set. For example, analytic continuations from positive dimensions [2–5] can define negative dimensions [6]. Fractional (or fractal), including negative [7], dimensions are consistent with experimental results and enable the examination of transport parameters in multiphase fractal media [8, 9]. This renders dimension n real, or at least a rational number. Complex [2], including complex fractional [10], dimensions can also be considered. Complex geodesic paths, for example, emerge in the presence of black hole singularities [11] and when studying entropic dynamics on curved statistical manifolds [12]. Complex wavelengths occur in Maxwell-Boltzmann, and Fermi-Dirac statistics on black hole event horizons [13]. Fractional derivatives of complex functions can describe different physical phenomena [14]. For example, it was recently shown [15] that magnetic monopole motion in 3-simplicial spin ice crystal lattice is limited to a fractal cluster.

In n -dimensional space, n -dimensional objects have $(n - 1)$ -dimensional surfaces which have a dimension of volume in $(n - 1)$ -dimensional space. However, this sequence has a singularity at $n = -1$. A 0-dimensional point in 0-dimensional space has a vanishing (-1) -surface

being a vanishing volume of the (-1) -dimensional void. But the surface of the (-1) -dimensional void is not (-2) -dimensional. It is undefined. This discontinuity hints that thinking about dimension in terms of a point on a number axis, where negative dimensions are analytic continuations from positive ones, may be misleading. Thinking of the dimension as a point on a number semiaxis, similar to a point on a radius with $(-n)$ -dimension corresponding to n -dimension, seems more appropriate. Considering the dimension of a set as the length exponent at which that set can be measured makes the negative dimensions refer to densities as positive ones refer to quantities [5]. Thus, the (-2) -dimensional pressure, for example, considered in terms of density (e.g., in units of N/m^2) corresponds to a 2-dimensional area (in units of m^2) that it acts upon. Following the same logic, gravitational force $F = GMm/R^2$ acting towards an *inside* of a 2-dimensional equipotential surface is (-2) -dimensional, whereas centripetal force $F = mV^2/R^1$ acting towards an *outside* of a 1-dimensional curve is (-1) -dimensional.

The notion of distance intuitively defines how far apart two points are. Thus, intuition suggests that it is a non-negative quantity. However, intuition can be misleading¹, and the Euclidean distance admits not only the principal, non-negative, square root but also a negative one. This fact, taken plainly, violates the nonnegativity axiom of the metric. However, diffuse metrics

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¹ E.g., in aviation, where relying on a sense of orientation (aka intuition) can be fatal.

[16], including the Łukaszyk-Karmowski metric [17], are known to violate the identity of indiscernibles axiom of the metric. This hints that axiomatizing distance as a non-negative quantity may also be misleading. Complex effects (bivalence) extend real effects (classically one value), just as quantum theory extends classical physics.

The study shows that the recurrence relations of the prior research [18] allow removing indefiniteness present in known formulas for volumes and surfaces of the three polytopes present in all natural dimensions [19], thus making them holomorphic functions of a complex dimension n and omnidimensional (i.e., well defined $\forall n \in \mathbb{C}$).

The paper is structured as follows. Section II summarizes known formulas and recurrence relations for volumes and surfaces of n -balls and regular, convex n -polytopes. Section III shows that these formulas and recurrence relations can be extended to all complex dimensions. Section IV examines the properties of the omnidimensional, regular, convex polytopes inscribed in and circumscribed about n -balls and their reflection functions. Section V presents metric-independent relations between volumes and surfaces of these polytopes and n -balls, whereas Section VI presents particular symmetries between volumes and surfaces for $|n| = 1/2$. Section VII:discussion concludes and discusses the findings of this study, hinting at their possible applications.

II. KNOWN FORMULAS

It is known that the volume of an n -ball (B) is

$$V(n)_B = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} R^n, \quad (1)$$

where $\Gamma : \mathbb{C} \rightarrow \mathbb{C}$ is the Euler's gamma function, $n \in \mathbb{C} \setminus \{n = -2k - 2, k \in \mathbb{N}_0\}$ [18], and R denotes the n -ball radius. As the gamma function is meromorphic, volumes of n -balls are complex in complex dimensions. The volume of an n -ball can be expressed [20] in terms of the volume of an $(n-2)$ -ball of the same radius as a recurrence relation

$$V(n)_B = \frac{2\pi R^2}{n} V(n-2)_B, \quad (2)$$

where $V(0)_B := 1$ and $V_1(R)_B := 2R$. The relation (2) can be extended [18] into negative dimensions as

$$V(n)_B = \frac{n+2}{2\pi R^2} V(n+2)_B, \quad (3)$$

solving (2) for $V(n-2)$ and assigning new n as the previous $n-2$. A radius recurrence relation² [18]

$$f(n) := \frac{2}{n} f(n-2), \quad (4)$$

² We choose the notation $f(n)$ over f_n , as $n \in \mathbb{C}$.

where $n \in \mathbb{C}$, $f(0) := 1$ and $f(1) := 2$, allow for expressing the volume n -ball as

$$V(n)_B := f(n) \pi^{\lfloor n/2 \rfloor} R^n, \quad (5)$$

where " $\lfloor x \rfloor$ " is the floor function that yields the greatest integer less than or equal to its argument x . The relation (4) can be, analogously as formula (2), extended [18] into complex dimensions with a negative real part as

$$f(n) = \frac{n+2}{2} f(n+2), \quad (6)$$

which allows to define $f(-1) = f(0) := 1$ to initiate the sequences (4) or (6). Known [20] surface of an n -ball³ is

$$S(n)_B = \frac{n}{R} V(n)_B. \quad (7)$$

Known volume of n -cube (C) is

$$V(n)_C = A^n, \quad (8)$$

where $n \in \mathbb{C}$ and A is the edge length. The known surface of n -cube is

$$S(n)_C = 2nA^{n-1}, \quad (9)$$

where $n \in \mathbb{C}$.

Known [21, 22] volume of a regular n -simplex (S) is

$$V(n)_S = \frac{\sqrt{n+1}}{n! \sqrt{2^n}} A^n, \quad (10)$$

where $n \in \mathbb{N}$. The formula (10) can be written [18] as a recurrence relation

$$V(n)_S = AV(n-1)_S \sqrt{\frac{n+1}{2n^3}}, \quad (11)$$

with $V(0)_S := 1$, to remove the indefiniteness of the factorial for $n < 1$. Formula (11) can be solved for $V(n-1)$. Assigning new n as the previous $n-1$, yields [18]

$$V(n)_S = \frac{V(n+1)_S}{A} \sqrt{\frac{2(n+1)^3}{n+2}}, \quad (12)$$

which also removes the singularity for $n = 0$ present in the volume (10). n -simplex has $n+1$ ($n-1$)-facets [20]. Therefore, its surface is

$$S(n)_S = (n+1)V(n-1)_S. \quad (13)$$

³ Commonly, the surface $S(n)$ is defined to correspond to $(n+1)$ -dimensional object. However, the authors find it confusing: $(n-1)$ -dimensional surface remains the property of the n -dimensional object. Thus, in this study $S(n)$ denotes $(n-1)$ -dimensional surface of the n -dimensional object, not n -dimensional surface of the $(n+1)$ -dimensional object.

Known [20] volume of an n -orthoplex (O) is

$$V(n)_O = \frac{\sqrt{2^n}}{n!} A^n, \quad (14)$$

where $n \in \mathbb{N}_0$.

Formula (14) can be written [18] as a recurrence relation

$$V(n)_O = AV(n-1)_O \frac{\sqrt{2}}{n}, \quad (15)$$

where $n \neq 0$ and $V(0)_O := 1$, to remove the indefiniteness of the factorial for $n < 0$. Solving (15) for $V(n-1)$ and assigning new n as the previous $n-1$, yields [18]

$$V(n)_O = V(n+1)_O \frac{n+1}{A\sqrt{2}}, \quad (16)$$

which also removes singularity for $n = 0$ present in formula (15). Any n -orthoplex has 2^n facets [20], which are regular $(n-1)$ -simplices. Therefore, its surface is

$$S(n)_O = 2^n V(n-1)_S. \quad (17)$$

III. HOLOMORPHIZING KNOWN FORMULAS

Known formulas presented in the preceding section can be extended to all complex dimensions. We note that volumes and surfaces of n -cubes are already defined $\forall n \in \mathbb{C}$ and thus holomorphic. We also note that a square root is bivalued in the complex domain⁴. Therefore, volumes and surfaces of n -balls and regular, convex n -polytopes are also bivalued in complex dimensions n . We call *principal branch* the branch with positive values in positive real dimensions.

Theorem 1. *The volume of an n -ball is a holomorphic, bivalued function of a complex dimension n .*

Proof. First, we note that the initial bivalued values for the radius recurrence relations (4), (6) are $f(-1) = f(0) := \pm 1$, $V(0)_B := \pm 1$ and $V_1(R)_B := \pm 2R$. Then, we note that the recurrence relations (3) and (5) correspond to each other

$$\begin{aligned} V(n)_B &= \frac{n+2}{2\pi R^2} V(n+2)_B = \frac{n+2}{2} f(n+2) \pi^{\lfloor n/2 \rfloor} R^n \\ V(n+2)_B &= \pi f(n+2) \pi^{\lfloor n/2 \rfloor} R^{n+2}, \end{aligned} \quad (18)$$

which, after setting $\tilde{n} := n+2$, yields

$$V(\tilde{n})_B = f(\tilde{n}) \pi^{1+\lfloor (\tilde{n}-2)/2 \rfloor} R^{\tilde{n}} = f(\tilde{n}) \pi^{\lfloor \tilde{n}/2 \rfloor} R^{\tilde{n}}. \quad (19)$$

⁴ Furthermore $A^n = A^{2n/2} = \sqrt{A^{2n}} \stackrel{?}{=} \pm A^n$. By choosing complex analysis, we enter into bivalence by its very nature.

Comparing the non-recurrence general n -ball volume formula (1), which is bivalued for $n \in \mathbb{C}$ and valid within the domain of the gamma function, with the recurrence relation (3)

$$V(n)_B = \pm \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n = \frac{n+2}{2\pi R^2} V(n+2)_B \quad (20)$$

yields

$$V(n+2)_B = \pm \frac{2\pi^{n/2+1}}{(n+2)\Gamma(n/2+1)} R^{n+2}. \quad (21)$$

Setting $\tilde{n} := n+2$ yields

$$\begin{aligned} V(\tilde{n})_B &= \pm \frac{2\pi^{\tilde{n}/2}}{m\Gamma(\tilde{n}/2)} R^m = \pm \frac{\pi^{\tilde{n}/2}}{\frac{\tilde{n}}{2}\Gamma(\tilde{n}/2)} R^{\tilde{n}} = \\ &= \pm \frac{\pi^{\tilde{n}/2}}{\Gamma(m/2+1)} R^{\tilde{n}}, \end{aligned} \quad (22)$$

which recovers (1), as $\frac{n}{2}\Gamma(\frac{n}{2}) = \Gamma(\frac{n}{2}+1)$, $\forall n \in \mathbb{C} \setminus \{n = -2k, k \in \mathbb{N}_0\}$. Thus we have proved that the recurrence relations (2), (3), and (5) correspond to the general n -ball volume formula (22) within this domain.

However, now we can use any of the *backward* recurrence relations (3) or (5) with (6) to determine the values of the n -ball volume outside this domain: $\forall n+2 \in \mathbb{C}$ we can find $V(n)_B$.

On the other hand

$$\lim_{n \rightarrow -2k-2, k \in \mathbb{N}_0} \pm \frac{\pi^{n/2} R^n}{\Gamma(n/2+1)} = \frac{a}{\infty} = 0, \quad (23)$$

so the poles of the meromorphic gamma function $\Gamma(n/2+1)$ present in (22), now defined in the sense of a limit of a function, vanish, which completes the proof. $\square$

Corollary 1.1. *The surface of an n -ball is a holomorphic, bivalued function of a complex dimension n .*

Proof. If the volume of an n -ball is a holomorphic, bivalued function by Theorem 1, then, using formula (7), the surface of an n -ball is also a holomorphic function.

Using (22) and (7) the surface of an n -ball is given by

$$S(n)_B = \pm \frac{n\pi^{n/2}}{\Gamma(n/2+1)} R^{n-1}. \quad (24)$$

$\square$

Theorem 2. *The volume of a regular n -simplex is a holomorphic, bivalued function of a complex dimension n .*

Proof. Expressing the factorial in the volume of a regular n -simplex formula (10) by the gamma function extends the domain of applicability of (10) to complex dimensions

$$V(n)_S = \pm \frac{\sqrt{n+1}}{n!\sqrt{2^n}} A^n = \pm \frac{(1+n)^{1/2}}{\Gamma(n+1)2^{n/2}} A^n. \quad (25)$$

On the other hand, comparing the volume (25) with the recurrence relation (12)

$$\pm \frac{\sqrt{n+1}}{\Gamma(n+1)2^{n/2}} A^n = \frac{V(n+1)_S}{A} \sqrt{\frac{2(n+1)^3}{n+2}} \quad (26)$$

yields

$$V(n+1)_S = \pm \frac{\sqrt{n+1}\sqrt{n+2}}{\Gamma(n+1)2^{(n+1)/2}\sqrt{(n+1)^3}} A^{n+1}, \quad (27)$$

which, after setting $\tilde{n} := n+1$, yields

$$\begin{aligned} V(\tilde{n})_S &= \pm \frac{\sqrt{\tilde{n}}\sqrt{\tilde{n}+1}}{\Gamma(\tilde{n})2^{\tilde{n}/2}\sqrt{\tilde{n}^3}} A^{\tilde{n}} = \\ &= \pm \frac{\sqrt{\tilde{n}+1}}{\Gamma(\tilde{n}+1)2^{\tilde{n}/2}} A^{\tilde{n}} \frac{\tilde{n}\sqrt{\tilde{n}}}{\sqrt{\tilde{n}^3}} = \\ &= \pm \frac{\sqrt{\tilde{n}+1}}{\Gamma(\tilde{n}+1)2^{\tilde{n}/2}} A^{\tilde{n}}, \end{aligned} \quad (28)$$

and recovers (25) as $n\Gamma(n) = \Gamma(n+1) \forall n \in \mathbb{C} \setminus \{-k, k \in \mathbb{N}_0\}$ and $n\sqrt{n}/\sqrt{n^3} = \pm 1$ (cf. Appendix A). Thus, we have proved that the recurrence relation (12) corresponds to the generalized n -simplex volume formula (25) within this domain.

However, now we can use the recurrence relation (12) to determine the values of the n -simplex volume outside this domain: $\forall n+1 \in \mathbb{C}$ we can find $V(n)_S$. For example, even though $n\sqrt{n}/\sqrt{n^3}$ is undefined for $n = 0$, we can determine that $V(0)_S = \pm 1$ using the recurrence relation (12) with $V(1)_S = \pm A$ obtained from (25).

On the other hand

$$\lim_{n \rightarrow -k-1, k \in \mathbb{N}_0} \pm \frac{2^{-n/2} A^n \sqrt{n+1}}{\Gamma(n+1)} = \frac{a}{\infty} = 0, \quad (29)$$

so the poles of the meromorphic gamma function $\Gamma(n+1)$ present in (25), now defined in the sense of a limit of a function, vanish - which completes the proof. $\square$

For $n \in \mathbb{R} \setminus \mathbb{Z} \cup \{n \in \mathbb{R} : n < -1\}$, n -simplex volume formula (25) is imaginary.

Corollary 2.1. *The surface of a regular n -simplex is a holomorphic, bivalued function of a complex dimension n .*

Proof. If the volume of a regular n -simplex is a holomorphic, bivalued function by Theorem 2, then its surface, using formula (13), is also a holomorphic, bivalued function.

Using (13) and (25) the surface of a regular n -simplex is given by (cf. Appendix A)

$$\begin{aligned} S(n)_S &= \pm \frac{n^{3/2}(n+1)}{\Gamma(n+1)2^{(n-1)/2}} A^{n-1} \frac{(n-1)\sqrt{n-1}}{\sqrt{(n-1)^3}} = \\ &= \pm \frac{n^{3/2}(1+n)}{\Gamma(n+1)2^{(n-1)/2}} A^{n-1}. \end{aligned} \quad (30)$$

$\square$

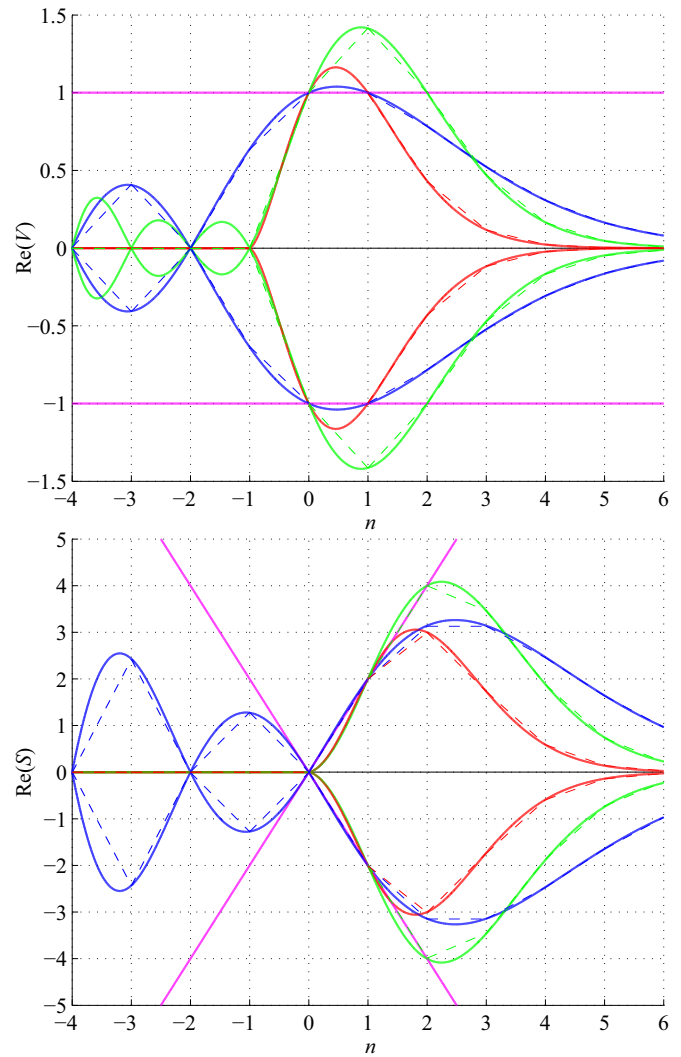


Figure 1. Graphs of volumes (V) and surfaces (S) of unit edge length regular n -simplices (red), n -orthoplices (green), n -cubes (pink), and unit diameter n -balls (blue), along with the integer recurrence relations (dashed lines) for $n \in [-4, 6]$.

Again, even though $(n-1)\sqrt{n-1}/\sqrt{(n-1)^3}$ is undefined for $n = 1$, we can determine that $S(1)_S = \pm 2$ directly from formula (13) knowing that $V(0)_S = \pm 1$.

For $n \in \mathbb{R} \setminus \mathbb{Z} : n < 0$, n -simplex surface formula (30) is imaginary.

Theorem 3. *The volume of an n -orthoplex is a bivalued, holomorphic function of a complex dimension n .*

Proof. Expressing the factorial in the volume of an n -orthoplex formula (14) by the gamma function extends the domain of applicability of (14) to complex dimensions

$$V(n)_O = \pm \frac{\sqrt{2^n}}{n!} A^n = \pm \frac{\sqrt{2^n}}{\Gamma(n+1)} A^n. \quad (31)$$

On the other hand, comparing (31) with the recurrence

relation (16)

$$\pm \frac{\sqrt{2^n}}{\Gamma(n+1)} A^n = V(n+1)_O \frac{n+1}{A\sqrt{2}}, \quad (32)$$

yields

$$V(n+1)_O = \pm \frac{2^{(n+1)/2}}{(n+1)\Gamma(n+1)} A^{n+1}, \quad (33)$$

whereas setting $\tilde{n} = n+1$ in (33) yields

$$V(\tilde{n})_O = \pm \frac{\sqrt{2^{\tilde{n}}}}{\tilde{n}\Gamma(\tilde{n})} A^{\tilde{n}} = \pm \frac{2^{\tilde{n}/2}}{\Gamma(\tilde{n}+1)} A^{\tilde{n}}, \quad (34)$$

which recovers n -orthoplex volume (14), as $n\Gamma(n) = \Gamma(n+1) \forall n \in \mathbb{C} \setminus \{n = -k, k \in \mathbb{N}_0\}$. Thus, we have proved that the recurrence relation (16) corresponds to the general n -orthoplex volume formula (31) within this domain.

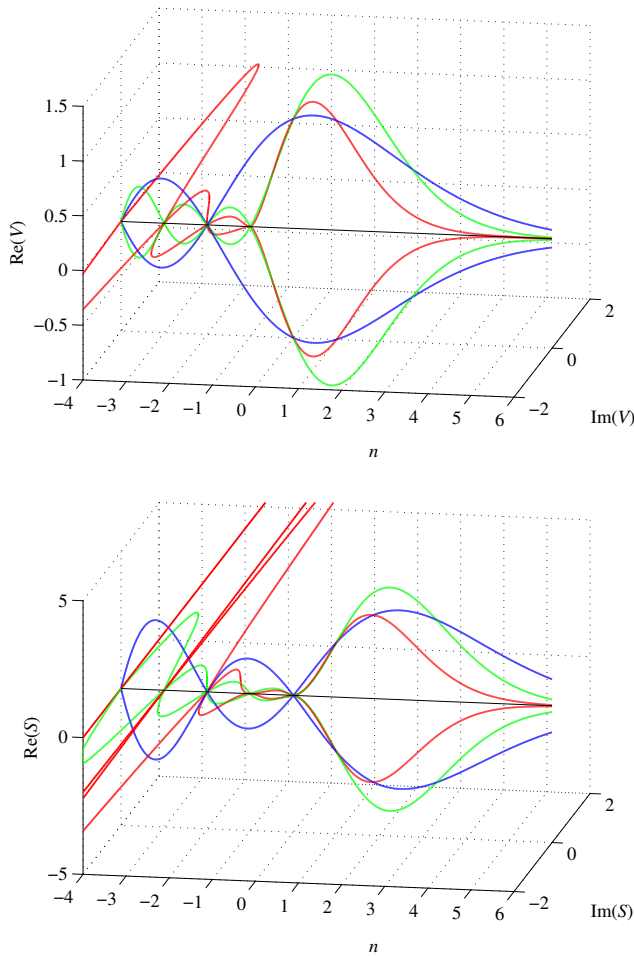


Figure 2. Graphs of volumes (V) and surfaces (S) of unit edge length regular n -simplices (red), n -orthoplices (green), and unit diameter n -balls (blue) for $n \in \mathbb{R} : n \in [-4, 6]$.

However, now we can use the recurrence relation (16) to determine the values of the n -orthoplex volume outside

this domain: $\forall n+1 \in \mathbb{C}$ we can find $V(n)_O$. On the other hand

$$\lim_{n \rightarrow -k-1, k \in \mathbb{N}_0} \pm \frac{2^{n/2} A^n}{\Gamma(n+1)} = \frac{a}{\infty} = 0, \quad (35)$$

so the poles of the meromorphic gamma function $\Gamma(n+1)$ present in (31), now defined in the sense of a limit of a function, vanish - which completes the proof. $\square$

Corollary 3.1. *The surface of an n -orthoplex is a holomorphic, bivalued function of a complex dimension n .*

Proof. If the volume of a regular n -simplex is a holomorphic, bivalued function by Theorem 2, then, using formula (17), the surface of an n -orthoplex is a holomorphic, bivalued function.

Using (17) and (31) the surface of an n -orthoplex is given by

$$S(n)_O = \pm \frac{n^{3/2} 2^{(n+1)/2}}{\Gamma(n+1)} A^{n-1}. \quad (36)$$

$\square$

For $n \in \mathbb{R} \setminus \mathbb{Z} : n < 0$, n -orthoplex bivalued surface formula (36) is imaginary.

Thus, the volumes and surfaces of n -balls, n -simplices, and n -orthoplices, shown in Figs. 1 and 2 for $n \in \mathbb{R}$, are holomorphic functions of n , which makes those objects omnidimensional. They can be explicitly expressed in terms of the complex dimension (cf. Appendix B).

IV. OMNIDIMENSIONAL POLYTOPES INSCRIBED IN AND CIRCUMSCRIBED ABOUT n -BALLS

First, we will introduce the definition of a reflection function.

Definition 1. *For each function $F : \mathbb{C} \rightarrow \mathbb{C}$ of a complex argument n given by the formula*

$$F(n) = \pm \prod_{j=1}^l (a_j + b_j n)^{(c_j + d_j n)/2} g(n), \quad (37)$$

where $j = 1, 2, \dots, l \in \mathbb{N}$, $a_j, b_j, c_j, d_j \in \mathbb{R} : a_j \geq 0, b_j > 0$, and $g(n)$ is some nonzero function of n , we define its reflection function

$$\tilde{F}(n) := \pm i^{c+dn} \prod_{j=1}^l (-a_j - b_j n)^{(c_j + d_j n)/2} g(n), \quad (38)$$

where

$$c := \sum_{j=1}^l c_j \quad \text{and} \quad d := \sum_{j=1}^l d_j \neq 0. \quad (39)$$

Table I. Particular volumes of omnidimensional polytopes inscribed in and circumscribed about unit diameter n -balls (principal branch).

n	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$
V_B	0.331	$2/\pi \approx 0.637$	0.867	1	1.039	1	0.908
V_{SIB}	$\frac{i3^{3/4}}{\sqrt{\pi}} \approx i1.286$	0	$\frac{1-i}{\sqrt{2\pi}} \approx 0.399(1-i)$	1	$\frac{3^{3/4}}{\sqrt{\pi}} \approx 1.286$	1	$\frac{5^{5/4}}{3^{7/4}\sqrt{\pi}} \approx 0.617$
V_{SCB}	$\frac{2^{3/2}}{3^{3/4}\sqrt{\pi}} \approx 0.7$	0	$\frac{1-i}{\sqrt{\pi}} \approx 0.564(1-i)$	1	$\frac{3^{3/4}}{\sqrt{2\pi}} \approx 0.909$	1	$\frac{5^{5/4}}{3^{1/4}2^{3/2}\sqrt{\pi}} \approx 1.133$
V_{OIB}	$\frac{-1}{2\sqrt{\pi}} \approx -0.282$	0	$\frac{1}{\sqrt{\pi}} \approx 0.564$	1	$\frac{2}{\sqrt{\pi}} \approx 1.128$	1	$\frac{4}{3\sqrt{\pi}} \approx 0.752$
V_{OCB}	$\frac{1+i}{2^{3/4}3^{3/4}\sqrt{\pi}} \approx 0.147(1+i)$	0	$\frac{1-i}{2^{1/4}\sqrt{\pi}} \approx 0.474(1-i)$	1	$\frac{2^{3/4}}{\sqrt{\pi}} \approx 0.949$	1	$\frac{2^{5/4}}{3^{1/4}\sqrt{\pi}} \approx 1.02$
V_{CIB}	$\frac{(i-1)3^{3/4}}{2^{5/4}} \approx 0.958(i-1)$	i	$\frac{(1+i)}{2^{3/4}} \approx 0.595(1+i)$	1	$2^{1/4} \approx 1.189$	1	$\frac{2^{3/4}}{3^{3/4}} \approx 0.738$
V_{CCB}	1	1	1	1	1	1	1

Theorem 4. The same branches of functions $F(n)$ (37) and $\tilde{F}(n)$ (38) are equal to each other for $n \leq 0$ or for $n \in \mathbb{Z}$ such that $\sin\left(\frac{c+dn}{2}\pi\right) = 0$, with c, d given by (39).

Proof. For $n \in \mathbb{R}$, the functions (37) and (38) can be expressed as

$$\begin{aligned}
 F(n) &= \pm \prod_{j=1}^l e^{(c_j+d_jn) \ln(a_j+b_jn)/2 + i\theta_n(c_j+d_jn)/2} g(n) = \\
 &= \pm e^{i\theta_n(\sum_{j=1}^l c_j + n \sum_{j=1}^l d_j)/2} \\
 &\prod_{j=1}^l e^{(c_j+d_jn) \ln(a_j+b_jn)/2} g(n) = \\
 &= \pm e^{i\theta_n(c+dn)/2} \prod_{j=1}^l e^{(c_j+d_jn) \ln(a_j+b_jn)/2} g(n),
 \end{aligned} \tag{40}$$

$$\begin{aligned}
 \tilde{F}(n) &= \pm i^{c+dn} \\
 &\prod_{j=1}^l e^{(c_j+d_jn) \ln(a_j+b_jn)/2 - i\theta_{-n}(c_j+d_jn)/2} g(n) = \\
 &= \pm e^{-i\theta_{-n}(\sum_{j=1}^l c_j + n \sum_{j=1}^l d_j)/2} \\
 &\prod_{j=1}^l e^{(c_j+d_jn) \ln(a_j+b_jn)/2} g(n) = \\
 &= \pm i^{c+dn} e^{-i\theta_{-n}(c+dn)/2} \\
 &\prod_{j=1}^l e^{(c_j+d_jn) \ln(a_j+b_jn)/2} g(n),
 \end{aligned} \tag{41}$$

where $\theta_n = \text{atan2}(0, n)$, and $\theta_{-n} = \text{atan2}(-0, -n)$ ⁵.

Without loss of generality, we consider positive

branches of $F(n)$ (40) and $\tilde{F}(n)$ (41)

$$\begin{aligned}
 &e^{i\theta_n(c+dn)/2} \prod_{j=1}^l e^{(c_j+d_jn) \ln(a_j+b_jn)/2} g(n) = \\
 &= i^{c+dn} e^{-i\theta_{-n}(c+dn)/2} \prod_{j=1}^l e^{(c_j+d_jn) \ln(a_j+b_jn)/2} g(n), \tag{42} \\
 &e^{i\theta_n(c+dn)/2} = i^{c+dn} e^{-i\theta_{-n}(c+dn)/2} \Leftrightarrow \\
 &e^{i(\theta_n+\theta_{-n})(c+dn)/2} = i^{c+dn}.
 \end{aligned}$$

We note here that $\forall n \in \mathbb{R}$, such that $n \neq 0$

$$\theta_n + \theta_{-n} = \begin{cases} \pi, & n < 0 \\ 0, & n = 0 \\ -\pi, & n > 0 \end{cases}, \tag{43}$$

(with $\text{atan2}(0, 0) := 0$) so that

$$\begin{aligned}
 &e^{i(\theta_n+\theta_{-n})(c+dn)/2} = \\
 &= \begin{cases} e^{i\pi(c+dn)/2} = \cos\left(\frac{c+dn}{2}\pi\right) + i\sin\left(\frac{c+dn}{2}\pi\right), & n < 0 \\ 1, & n = 0 \\ e^{-i\pi(c+dn)/2} = \cos\left(\frac{c+dn}{2}\pi\right) - i\sin\left(\frac{c+dn}{2}\pi\right), & n > 0, \end{cases} \tag{44}
 \end{aligned}$$

and that $\forall n \in \mathbb{C}$

$$i^n = \cos\left(\frac{n}{2}\pi\right) + i\sin\left(\frac{n}{2}\pi\right). \tag{45}$$

Therefore,

$$\begin{aligned}
 &i^{c+dn} = \cos\left(\frac{c+dn}{2}\pi\right) + i\sin\left(\frac{c+dn}{2}\pi\right) = \\
 &= \begin{cases} \cos\left(\frac{c+dn}{2}\pi\right) + i\sin\left(\frac{c+dn}{2}\pi\right), & n < 0 \\ 1, & n = 0 \\ \cos\left(\frac{c+dn}{2}\pi\right) - i\sin\left(\frac{c+dn}{2}\pi\right), & n > 0 \end{cases} \Leftrightarrow \tag{46}
 \end{aligned}$$

$\Leftrightarrow F(n) = \tilde{F}(n)$ under the condition

$$\forall n \leq 0 \vee \forall n \in \mathbb{Z} : \sin\left(\frac{c+dn}{2}\pi\right) = 0.$$

⁵ $\lim_{y \rightarrow 0^-} \text{atan2}(y, x) = \pi$, $\lim_{y \rightarrow 0^+} \text{atan2}(y, x) = -\pi$

Each regular, omnidimensional polytope discussed in Section III can be inscribed in and circumscribed about an n -ball. This section presents their volumes, surfaces, and reflection functions (38). Particular values of their volumes and surfaces are listed in Tables I and II. Unless stated otherwise, $n \in \mathbb{R}$ and $k \in \mathbb{N}$. Unit diameter n -balls are assumed.

A. Regular n -Simplexes Inscribed in n -Balls

The diameter D_{BCS} of an n -ball circumscribing a regular n -simplex (BCS) is known [22] to be

$$D_{BCS} = \pm \frac{\sqrt{2n}}{\sqrt{n+1}} A, \quad (47)$$

where A is the edge length. Hence, the edge length A_{SIB} of a regular n -simplex inscribed (SIB) in an n -ball (B) with diameter D is

$$A_{SIB} = \pm \frac{\sqrt{n+1}}{\sqrt{2n}} D, \quad (48)$$

so that the regular n -simplex volume (25) becomes

$$V(n)_{SIB} = \pm \frac{(1+n)^{(1+n)/2} n^{-n/2} 2^{-n}}{\Gamma(1+n)} D^n. \quad (49)$$

For $n \notin \mathbb{Z} : n < -1$ the inscribed n -simplex volume (49) is imaginary and divergent with n approaching negative infinity. It is complex for $-1 < n < 0$, with the real part equaling the imaginary part for $n = -1/2$. It is zero for $n = -k$, and for $0 < n < 1$, it is larger than the volume of the circumscribing n -ball. The volume (49) does not have a reflection function (38), as d in (39) vanishes.

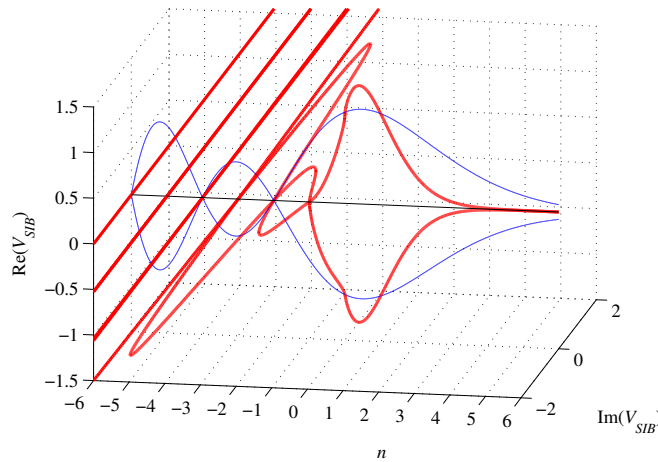


Figure 3. Graphs of $V(n)_{SIB}$ (red) and $V(n)_B$ (blue) for $n \in [-6, 6]$.

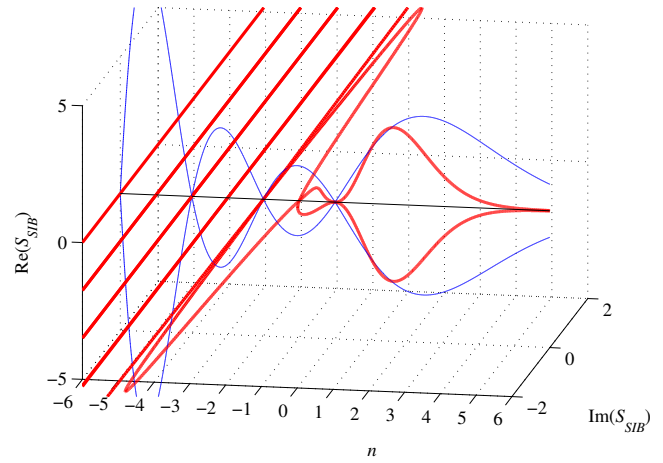


Figure 4. Graphs of $S(n)_{SIB}$ (red) and $S(n)_B$ (blue) for $n \in [-6, 6]$.

Similarly, the surface (30) of a regular inscribed n -simplex with edge length A given by (48) is

$$S(n)_{SIB} = \pm \frac{(1+n)^{(1+n)/2} n^{(4-n)/2} 2^{1-n}}{\Gamma(1+n)} D^{n-1}, \quad (50)$$

as shown in Fig. 4. For $n \notin \mathbb{Z} : n < -1$ the inscribed n -simplex surface (50) is imaginary and divergent with n approaching negative infinity. It is complex for $-1 < n < 0$, with the real part equaling the imaginary part for $n = -1/2$. It is zero for $n = -k$, $k \in \mathbb{N}_0$. The surface (50) does not have a reflection function (38), as d in (39) vanishes.

Volume (49) and surface (50) of n -simplices inscribed in n -balls are shown in Figs. 3 and 4.

B. Regular n -Simplexes Circumscribed About n -Balls

The diameter D_{BIS} of an n -ball inscribed in a regular n -simplex (BIS) is known [22] to be

$$D_{BIS} = \pm \frac{\sqrt{2}}{\sqrt{n}\sqrt{n+1}} A, \quad (51)$$

where A is the edge length. Hence, the edge length A_{SCB} of a regular n -simplex circumscribed (SCB) about an n -ball (B) with diameter D is

$$A_{SCB} = \pm \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{2}} D, \quad (52)$$

so that its volume (25) becomes

$$V(n)_{SCB} = \pm \frac{(1+n)^{(1+n)/2} n^{n/2} 2^{-n}}{\Gamma(1+n)} D^n. \quad (53)$$

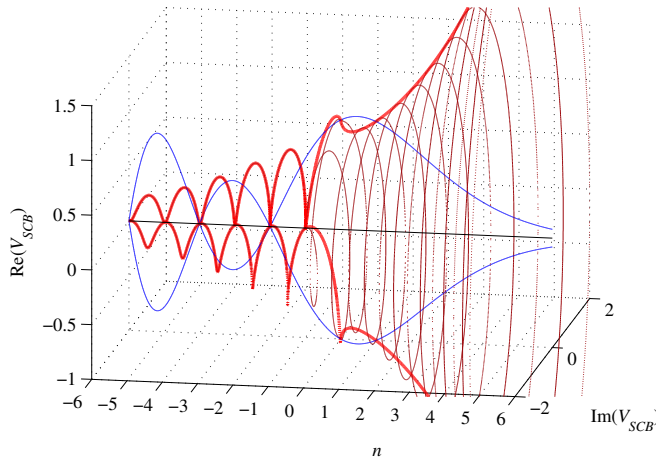


Figure 5. Graphs of $V(n)_{SCB}$ (red), $V(n)_B$ (blue), and $\tilde{V}(n)_{SCB}$ (dark red), for $n \in [-6, 6]$.

For $n < 0$, the volume (53) is complex with a nonzero imaginary part, whereas both branches are left-handed spirals converging to zero with n approaching negative infinity. For $0 < n < 1$, the volume (53) is smaller than the volume of the inscribed n -ball. It is zero for $n = -k$, and real for $n = -(2k+1)/2$, where it amounts

$$\begin{aligned} V\left(\frac{-(2k+1)}{2}\right)_{SCB} &= \\ &= \pm \frac{i^{2k} 2^{(4k+1)/2} (2k-1)^{(1-2k)/4}}{\Gamma\left(\frac{1-2k}{2}\right) (2k+1)^{(1+2k)/4}} \approx \\ &\approx \pm\{0.7, 0.5618, 0.4251, 0.3172, 0.2353, \dots\}. \end{aligned} \quad (54)$$

Furthermore, for $n = -1/2$ and $n = -(2k+3)/4$, the real part of the volume (53) equals the imaginary part up to a modulus. For $n = -1/2$, $V(-1/2)_{SCB} = \pm(1-i)/\sqrt{\pi} \approx \pm 0.5642(1-i)$. Otherwise, it amounts

$$\begin{aligned} V\left(\frac{-(2k+3)}{4}\right)_{SCB} &= \\ &= \pm \frac{(1+i)i^k 2^{(6k+3)/4} (2k-1)^{(1-2k)/8}}{\Gamma\left(\frac{1-2k}{4}\right) (2k+3)^{(2k+3)/8}} \approx \\ &\approx \pm\{-0.3549, -0.3359, 0.2996, 0.2626, -0.2283, \dots\} \\ &(1+i)i^k. \end{aligned} \quad (55)$$

The reflection function (38) of the volume (53) is

$$\tilde{V}(n)_{SCB} = \pm \frac{i^{1+2n} (-1-n)^{(1+n)/2} (-n)^{n/2} 2^{-n}}{\Gamma(1+n)} D^n, \quad (56)$$

and the volumes (56) and (53) are equal for $n = (2k+1)/2, k \in \mathbb{N}_0$.

Similarly, the surface (30) of a regular circumscribed

n -simplex with edge length A_{SCB} (52) is

$$S(n)_{SCB} = \pm \frac{(1+n)^{(1+n)/2} n^{(2+n)/2} 2^{1-n}}{\Gamma(1+n)} D^{n-1}. \quad (57)$$

The surface (57) is zero for $n = -k, k \in \mathbb{N}_0$, and for $0 < n < 1$, it is smaller than the surface of the inscribed n -ball. The surface (57) is complex for $n < 0$, with both branches being left-handed spirals converging towards zero with n approaching negative infinity. However, it is initially divergent to achieve a modulus maximum of about 2.9757 at $n \approx -3.4997$ (numerically calculated) and a real maximum of about 2.976 at $n = -3.5$. For $n = -(2k+1)/2$ the imaginary part of the surface (57) vanishes and it amounts

$$\begin{aligned} S\left(\frac{-(2k+1)}{2}\right)_{SCB} &= \\ &= \pm \frac{i^{2k} 2^{(4k+1)/2} (2k-1)^{(1-2k)/4}}{\Gamma\left(\frac{1-2k}{2}\right) (2k+1)^{(2k-3)/4}} \approx \\ &\approx \pm\{2.1, 2.809, \underline{2.976}, 2.854, 2.588, \dots\}. \end{aligned} \quad (58)$$

The real part of the surface (57) is equal to the imaginary part up to a modulus for $n = -1/2$ ($S(-1/2)_{SCB} = \pm(-1+i)/\sqrt{\pi}$) and $n = -(2k+3)/4$

$$\begin{aligned} S\left(\frac{-(2k+3)}{4}\right)_{SCB} &= \\ &= \pm \frac{(1+i)i^k 2^{(6k-1)/4} (2k-1)^{(1-2k)/8}}{\Gamma\left(\frac{1-2k}{4}\right) (2k+3)^{(2k-5)/8}} \approx \\ &\approx \pm\{-0.8973, -1.1755, 1.3484, 1.4443, \\ &\quad -\underline{1.4842}, -1.4828, \dots\} (1+i)i^k. \end{aligned} \quad (59)$$

The reflection function (38) of the surface (57) is

$$\begin{aligned} \tilde{S}(n)_{SCB} &= \\ &\pm \frac{i^{3+2n} (-n)^{(2+n)/2} (-1-n)^{(1+n)/2} 2^{1-n}}{\Gamma(1+n)} D^{n-1}, \end{aligned} \quad (60)$$

and it equals (53) for $n = (2k+1)/2, k \in \mathbb{N}_0$.

Volumes (53) and surfaces (57) of n -simplices circumscribed about n -balls are shown in Figs. 5 and 6.

C. n -Orthoplices Inscribed in n -Balls

The diameter D_{BCO} of an n -ball circumscribing an n -orthoplex (BCO) is known [23] to be

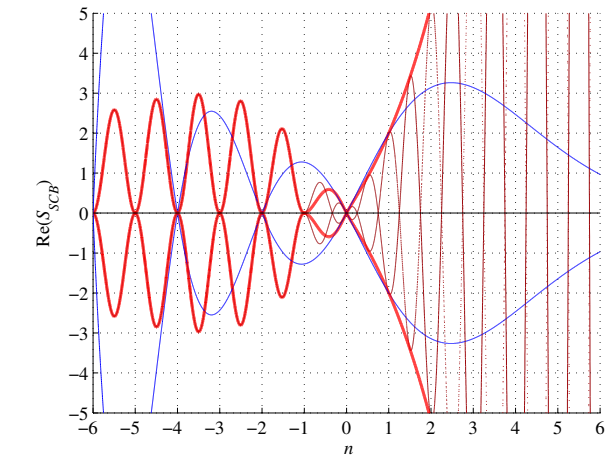
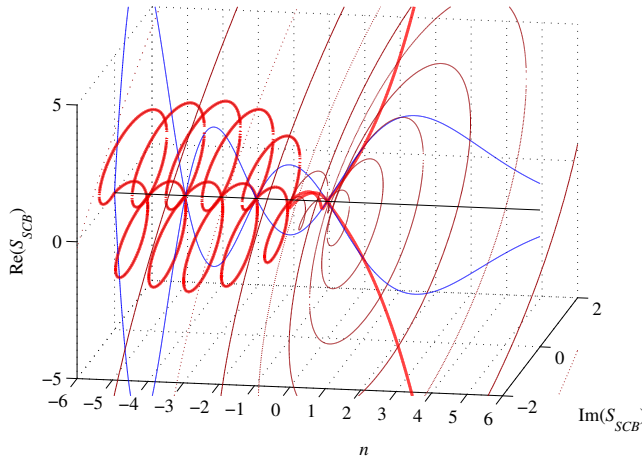
$$D_{BCO} = \pm\sqrt{2}A, \quad (61)$$

where A is the edge length. Hence, the edge length A_{OIB} of an n -orthoplex inscribed in an n -ball (OIB) with diameter D is

$$A_{OIB} = \pm \frac{1}{\sqrt{2}}D, \quad (62)$$

Table II. Particular surfaces of omnidimensional polytopes inscribed in and circumscribed about unit diameter n -balls (principal branch).

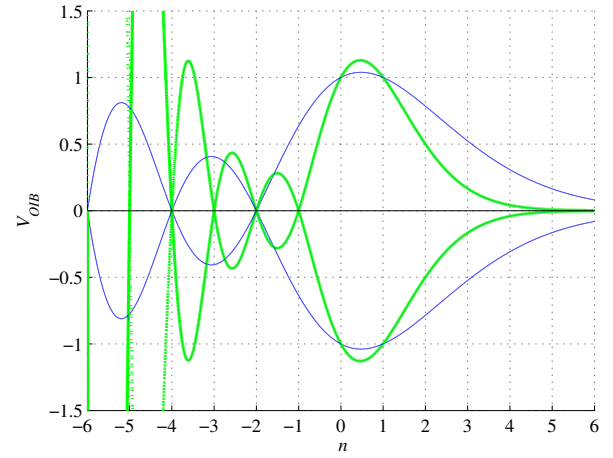
n	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$
S_B	-0.992	$-4/\pi \approx -1.273$	-0.867	0	1.039	2	2.723
S_{SIB}	$\frac{-i3^{11/4}}{2\sqrt{\pi}} \approx -5.787i$	0	$\frac{1+i}{\sqrt{\pi}2^{3/2}} \approx 0.199(1+i)$	0	$\frac{3^{3/4}}{2\sqrt{\pi}} \approx 0.643$	2	$\frac{3^{1/4}5^{5/4}}{2\sqrt{\pi}} \approx 2.776$
S_{SCB}	$\frac{-2^{3/2}3^{1/4}}{\sqrt{\pi}} \approx -2.1$	0	$\frac{i-1}{\sqrt{\pi}} \approx 0.564(i-1)$	0	$\frac{3^{3/4}}{\sqrt{2\pi}} \approx 0.909$	2	$\frac{3^{3/2}5^{5/4}}{2^{3/2}\sqrt{\pi}} \approx 3.4$
S_{OIB}	$\frac{3^{3/2}i}{2^{3/2}\sqrt{\pi}} \approx 1.036i$	0	$\frac{-i}{\sqrt{2\pi}} \approx -0.399i$	0	$\frac{\sqrt{2}}{\sqrt{\pi}} \approx 0.798$	2	$\frac{2^{3/2}\sqrt{3}}{\sqrt{\pi}} \approx 2.764$
S_{OCB}	$\frac{-(1+i)3^{1/4}}{2^{3/4}\sqrt{\pi}} \approx -0.442(1+i)$	0	$\frac{(i-1)}{2^{1/4}\sqrt{\pi}} \approx 0.474(i-1)$	0	$\frac{2^{3/4}}{\sqrt{\pi}} \approx 0.949$	2	$\frac{2^{5/4}3^{3/4}}{\sqrt{\pi}} \approx 3.059$
S_{CIB}	$\frac{(1+i)3^{9/4}}{2^{7/4}} \approx 3.521(1+i)$	2	$\frac{1-i}{2^{5/4}} \approx 0.42(1-i)$	0	$2^{-1/4} \approx 0.841$	2	$3^{3/4}2^{1/4} \approx 2.711$
S_{CCB}	-3	-2	-1	0	1	2	3

Figure 6. Graphs of $S(n)_{SCB}$ (red), $S(n)_B$ (blue), and $\tilde{S}(n)_{SCB}$ (dark red), for $n \in [-6, 6]$.

so that its volume (31) becomes

$$V(n)_{OIB} = \pm \frac{1}{\Gamma(n+1)} D^n. \quad (63)$$

The inscribed n -orthoplex volume (63) is real, vanishes for $n = -k$, and for $0 < n < 1$ it is larger than the volume

Figure 7. Graphs of $V(n)_{OIB}$ (green) and $V(n)_B$ (blue), for $n \in [-6, 6]$.

of the circumscribing n -ball. It does not have a reflection function.

Similarly, the surface (36) of the inscribed n -orthoplex with edge length A given by (62) becomes

$$S(n)_{OIB} = \pm \frac{2n^{3/2}}{\Gamma(n+1)} D^{n-1}. \quad (64)$$

For $n \notin \mathbb{Z} : n < 0$, inscribed n -orthoplex surface (64) is imaginary and oscillatory divergent with n approaching negative infinity, and for $n = -k$ it vanishes. It does not have a reflection function.

Volumes (63) and surfaces (64) of n -orthoplices inscribed in n -balls are shown in Figs. 7 and 8.

D. n -Orthoplices Circumscribed About n -Balls

The diameter D_{BIO} of an n -ball inscribed in an n -orthoplex (BIO) is known [23] to be

$$D_{BIO} = \pm \sqrt{\frac{2}{n}} A, \quad (65)$$

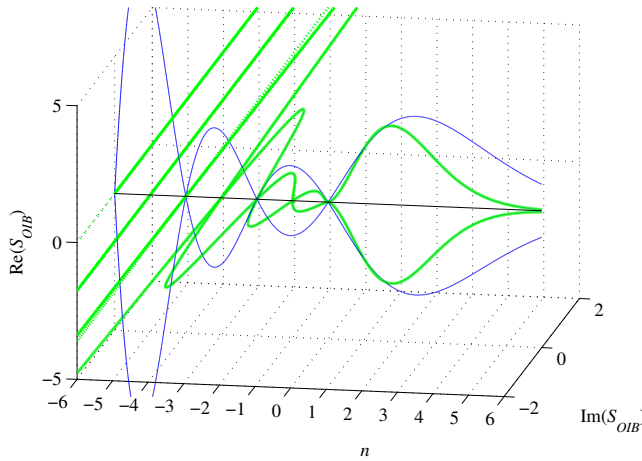


Figure 8. Graphs of $S(n)_{OIB}$ (green) and $S(n)_B$ (blue), for $n \in [-6, 6]$.

where A is the edge length. Hence, the edge length A_{OCB} of an n -orthoplex circumscribed about an n -ball (OCB) with diameter D is

$$A_{OCB} = \pm \sqrt{\frac{n}{2}} D, \quad (66)$$

so that its volume (31) becomes

$$V(n)_{OCB} = \pm \frac{n^{n/2}}{\Gamma(n+1)} D^n, \quad (67)$$

as shown in Fig. 9.

Circumscribed n -orthoplex volume (67) is complex for $n < 0$. It oscillates and is initially convergent to achieve a modulus minimum of about 0.1181 at $n \approx -3.4976$ (numerically computed) and then becomes divergent with n approaching negative infinity. For $n = -k$, it vanishes. For $0 < n < 1$, it is smaller than the volume of the inscribed n -ball. For $n = -(2k+1)/2$, $k \in \mathbb{N}_0$ the real part of the volume (67) equals the imaginary part up to a modulus, achieving a local minimum at $n = -3.5$ and amounts

$$\begin{aligned} V\left(\frac{-(2k+1)}{2}\right)_{OCB} &= \\ &= \pm \frac{(1+i)^k 2^{(2k-1)/4}}{\Gamma\left(\frac{1-2k}{2}\right) (2k+1)^{(2k+1)/4}} \approx \\ &\approx \pm \{0.4744, -0.1472, 0.0952, -0.0835, \\ &0.0888, -0.1084, \dots\} (1+i)^k. \end{aligned} \quad (68)$$

The reflection function (38) of the volume (67) is

$$\tilde{V}(n)_{OCB} = \pm i^n \frac{(-n)^{n/2}}{\Gamma(n+1)} D^n, \quad (69)$$

and it equals the volume (67) for $n = 2k$, $k \in \mathbb{N}_0$.

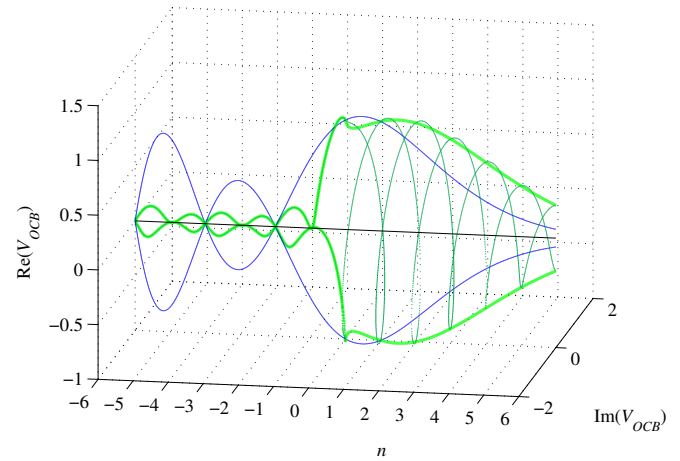


Figure 9. Graphs of $V(n)_{OCB}$ (green), $V(n)_B$ (blue), and $\tilde{V}(n)_{OCB}$ (dark green), for $n \in [-6, 6]$.

Similarly, the surface (36) of the circumscribed n -orthoplex with edge length A given by (66) becomes

$$S(n)_{OCB} = \pm \frac{2n^{(2+n)/2}}{\Gamma(n+1)} D^{n-1}, \quad (70)$$

as shown in Fig. 10.

Circumscribed n -orthoplex surface (70) is complex for $n < 0$. It oscillates and is initially convergent to achieve a modulus minimum of about 0.6244 at $n \approx -1.5$ (numerical) and then becomes divergent with n approaching negative infinity. For $n = -k$, it vanishes. For $0 < n < 1$, it is smaller than the surface of the inscribed n -ball. Furthermore, its real part equals the imaginary part up to a modulus for $n = -(2k+1)/2$, $k \in \mathbb{N}_0$. It achieves a local

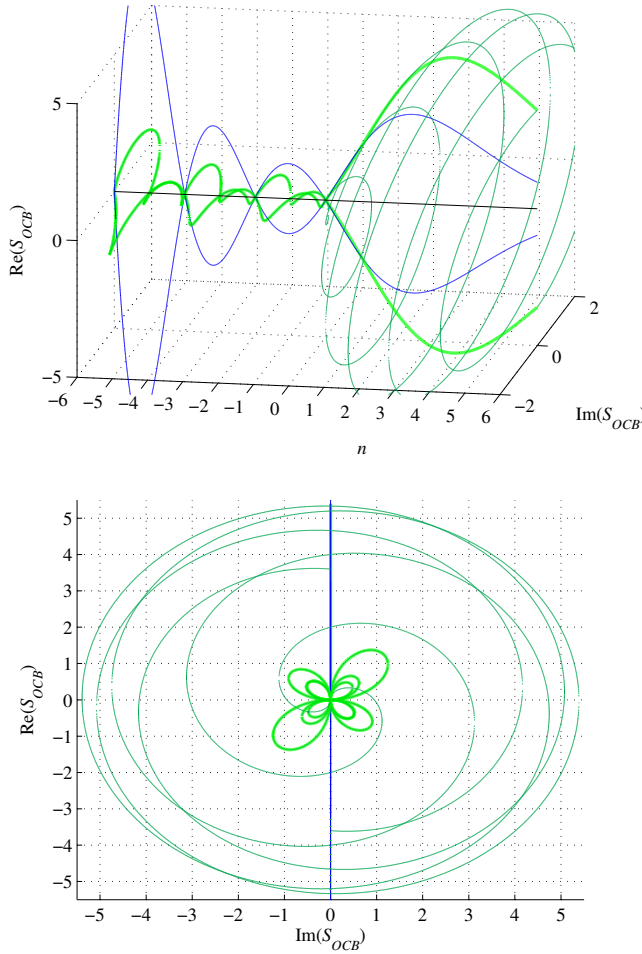


Figure 10. Graphs of $S(n)_{OCB}$ (green), $S(n)_B$ (blue), and $\tilde{S}(n)_{OCB}$ (dark green), for $n \in [-6, 6]$.

minimum at $n = -1.5$ and amounts

$$\begin{aligned} S\left(\frac{-(2k+1)}{2}\right)_{OCB} &= \\ &= \pm \frac{-(1+i)^k 2^{(2k-1)/4} (2k+1)^{(3-2k)/4}}{\Gamma\left(\frac{1-2k}{2}\right)} \approx \\ &\approx \pm \{-0.4744, 0.4415, -0.4759, 0.5846, \\ &\quad -0.7989, \dots\} (1+i)^k. \end{aligned} \quad (71)$$

The reflection function (38) of the surface (70) is

$$\tilde{S}(n)_{OCB} = \pm \frac{i^{2+n} 2(-n)^{(2+n)/2} D^{n-1}}{\Gamma(n+1)}, \quad (72)$$

and it equals (70) for $n = 2k$, $k \in \mathbb{N}_0$.

Volumes (67), (69) and surfaces (70), (72) of n -orthoplices inscribed in n -balls are shown in Figs. 9 and 10.

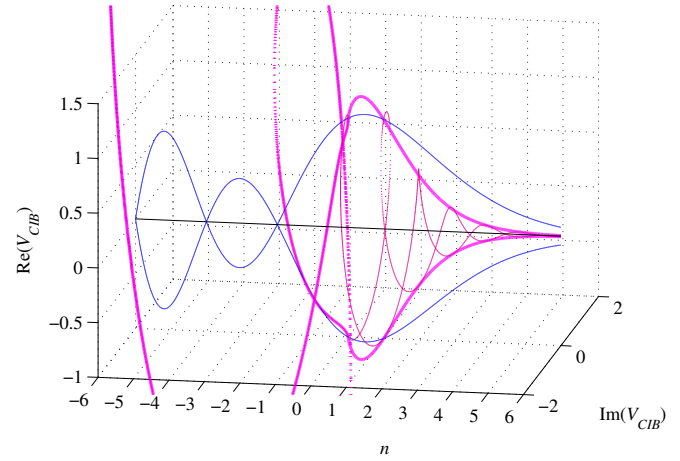


Figure 11. Graphs of $V(n)_{CIB}$ (pink), $V(n)_B$ (blue), and $\tilde{V}(n)_{CIB}$ (dark pink), for $n \in [-6, 6]$.

E. n -Cubes Inscribed in and Circumscribed About n -Balls

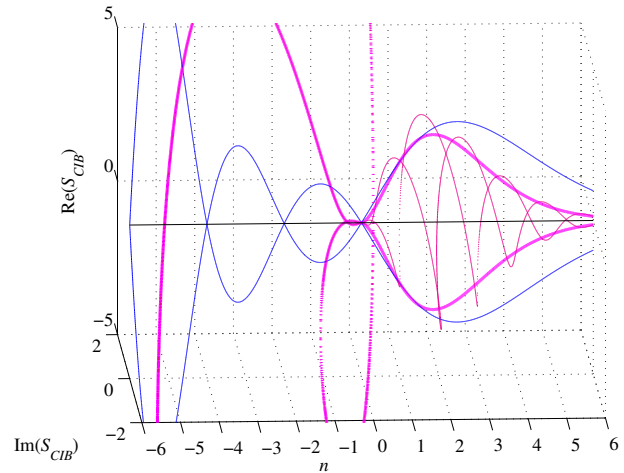


Figure 12. Graphs of $S(n)_{CIB}$ (pink), $S(n)_B$ (blue), and $\tilde{S}(n)_{CIB}$ (dark pink), for $n \in [-6, 6]$.

The edge length ACC_B of an n -cube circumscribed about an n -ball (CCB) corresponds to the diameter D of this n -ball. Thus, the volume of this cube is simply

$$V(n)_{CCB} = \pm D^n, \quad (73)$$

and the surface is

$$S(n)_{CCB} = \pm 2nD^{n-1}. \quad (74)$$

However, the edge length $ACIB$ of an n -cube inscribed in an n -ball (CIB) of diameter D is

$$A(n)_{CIB} = \pm Dn^{-1/2}, \quad (75)$$

which is singular for $n = 0$ and complex for $n < 0$, rendering [18] the volume

$$V(n)_{CIB} = \pm n^{-n/2} D^n, \quad (76)$$

and the surface

$$S_{CIB} = \pm 2n^{(3-n)/2} D^{n-1}, \quad (77)$$

of an n -cube inscribed in an n -ball.

The reflection functions (38) of the volume (76) and surface (77) are [18]

$$\tilde{V}(n)_{CIB} = \pm i^{-n} (-n)^{-n/2} D^n, \quad (78)$$

$$\tilde{S}(n)_{CIB} = \pm 2i^{3-n} (-n)^{(3-n)/2} D^{n-1}. \quad (79)$$

The volumes (76) and (78) are equal [18] for $n = 2k$, $k \in \mathbb{N}_0$, and the surfaces (77) and (79) are equal for $n = 2k - 1$, $k \in \mathbb{N}$, as shown in Figs. 11 and 12.

The volume (76) and the surface (77) are complex if $n < 0$ and divergent, with n approaching negative infinity. For $0 < n < 1$, the volume (76) is larger than the volume of the circumscribing n -ball.

V. METRIC-INDEPENDENT RELATIONS

The following metric-independent relations hold, for $n \in \mathbb{C} : n \neq 0$, between volumes of n -balls (22)

$$V(n)_B V(-n)_B = \pm \frac{2 \sin(\pi n/2)}{\pi n}, \quad (80)$$

n -simplices (25)

$$V(n)_S V(-n)_S = \pm \frac{(1+n)^{1/2} (1-n)^{1/2} \sin(\pi n)}{\pi n}, \quad (81)$$

n -orthoplices (31) and n -orthoplices inscribed in n -balls (63)

$$V(n)_O V(-n)_O = V(n)_{OIB} V(-n)_{OIB} = \pm \frac{\sin(\pi n)}{\pi n}, \quad (82)$$

n -simplices inscribed in n -balls (49)

$$V(n)_{SIB} V(-n)_{SIB} = \pm \frac{(1+n)^{(1+n)/2} (1-n)^{(1-n)/2} (n)^{(-n)/2} (-n)^{(n)/2} \sin(\pi n)}{\pi n}, \quad (83)$$

n -simplices circumscribed about n -balls (53)

$$V(n)_{SCB} V(-n)_{SCB} = \pm \frac{(1+n)^{(1+n)/2} (1-n)^{(1-n)/2} (n)^{(n)/2} (-n)^{(-n)/2} \sin(\pi n)}{\pi n}, \quad (84)$$

and n -orthoplices circumscribed about n -balls (67)

$$V(n)_{OCB} V(-n)_{OCB} = \pm \frac{(n)^{(n)/2} (-n)^{(-n)/2} \sin(\pi n)}{\pi n}, \quad (85)$$

where we used Euler's reflection formula, $\Gamma(\tilde{n})\Gamma(1-\tilde{n}) = \pi/\sin \pi\tilde{n}$, with $\tilde{n} := n+1$, and $\Gamma(n) = (n-1)\Gamma(n-1)$

$$\begin{aligned} \Gamma(n+1)\Gamma(-n+1) &= \Gamma(\tilde{n})\Gamma(2-\tilde{n}) = \Gamma(\tilde{n})\Gamma(1-\tilde{n})(1-\tilde{n}) = \\ &= \frac{\pi}{\sin(\pi\tilde{n})}(1-\tilde{n}) = -\frac{\pi n}{\sin[\pi(n+1)]} = \frac{\pi n}{\sin(\pi n)}, \end{aligned} \quad (86)$$

as well as (for $n \in \mathbb{C} : n \neq 1$), between surfaces of n -balls (24)

$$S(n)_B S(2-n)_B = \pm 4 \sin(\pi n/2), \quad (87)$$

n -simplices (30),

$$S(n)_S S(2-n)_S = \pm \frac{n^{1/2} (n+1) (2-n)^{1/2} (3-n) \sin(\pi n)}{\pi (1-n)}, \quad (88)$$

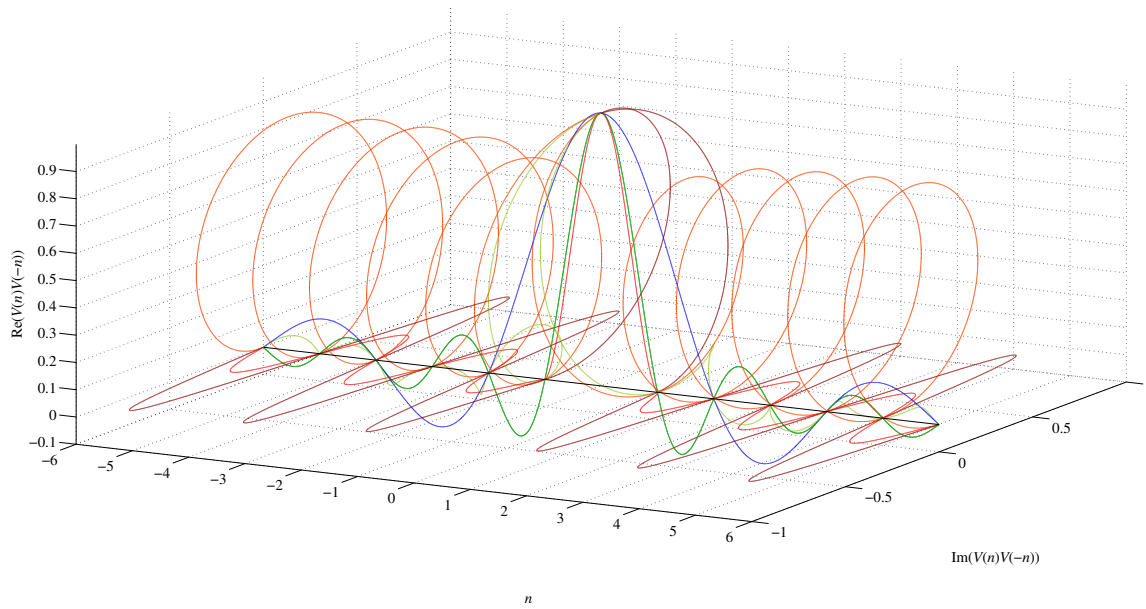


Figure 13. Volume metric and gamma function independent relations. $V(n)_B V(-n)_B$ (blue), $V(n)_S V(-n)_S$ (red), $V(n)_{SIB} V(-n)_{SIB}$ (dark red), $V(n)_{SCB} V(-n)_{SCB}$ (orange red), $V(n)_O V(-n)_O = V(n)_{OIB} V(-n)_{OIB}$ (lime green), $V(n)_{OCB} V(-n)_{OCB}$ (yellow green), for $n \in \mathbb{R} : n \in [-6, 6]$.

n -orthoplices (36)

$$S(n)_O S(2-n)_O = \pm \frac{4n^{1/2} (2-n)^{1/2} \sin(\pi n)}{\pi (1-n)}, \quad (89)$$

n -simplices inscribed in n -balls (50)

$$S(n)_{SIB} S(2-n)_{SIB} = \pm \frac{n^{(2-n)/2} (n+1)^{(n+1)/2} (2-n)^{n/2} (3-n)^{(3-n)/2} \sin(\pi n)}{\pi (1-n)}, \quad (90)$$

n -simplices circumscribed about n -balls (57)

$$S(n)_{SCB} S(2-n)_{SCB} = \pm \frac{n^{n/2} (n+1)^{(n+1)/2} (2-n)^{(2-n)/2} (3-n)^{(3-n)/2} \sin(\pi n)}{\pi (1-n)}, \quad (91)$$

n -orthoplices inscribed in n -balls (64)

$$S(n)_{OIB} S(2-n)_{OIB} = \pm \frac{4n^{1/2} (2-n)^{1/2} \sin(\pi n)}{\pi (1-n)}, \quad (92)$$

n -orthoplices circumscribed about n -balls (70)

$$S(n)_{OCB} S(2-n)_{OCB} = \pm \frac{4n^{n/2} (2-n)^{(2-n)/2} \sin(\pi n)}{\pi (1-n)}, \quad (93)$$

and n -cubes inscribed in n -balls (77) ($\forall n \in \mathbb{C}$)

$$S(n)_{CIB} S(2-n)_{CIB} = \pm 4n^{(3-n)/2} (2-n)^{(1+n)/2}, \quad (94)$$

where we again used $\tilde{n} = n + 1$ and Euler's reflection formula

$$\begin{aligned} \Gamma(n+1) \Gamma(3-n) &= \Gamma(\tilde{n}) \Gamma(1-\tilde{n}) (1-\tilde{n}) (2-\tilde{n}) (3-\tilde{n}) = \frac{\pi (1-\tilde{n}) (1-\tilde{n}) (2-\tilde{n}) (3-\tilde{n})}{\sin(\pi \tilde{n})} \\ &= -\frac{\pi n (1-n) (2-n)}{\sin[\pi (n+1)]} = \frac{\pi n (1-n) (2-n)}{\sin(\pi n)}. \end{aligned} \quad (95)$$

These relations are independent of the distance value D and its particular form, whether negative ($D < 0$) or complex ($D = a + bi$, $a, b \in \mathbb{R}$), including purely imaginary ($a = 0$). Furthermore, the relations (80)-(85) and (87)-(93) are independent of the gamma function. Relations (80)-(85) are shown in Fig. 13. The imaginary part of the surface relations (88)-(94) vanishes for $0 < n < 2$.

Also, for $n \in \mathbb{C} : n \neq 0$, the following both metric and gamma function independent relations can be obtained between volumes (73), (76) and surfaces (74), (77) of circumscribed and inscribed n -cubes

$$V(n)_{CCB} = \pm n^{n/2} V(n)_{CIB}, \quad S(n)_{CCB} = \pm n^{(n-1)/2} S(n)_{CIB}, \quad (96)$$

between volumes (53), (49) and surfaces (57), (50) of circumscribed and inscribed n -simplices

$$V(n)_{SCB} = \pm n^n V(n)_{SIB}, \quad S(n)_{SCB} = \pm n^{n-1} S(n)_{SIB}, \quad (97)$$

and between volumes (67), (63) and surfaces (70), (64) of circumscribed and inscribed n -orthoplices

$$V(n)_{OCB} = \pm n^{n/2} V(n)_{OIB}, \quad S(n)_{OCB} = \pm n^{(n-1)/2} S(n)_{OIB}. \quad (98)$$

Notably, the ratio of the volume of n -cube circumscribed about n -ball $V(n)_{CCB}$ to the volume of n -cube inscribed in n -ball $V(n)_{CIB}$ (96) is the same as the ratio of the volume of n -orthoplex circumscribed about n -ball $V(n)_{OCB}$ to the volume of n -orthoplex inscribed in n -ball $V(n)_{OIB}$ (98), and the same holds for the ratio of their surfaces, $S(n)_{CCB}/S(n)_{CIB} = S(n)_{OCB}/S(n)_{OIB}$. This is unsurprising: as n -cube is dual to n -orthoplex, these ratios remain invariant.

VI. VOLUME-SURFACE SYMMETRIES

Furthermore, the following particular symmetries between $n = -1/2$ and $n = 1/2$ hold for (22), (24); (25), (30); (31), (36); (49), (50); (53), (57); (63), (64); (67), (70); (76), (77); and (73), (74)

$$V(-1/2)_B = \mp S(-1/2)_B D, \quad V(1/2)_B = \pm S(1/2)_B D, \quad (99)$$

$$V(-1/2)_S = \pm S(-1/2)_S A i 2\sqrt{2}, \quad V(1/2)_S = \pm S(1/2)_S A 2\sqrt{\frac{2}{3}}, \quad (100)$$

$$V(-1/2)_O = \pm 2i S(-1/2)_O A, \quad V(1/2)_O = \pm 2S(1/2)_O A, \quad (101)$$

$$V(-1/2)_{SIB} = \pm 2S(-1/2)_{SIB} D, \quad V(1/2)_{SIB} = \pm 2S(1/2)_{SIB} D, \quad (102)$$

$$V(-1/2)_{SCB} = \mp S(-1/2)_{SCB} D, \quad V(1/2)_{SCB} = \pm S(1/2)_{SCB} D, \quad (103)$$

$$V(-1/2)_{OIB} = \pm i\sqrt{2} S(-1/2)_{OIB} D, \quad V(1/2)_{OIB} = \pm \sqrt{2} S(1/2)_{OIB} D, \quad (104)$$

$$V(-1/2)_{OCB} = \mp S(-1/2)_{OCB} D, \quad V(1/2)_{OCB} = \pm S(1/2)_{OCB} D, \quad (105)$$

$$V(-1/2)_{CIB} = \pm \sqrt{2} S(-1/2)_{CIB}^* D, \quad V(1/2)_{CIB} = \pm \sqrt{2} S(1/2)_{CIB} D, \quad (106)$$

$$V(-1/2)_{CCB} = \mp S(-1/2)_{CCB} D, \quad V(1/2)_{CCB} = \pm S(1/2)_{CCB} D, \quad (107)$$

where "*" denotes a complex conjugate.

Furthermore, if $A = D$

$$S(3)_{OIB} = \pm S(3)_S, \quad S(2)_{OIB} = \pm S(2)_{CIB}. \quad (108)$$

VII. DISCUSSION

The volumes and surfaces of n -balls, n -simplices, and n -orthoplices have been defined in any complex dimen-

sion. As these geometric objects occur, along with n -

cubes, in all natural dimensions [19], this result makes them omnidimensional, that is, present in any complex dimension.

Applications of these formulas to the omnidimensional polytopes inscribed in and circumscribed about n -balls revealed previously unknown properties of these geometric objects. In particular, for $0 < n < 1$, the omnidimensional polytopes' volumes are larger than those of circumscribing n -balls, while their volumes and surfaces are smaller than those of inscribed n -balls.

Reflection functions for volumes and surfaces of these polytopes inscribed in and circumscribed about n -balls are proposed.

Symmetries of products (80)-(85), (87)-(94) and quotients (96)-(98) of volumes of these circumscribed and inscribed omnidimensional polytopes and n -balls in complex dimensions n and $-n$ and of surfaces in complex dimensions n and $2-n$ are shown to be independent of the metric factor and the gamma function.

Specific symmetries also hold between volumes and surfaces in dimensions $n = -1/2$ and $n = 1/2$.

The results of this study could be applied in linguistic statistics, where the dimension in the distribution for frequency dictionaries is chosen to be negative [3], in fog computing, where n -simplex is related to a full mesh pattern, n -orthoplex is linked to a quasi-full mesh structure, and n -cube is referred to as a certain type of partial mesh layout [24]. Further possible applications include molecular physics and crystallography. Perhaps the results of this study are also related to the 2-dimensional quantum hall effect.

One can also investigate the properties of the examined geometric objects extending these concepts to quaternions. This can help us discover many interesting mathematical properties and physical phenomena. One of the challenges certainly involves characterizing all types of symmetries introduced by Definition 1.

Data Availability Statement

The public repository for the code written in Matlab computational environment is given under the link <https://github.com/szluk/Omnidimensional-Polytopes>

Authors' Contribution

AT: Proof that $n\sqrt{n}/\sqrt{n^3} = \pm 1 \quad \forall n \in \mathbb{C}$; the proposition of the affine exponential function $(a_j + b_j n)^{(c_j + d_j n)/2}$ (37); remarks on the square root properties in the complex domain and the general bivalence that the latter brings (which renders volumes and surfaces bivalued) in the context of classical and quantum physics; numerous clarity and formal corrections and improvements; hinting at possible applications of the results of this study to quaternions.

SL: Remaining part of the study.

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Appendix A: Bivalence of the complex square root

It is known that complex number $n = |n|e^{i\theta} = |n|(\cos \theta + i \sin \theta) \neq 0$ has two square roots

$$\sqrt{n} = \sqrt{|n|} \left(\cos \frac{\theta + 2k\pi}{2} + i \sin \frac{\theta + 2k\pi}{2} \right), \quad (A1)$$

for $k = 0, 1$, that is

$$\begin{aligned} \sqrt{n} &= \sqrt{|n|} \left\{ \begin{array}{l} \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \\ \cos \frac{\theta+2\pi}{2} + i \sin \frac{\theta+2\pi}{2} \end{array} \right\} = \\ &= \sqrt{|n|} \left\{ \begin{array}{l} \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \end{array} \right\} = \\ &= \sqrt{|n|} \left\{ \begin{array}{l} \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \\ -(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}) \end{array} \right\} = \\ &= \pm \sqrt{|n|} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) = \pm \sqrt{|n|} e^{i\frac{\theta}{2}}. \end{aligned} \quad (A2)$$

Thus,

$$\frac{n\sqrt{n}}{\sqrt{n^3}} = \frac{|n|e^{i\theta} (\pm \sqrt{|n|} e^{i\frac{\theta}{2}})}{\pm \sqrt{|n|^3} e^{i\frac{3\theta}{2}}} = \pm 1. \quad (A3)$$

The same holds for $(n-1)\sqrt{n-1}/\sqrt{(n-1)^3} = \pm 1$, setting $\tilde{n} := n-1$.

Appendix B: Volumes and Surfaces in terms of the real and imaginary parts of the complex dimension

The volumes and surfaces of the omnidimensional objects can be explicitly expressed in terms of the real and imaginary part of the Complex Dimension $n = a+ib \in \mathbb{C}$.

In the case of n -balls, for example

$$\begin{aligned} \pi^{n/2} &= \pi^{(a+ib)/2} = \\ &= \pi^{a/2} \left[\cos \left(\frac{b}{2} \ln \pi \right) + i \sin \left(\frac{b}{2} \ln \pi \right) \right], \end{aligned} \quad (B1)$$

$$R^n = R^{a+ib} = R^a [\cos(b \ln R) + i \sin(b \ln R)], \quad (B2)$$

and the volume (22) and surface (24) become

$$\begin{aligned} V(n)_B &= \frac{\pi^{a/2} R^a}{\Gamma\left(\frac{n}{2} + 1\right)} \\ &\quad \{ \cos [b \ln (R\sqrt{\pi})] + i \sin [b \ln (R\sqrt{\pi})] \}, \end{aligned} \quad (B3)$$

$$\begin{aligned} S(n)_B &= \frac{n\pi^{a/2} R^{a-1}}{\Gamma\left(\frac{n}{2} + 1\right)} \\ &\quad \{ \cos [b \ln (R\sqrt{\pi})] + i \sin [b \ln (R\sqrt{\pi})] \}, \end{aligned} \quad (B4)$$

where we used $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and similarly $\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$.

In particular for $n = 3 + bi$, $b \in \mathbb{R}$ (spacetime dimensionality) the volume (B3) and the surface (B4) become

$$V(n)_B = \frac{\pi^{3/2} R^3}{\Gamma\left(\frac{5+ib}{2}\right)} \left\{ \cos[b \ln(R\sqrt{\pi})] + i \sin[b \ln(R\sqrt{\pi})] \right\}, \quad (\text{B5})$$

$$S(n)_B = \frac{n\pi^{a/2} R^{a-1}}{\Gamma\left(\frac{5+ib}{2}\right)} \left\{ \cos[b \ln(R\sqrt{\pi})] + i \sin[b \ln(R\sqrt{\pi})] \right\}, \quad (\text{B6})$$

which forms reduce to familiar $V(3)_B = 4\pi R^3/3$ and $S(3)_B = 4\pi R^2$ for $n = 3 + 0i$ (i.e., at the present moment of perception). The trigonometric member also vanishes in (B5) and (B6) for $R = 1/\sqrt{\pi}$. Notably, $R = \ell_P/\sqrt{\pi}$, where ℓ_P is the Planck length, is the radius of a 4-bit black hole [13], and one unit of a black hole entropy [25].

In the case of n -cubes, for example, the volume (8) is equal to

$$V(n)_C = A^a \left\{ \cos[b \ln(A)] + i \sin[b \ln(A)] \right\}, \quad (\text{B7})$$

and it reduces to $V(3)_C = A^a$ for $b = 0$. Also, $V(n)_C = 1$ $\forall n \in \mathbb{C}$ if $A = 1$.

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