THE FRACTIONAL HILBERT TRANSFORM OF GENERALISED FUNCTIONS

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ABSTRACT. The fractional Hilbert transform, a generalization of the Hilbert transform, has been extensively studied in the literature because of its extensive use in optics, engineering, and signal processing. In the present work, we aimed to expand the fractional Hilbert transform to a space of generalized functions known as Boehmians. We introduce a new fractional convolution operator for the fractional Hilbert transform to prove a convolution theorem similar to the classical Hilbert transform and also to extend the fractional Hilbert transform to Boehmians. We also construct a suitable Boehmian space on which the fractional Hilbert transform exists. Further, we investigate convergence of the fractional Hilbert transform for the class of Boehmians and discuss the continuity of the extended fractional Hilbert transform.

1. INTRODUCTION

The space of Boehmians is a class of generalized functions that includes all regular operators and generalized functions or distributions as well as other objects. The theory of Boehmians with two convergences, introduced by Mikusinski and Mikusinski (MIKUSINSKI, 1983), is a broadening of the concept of Boehme's regular operators (Boehme, 1973). In contrast to the theory of distributions in which generalized functions are treated as members of the dual space of any space of testing-function, the space of Boehmians treats distributions more as algebraic objects. Several integral transforms have been studied for various spaces of Boehmians and their properties are investigated (Al-Omari and Kılıçman, 2012; 2013; Al-Omari, 2015; Al-Omari and Agarwal, 2016; Karunakaran and Kalpakam, 2000; Karunakaran and Roopkumar, 2002; Loonker and Banerji, 2013; Roopkumar, 2007; 2009; 2020; I. Zayed, 1998). Currently a large amount of literature is available on the extension of classical integral transforms to Boehmians. Karunakaran and Roopkumar introduced the Hilbert transform as a continuous linear mapping defined on some space of Boehmians into another space of Boehmians (Karunakaran and Kalpakam, 2000). They also studied Hilbert transform for the space of ultradistributions (Karunakaran and Roopkumar, 2002). The pioneer work of Zayed (I. Zayed, 1998), Al-Omari and Agarwal (Al-Omari and Agarwal, 2016)] introduced extension of fractional integral transform to Boehmians by extending fractional Fourier transform and fractional Sumudu transform to space of integrable Boehmians respectively. In the recent years the extension of fractional integral transforms to the space of Boehmians has been an active area of research. A number of well-known fractional integral transforms have been extended to the space of Boehmians,

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but an extension of fractional Hilbert transform (FHT) has not yet been reported. So the goal of this paper is to extend the FHT on some space of Boehmians. Different definitions of FHT exist in literature (Abdullah et al., 2022; Cusmariu, 2002; Lohmann et al., 1996; Zayed, 1998), but the generalization of classical Hilbert transform that might rightly be said the fractionalization of Hilbert transform is given by Zayed. The fractional Hilbert transform of a function $f(x)$, denoted by $H_\alpha[f(x)]$, is defined as (Zayed, 1998)

$$H_\alpha[f(x)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\frac{x^2-t^2}{2}\cot\alpha}}{x-t} f(t)dt \quad \text{for } \alpha \neq 0,\ \pi/2,\ \pi, \tag{1.1}$$

where the integral is taken in the sense of Cauchy principal value. The special case $\alpha = \pi/2$ reduces FHT into the ordinary Hilbert transform. Indeed, the FHT has many uses in optics, signal processing, and image processing (Davis et al., 1998; Deng et al., 2019; Lohmann et al., 1997; Sharma, 2019). To achieve our goals, we need to extend the existing theory on FHT in terms of Boehmians. An extension of FHT to some space of Boehmians may have applications in engineering and other sciences as it may be applicable in converting functions with discontinuities into smooth functions that consequently lead to the description of various physical occurrences like point charges (Al-Omari, 2018).
The organization of the present paper is as follows. Section 1 covers the introduction. Section 2 covers the important definitions and theorems, we also discuss abstract construction of Boehmians to make the paper self-contained. Section 3 covers results and discussion which comprises a new convolution operator and new convolution theorem for FHT and prove auxiliary results required for the construction of two Boehmian spaces. Finally, we will extend the FHT to some space of Boehmians.

2. PRELIMINARIES

Let $\mathbb{R}$ be the set of all real numbers, $\mathcal{L}^1(\mathbb{R}) = \mathcal{L}^1$ be the collection of complex valued measurable functions f defined on $\mathbb{R}$ for which

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|dx < \infty,$$

and $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R})$ be the set of all infinitely differentiable functions defined on $\mathbb{R}$ such that functions and their derivatives converge uniformly on compact sets in $\mathbb{R}$.

Theorem 2.1. (Rudin, 1987, Theorem 9.5) *For any function f on $\mathbb{R}$ and for all $t \in \mathbb{R}$, let f_t is defined by*

$$f_t(x) = f(x-t).$$

If $p \geq 1$ and $f \in \mathcal{L}^p$, then the mapping $t \to f_t$ is a uniformly continuous mapping from $\mathbb{R}$ into $\mathcal{L}^p(\mathbb{R})$.

Definition 2.2. *Let f and g be any two functions on $\mathbb{R}$ their convolution, denoted by $f * g$, is defined as*

$$f * g = \int_{-\infty}^{\infty} f(t)g(x-t)dt. \tag{2.1}$$

The Hilbert transform of convolution operation $*$ is given as follows:

Theorem 2.3. *If $f,\ g \in \mathcal{L}^1(\mathbb{R})$ with Hilbert transforms Hf, Hg respectively, so that $Hf, Hg \in \mathcal{L}^1(\mathbb{R})$ then*

$$H[f * g] = Hf * g = f * Hg.$$

The FHT may not act as agreeably with the classical convolution operator as the classical Hilbert transform (Theorem 2.3).

2.1. **Boehmian Space.** The members of Boehmian spaces are called Boehmians, which are equivalence classes of "quotients of sequences". These equivalence classes are formulated from an integral domain of continuous functions. The integral domain operations for Boehmians are addition and convolution. This convolution operation may differ from the standard convolution operation given in Definition 2.1.

We now recall a brief introduction to Boehmians.
Let G be a complex linear space, $(H, .)$ is a commutative semigroup, and let $\otimes : G \times H \to G$ so that the conditions given below hold:

- $(f \otimes \phi) \otimes \psi = f \otimes (\phi.\psi), \quad \forall f \in G,\ \forall \phi, \psi \in H;$
- $(f + g) \otimes \phi = f \otimes \phi + g \otimes \phi, \quad \forall f, g \in G,\ \forall \phi \in H;$
- $\lambda(f \otimes \phi) = (\lambda f \otimes \phi)\ \forall f \in G, \quad \forall \phi \in H, \lambda \in \mathbb{C};$
- If $f_n \to f$ as $n \to \infty$ and $\phi \in H$ then $f_n \otimes \phi \to f \otimes \phi$ as $n \to \infty$.

Also let Δ be a collection of sequences on H in such a way that

- If $\{\phi_n\},\ \{\psi_n\} \in \Delta$ then $\{\phi_n.\psi_n\} \in \Delta$;
- If $f_n \to f$ as $n \to \infty$ and $\{\phi_n\} \in H$ then $f_n \otimes \phi_n \to f$ as $n \to \infty$.

A pair of sequences $\{f_n, \phi_n\}$ with $f_n \in G$ for all $n \in \mathbb{N}$ and $\{\phi_n\} \in \Delta$ is known as quotient of sequences, denoted by $\frac{f_n}{\phi_n}$, if

$$f_n \otimes \phi_m = f_m \otimes \phi_n \quad \forall m, n \in \mathbb{N}.$$

Two quotients of sequences $\frac{f_n}{\phi_n}$ and $\frac{g_n}{\psi_n}$ are equivalent $(\sim)$ if for every $n \in \mathbb{N}$

$$f_n \otimes \psi_n = g_n \otimes \phi_n.$$

The equivalence class of $\frac{f_n}{\phi_n}$ induced by $\sim$ will be denoted by $\left[\frac{f_n}{\phi_n}\right]$. Every equivalence class is called a Boehmian. The space of all Boehmians will be denoted by $\mathcal{B} = \mathcal{B}(G, H, \otimes, \Delta)$. $\mathcal{B}$ is a vector space under the operations of addition and scalar multiplication defined as follows:

- $\lambda \left[\frac{f_n}{\phi_n}\right] = \left[\frac{\lambda f_n}{\phi_n}\right];$
- $\left[\frac{f_n}{\phi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n \otimes \phi_n + g_n \otimes \psi_n}{\phi_n \otimes \psi_n}\right].$

If we define an isomorphism $f \to \left[\frac{f \otimes \phi_n}{\phi_n}\right]$ then G is a subspace of $\mathcal{B}$. Therefore every element of G can be expressed uniquely as a Boehmian.

3. RESULTS AND DISCUSSION

In this section we define a new convolution operation for FHT which yield generalized result for Theorem 2.3. Moreover, to extend the FHT to the class of Boehmians, we define two classes of Boehmians. Also, two convergences of FHT are proved on $\mathcal{C}^\infty$. Finally, an extension of FHT on Boehmians is introduced.

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3.1. **Convolution Structure for Fractional Hilbert Transform.** It is clear from the theory of convolution operation that given any integral transform we can associate a convolution operation to it (Zayed, 2019). So, we introduce a new fractional convolution operator that will help us to extend FHT to the space of Boehmians.

Definition 3.1. *Let $f, g \in \mathcal{L}^1(\mathbb{R})$, we define a fractional convolution $(f *_\alpha g)$ as*

$$(f *_\alpha g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)e^{-it(x-t)\cot\alpha}dt. \tag{3.1}$$

Lemma 3.2. *Let $f, g \in \mathcal{L}^1$, then $(f *_\alpha g)$ is also in $\mathcal{L}^1$.*

Proof: To prove that $f *_\alpha g \in \mathcal{L}^1$, we consider its $\mathcal{L}^1$ norm.

$$\begin{aligned}\|f *_\alpha g\|_1 &= \int_{-\infty}^{\infty} |f *_\alpha g|dx \\ &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-t)||g(t)|dtdx.\end{aligned}$$

By making use of Fubini's theorem, we have

$$\|f *_\alpha g\|_1 \leq \int_{-\infty}^{\infty} |f(x-t)|dx \int_{-\infty}^{\infty} |g(t)|dt.$$

Since $\mathcal{L}^1$ norm is translation invariance, so $\int_{-\infty}^{\infty} |f(x-t)|dx = \|f_t\|_1 = \|f\|_1$, therefore

$$\|f *_\alpha g\|_1 \leq \|f\|_1\|g\|_1.$$

Since $f, g \in \mathcal{L}^1$,

$$\|f *_\alpha g\|_1 \leq \|f\|_1\|g\|_1 < \infty$$

which proves that $f *_\alpha g \in \mathcal{L}^1$.
To extend the FHT to the case of Boehmians, the essential step is to prove the convolution theorem and then suitable Boehmian space(s) can be constructed by proving the supplementary results. Now we state and prove convolution theorem for FHT.

Theorem 3.3. *(Convolution Theorem) Assume that $f, g \in \mathcal{L}^1$ then*

$$H_\alpha[f *_\alpha g] = H_\alpha[f] *_\alpha g = f *_\alpha H_\alpha[g]. \tag{3.2}$$

Also $(f *_\alpha g) = -(H_\alpha[f] *_\alpha H_\alpha[g])$.

Proof:

$$\begin{aligned}H_\alpha[(f *_\alpha g)(x)] &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\frac{x^2-t^2}{2}\cot\alpha}}{x-t}(f *_\alpha g)(t)\,dt \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\frac{x^2-t^2}{2}\cot\alpha}}{x-t}\int_{-\infty}^{\infty} f(t-y)g(y)e^{-iy(t-y)\cot\alpha}dydt.\end{aligned}$$

By changing the variables $t - y = \nu$ the above equation can be simplified to

$$\begin{aligned} H_\alpha[(f *_\alpha g)(x)] &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i\frac{x^2-2xy+y^2-\nu^2}{2}\cot\alpha}}{(x-y)-\nu} f(\nu)g(y)e^{-i(yx-y^2)\cot\alpha}d\nu dy \\ &= \int_{-\infty}^{\infty} H_\alpha[f(x-y)]g(y)e^{-iy(x-y)\cot\alpha}dy \\ &= \left(H_\alpha[f] *_\alpha g\right)(x). \end{aligned}$$

Similarly,

$$H_\alpha[(f *_\alpha g)(x)] = H_\alpha[(g *_\alpha f)(x)] = \left(H_\alpha[g] *_\alpha f\right)(x) = \left(f *_\alpha H_\alpha[g]\right)(x). \tag{3.3}$$

If we substitute g by $H_\alpha[g]$ in Equation 3.2, we can write

$$\begin{aligned} H_\alpha[(f *_\alpha H_\alpha[g])(x)] &= \left(H_\alpha[f] *_\alpha H_\alpha[g]\right)(x), \\ (f *_\alpha H_\alpha[H_\alpha[g]])(x) &= \left(H_\alpha[f] *_\alpha H_\alpha[g]\right)(x), \quad \text{(by Equation 3.3)} \\ f *_\alpha g &= -\left(H_\alpha[f] *_\alpha H_\alpha[g]\right), \end{aligned}$$

where $H_\alpha^2 = -I$,
and this proves the theorem.

3.2. **Abstract Construction of Boehmians.** Now we construct the Boehmian space required for extending the theory of Fractional Hilbert transform to some space of Boehmians. Here we refer only two spaces of Boehmians needed in order to develop the theory of FHT. To define the space of Boehmians, we establish a family of identities as follows: Now to define the space of Boehmians, we introduce a class of identities as follows: Let the space $\mathcal{D}$ constitute of all infinitely differentiable functions with compact support in $\mathbb{R}$. Let

$$S = \{\phi \in \mathcal{D} : \phi \geq 0 \text{ and } \int \phi = 1\}$$

then the space of Boehmians is given by

$$\mathcal{B}_1 = \mathcal{B}_1(\mathcal{L}^1(\mathbb{R}), S, *_\alpha, \Delta),$$

where Δ is the collection of all sequences of real-valued functions $\{\phi_n(x)\} \subset S$ such that

(i) $\int_{\mathbb{R}} e^{it(x-t)\cot\alpha}\phi_n(x)dx = 1$, $\forall n \in \mathbb{N}$,
(ii) $\|\phi_n\|_1 \leq M$, $\forall n \in \mathbb{N}$ for some $M > 0$,
(iii) $\lim_{n\to\infty}\int_{|t|>\epsilon}|\phi_n(t)|dt = 0$, $\epsilon > 0$.

These sequences are known as *delta sequences.*
We now state and prove results which are needed to build the desired space of Boehmians.

Lemma 3.4. *The operation $*_\alpha$ is both commutative and associative.*

Proof: To prove that $*_\alpha$ is commutative, consider

$$\left(f *_\alpha g\right)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)e^{-i(x-t)\cot\alpha}dt.$$

By making change of variable $x - t = \tau$, we can simplify the above equation to

$$\left(f *_\alpha g\right)(x) = \int_{-\infty}^{\infty} f(\tau)g(x-\tau)e^{-i(x-\tau)\tau\cot\alpha}d\tau = \left(g *_\alpha f\right)(x).$$

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In order to prove the associativity, let us consider

$$\begin{aligned}((f *_\alpha g) *_\alpha h)(x) &= \int_{-\infty}^{\infty} (f *_\alpha g)(x-t) h(t) e^{-i(x-t)\cot\alpha} dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-t-u) g(u) h(t) e^{-iu(x-t-u)\cot\alpha} e^{-it(x-t)\cot\alpha} dt du.\end{aligned}$$

By changing the variables $t+u=y$, we can write the above equation as

$$((f *_\alpha g) *_\alpha h)(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y) g(y-t) h(t) e^{-i(y-t)(x-y)\cot\alpha} e^{-it(x-t)\cot\alpha} dt dy.$$

An application of Fubini's theorem, we have

$$\begin{aligned}((f *_\alpha g) *_\alpha h)(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y-t) h(t) e^{-i(-tx+yt+tx-t^2)\cot\alpha} f(x-y) e^{-iy(x-y)\cot\alpha} dt dy \\ &= \int_{-\infty}^{\infty} f(x-y)(g *_\alpha h)(y) e^{-iy(x-y)\cot\alpha} dy \\ &= (f *_\alpha (g *_\alpha h))(x).\end{aligned}$$

Thus,
$((f *_\alpha g) *_\alpha h)(x) = (f *_\alpha (g *_\alpha h))(x)$.

Lemma 3.5. *Assume that $\{\phi_n\}$ and $\{\psi_n\}$ are in Δ. Then their convolution $\{\phi_n *_\alpha \psi_n\}$ is also in Δ.*

Proof: In order to prove that $\{\phi_n *_\alpha \psi_n\} \in \Delta$, we have to show that the three conditions for delta sequences are fulfilled.

(i) $\int_{\mathbb{R}} e^{it(x-t)\cot\alpha}(\phi_n *_\alpha \psi_n)(x) dx = \int_{\mathbb{R}} e^{it(x-t)\cot\alpha} \int_{-\infty}^{\infty} \left(\phi_n(x-t)\psi_n(t) e^{-it(x-t)\cot\alpha}\right) dt dx,$
By making use of Fubini's theorem, we can write

$$\int_{\mathbb{R}} e^{it(x-t)\cot\alpha}(\phi_n *_\alpha \psi_n)(x) dx = \int_{\mathbb{R}} e^{it(x-t)\cot\alpha} e^{-it(x-t)\cot\alpha} \phi_n(x-t) dx \int_{-\infty}^{\infty} \psi_n(t) dt$$

Since $\{\phi_n\}, \{\psi_n\} \in \Delta$, therefore

$$\int_{\mathbb{R}} e^{it(x-t)\cot\alpha}(\phi_n *_\alpha \psi_n)(x) dx = 1$$

.

(ii)

$$\begin{aligned}\|\phi_n *_\alpha \psi_n\|_1 &= \int_{-\infty}^{\infty} |(\phi_n *_\alpha \psi_n)(x)| dx, \\ &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \phi_n(x-t)\psi_n(t) e^{-it(x-t)\cot\alpha} dt \right| dx, \\ &\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \phi_n(x-t)\psi_n(t) e^{-it(x-t)\cot\alpha} dt \right| dx \\ &= \|\phi_n\|_1 \|\psi_n\|_1, \\ &\le M^2, \ \forall n \in \mathbb{N}.\end{aligned}$$

Thus, $\|\phi_n *_\alpha \psi_n\|_1 \le M^2$.

(iii)

$$\begin{aligned}&\lim_{n\to\infty}\int_{|t|>\epsilon}|(\phi_n *_\alpha \psi_n)(x)|dx\\ &\leq \lim_{n\to\infty}\int_{|t|>\epsilon}\int_{-\infty}^{\infty}|\phi_n(x-t)\psi_n(t)|dtdx\\ &= \|\phi_n\|_1 \lim_{n\to\infty}\int_{|t|>\epsilon}|\psi_n(t)|dt.\end{aligned}$$

Since $\{\psi_n\} \in \Delta$ therefore

$$\lim_{n\to\infty}\int_{|t|>\epsilon}|\psi_n(t)|dt = 0 \qquad \text{for } \epsilon > 0.$$

Hence,

$$\int_{|t|>\epsilon}|(\phi_n *_\alpha \psi_n)(x)|dx \to 0 \quad \text{as } n\to\infty, \text{ for } \epsilon>0.$$

This completes the proof.

Lemma 3.6. *If $f \in \mathcal{L}^1$ and $\phi_n \in \Delta$ then the convolution $f *_\alpha \phi_n \in \mathcal{L}^1$.*

Proof: Let $f \in \mathcal{L}^1$ and $\phi_n \in \Delta$. To show that $f *_\alpha \phi_n \in \mathcal{L}^1$, we consider the $\mathcal{L}^1$-norm.

$$\begin{aligned}\|f *_\alpha \phi_n\|_1 &= \int_{\mathbb{R}}|(f *_\alpha \phi_n)(x)|dx\\ &= \int_{\mathbb{R}}\left|\int_{-\infty}^{\infty} f(x-t)\phi_n(t)e^{-it(x-t)\cot\alpha}dt\right|dx,\\ &\leq \int_{\mathbb{R}}\int_{-\infty}^{\infty}\left|f(x-t)\phi_n(t)e^{-it(x-t)\cot\alpha}\right|dtdx\\ &= \int_{-\infty}^{\infty}|f(x-t)|dx\int_{-\infty}^{\infty}|\phi_n(t)|dt\\ &= \|f\|_1\|\phi_n\|_1.\end{aligned}$$

Since $f \in \mathcal{L}^1$ and $\{\phi_n\} \in \Delta$,

$$\|f *_\alpha \phi_n\|_1 \leq \|f\|_1\|\phi_n\|_1 < \infty$$

which proves that $f *_\alpha \phi_n \in \mathcal{L}^1$.

Lemma 3.7. *If $f, g \in \mathcal{L}^1$, $\phi \in S$, then*

$$(f+g) *_\alpha \phi = f *_\alpha \phi + g *_\alpha \phi.$$

Proof of this lemma is straightforward. Therefore, we have omitted the details.

Lemma 3.8. *Let $f_n \to f$ in $\mathcal{L}^1$ as $n \to \infty$ and $\phi \in S$. Then $f_n *_\alpha \phi \to f *_\alpha \phi$ in $\mathcal{L}^1$.*

Proof: From Lemma 3.6 we can write

$$\begin{aligned}\|(f_n *_\alpha \phi) - (f *_\alpha \phi)\|_1 &= \|(f_n - f) *_\alpha \phi\|_1\\ &\leq \|f_n - f\|_1\|\phi\|_1\\ &\leq M\|f_n - f\|_1 \to 0 \text{ as } n\to\infty \text{ for } M>0.\end{aligned}$$

Hence, $f_n *_\alpha \phi \to f *_\alpha \phi$ in $\mathcal{L}^1$ whenever $f_n \to f$ in $\mathcal{L}^1$.

Lemma 3.9. *Let $f_n \to f$ in $\mathcal{L}^1$ and $\{\phi_n\} \in \Delta$. Then $f_n *_\alpha \phi_n \to f$ in $\mathcal{L}^1$.*

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Proof: Let $\{\phi_n\} \in \Delta$ then $\int_{-\infty}^{\infty} \phi_n(t) e^{it(x-t)} dt = 1$, therefore we can write

$$\begin{aligned}(f_n *_\alpha \phi_n)(x) - f(x) &= \int_{-\infty}^{\infty} f_n(x-t)\phi_n(t) e^{-it(x-t)\cot\alpha} dt - f(x) \int_{-\infty}^{\infty} \phi_n(t) e^{it(x-t)\cot\alpha} dt \\ &= \int_{-\infty}^{\infty} \left(f_n(x-t) e^{-2it(x-t)\cot\alpha} - f(x) \right) e^{it(x-t)\cot\alpha} \phi_n(t) dt.\end{aligned}$$

Now we consider the L^1-norm of above equation,

$$\begin{aligned}\|f_n *_\alpha \phi_n - f\|_1 &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \left(f_n(x-t) e^{-2it(x-t)\cot\alpha} - f(x) \right) e^{it(x-t)\cot\alpha} \phi_n(t) dt \right| dx, \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_n(x-t) e^{-2it(x-t)\cot\alpha} - f(x)| |\phi_n(t)| dt dx\end{aligned}$$

An application of Fubini's theorem, we have

$$\begin{aligned}\|f_n *_\alpha \phi_n - f\|_1 &\leq \int_{-\infty}^{\infty} |\phi_n(t)| dt \int_{-\infty}^{\infty} |f_n(x-t) e^{-2it(x-t)\cot\alpha} - f(x)| dx \\ &\leq M \|(f_n)_t e^{-2it(x-t)\cot\alpha} - f\|_1, \quad M > 0\end{aligned}$$

(by property 2 of delta sequences).

Using the triangular inequality of normed spaces,

$$\begin{aligned}\|f_n *_\alpha \phi_n - f\|_1 &\leq M \|(f_n)_t e^{-2it(x-t)\cot\alpha} - f_t e^{-2it(x-t)\cot\alpha}\|_1 \\ &\quad + \|f_t e^{-2it(x-t)\cot\alpha} - f\|_1, \\ &\leq M \|(f_n)_t e^{-2it(x-t)\cot\alpha} - f_t e^{-2it(x-t)\cot\alpha}\|_1 + M \|f_t e^{-2it(x-t)\cot\alpha} - f\|_1.\end{aligned}$$

By using the convergence of $f_n \in \mathcal{L}^1$ and Theorem 2.1,we have

$$\|(f_n)_t e^{-2it(x-t)\cot\alpha} - f_t e^{-2it(x-t)\cot\alpha}\|_1 \to 0 \text{ as } n \to \infty$$

and

$$\|f_t e^{-2it(x-t)\cot\alpha} - f\|_1 \to 0 \text{ as } y \to 0.$$

Therefore,

$$\|f_n *_\alpha \phi_n - f\|_1 \to 0 \text{ as } n \to \infty,$$

hence,

$$f_n *_\alpha \phi_n \to f \text{ in } \mathcal{L}^1.$$

In order to extend the FHT to the class of Boehmians, we define another class of Boehmians (as the co-domain of extended fractional Hilbert transform) $\mathcal{B}_2 = \mathcal{B}_2(\mathcal{C}^\infty, S, *_\alpha, \Delta)$ (Karunakaran and Kalpakam, 2000). The notion of delta sequences, quotients and their equivalence classes remain the same as in the prior case. We shall also retain the definitions of addition and scalar multiplication. Now we define,

$$D^m \left[\frac{f_n}{\phi_n} \right] = \left[\frac{D^m f_n}{\phi_n} \right], \text{ for any } \left[\frac{f_n}{\phi_n} \right] \in \mathcal{B}_2.$$

Also,

$$\left[\frac{f_n}{\phi_n} \right] *_\alpha \left[\frac{g_n}{\psi_n} \right] = \left[\frac{f_n *_\alpha g_n}{\phi_n *_\alpha \psi_n} \right].$$

Since a concept of convergence is required to construct a Boehmian space, we prove two convergences on $\mathcal{C}^\infty$.

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Lemma 3.10. *Let $f_n \to f$ as $n \to \infty$ in $\mathcal{C}^\infty$ then $f_n *_\alpha \phi \to f *_\alpha \phi$ in $\mathcal{C}^\infty$ for all $\phi \in D$ and further for each delta sequence $\{\delta_n\}$, $f_n *_\alpha \delta_n \to f$ as $n \to \infty$ in $\mathcal{C}^\infty$.*

Proof: Let $K \subset \mathbb{R}$ be any compact set such that $x \in K$. To prove the convergence of a sequence of functions in $\mathcal{C}^\infty$ we have to show that the functions and their derivatives converge uniformly on compact sets.
First of all, we prove that $f_n *_\alpha \phi \to f *_\alpha \phi$ in $\mathcal{C}^\infty$. For this consider

$$|\left(f_n *_\alpha \phi - f *_\alpha \phi\right)(x)| = |\left((f_n - f) *_\alpha \phi\right)(x)| \leq \int_{-\infty}^{\infty} |(f_n - f)(x-t)|\phi(t)dt.$$

Since t varies over the compact support of ϕ, therefore $x - t$ also varies over a compact set in $\mathbb{R}$. So $|\left((f_n - f) *_\alpha \phi\right)(x)| \to 0$ as $n \to \infty$ uniformly on compact sets. Then

$$|\left(f_n *_\alpha \phi - f *_\alpha \phi\right)(x)| \to 0 \text{ as } n \to \infty,$$

or we can write

$$f_n *_\alpha \phi \to f *_\alpha \phi \text{ as } n \to \infty, \tag{3.4}$$

uniformly on compact sets.
Also,

$$D^m\left((f_n *_\alpha \phi) - (f *_\alpha \phi)\right) = (D^m f_n *_\alpha \phi) - (D^m f *_\alpha \phi). \tag{3.5}$$

Replacing $D^m f_n$ by f_n and $D^m f$ by f in (3.5), we have

$$D^m\left((f_n *_\alpha \phi) - (f *_\alpha \phi)\right) = (f_n *_\alpha \phi) - (f *_\alpha \phi), \tag{3.6}$$

the right hand side of (3.6) approaches to zero by (3.4). Thus

$$D^m(f_n *_\alpha \phi) \to D^m(f *_\alpha \phi)$$

uniformly on compact sets. Hence, $f_n *_\alpha \phi \to f *_\alpha \phi$ as $n \to \infty$ in $\mathcal{C}^\infty$.
Next, without any loss of generality let us suppose that $(\delta_n) \in \Delta$ is such that it has a compact support. Then

$$\begin{aligned}|(f_n *_\alpha \delta_n - f)(x)| &= \left|\int_{-\infty}^{\infty} f_n(x-t)\delta_n(t)e^{-it(x-t)\cot\alpha}dt - f(x)\int_{-\infty}^{\infty} e^{it(x-t)\cot\alpha}\delta_n(t)dt\right| \\ &\leq \int_{-\infty}^{\infty} |f_n(x-t)e^{-2it(x-t)\cot\alpha} - f(x)|\delta_n(t)dt\end{aligned}$$

$$\leq \int_{-\infty}^{\infty}\left(|f_n(x-t)e^{-2it(x-t)\cot\alpha} - f(x-t)e^{-2it(x-t)\cot\alpha}| + |f(x-t)e^{-2it(x-t)\cot\alpha} - f(x)|\right)\delta_n(t)dt.$$

Now both x and t vary over compact sets therefore $x - t$ also vary over a compact set. Thus,

$$\int_{-\infty}^{\infty}\left(|f_n(x-t)e^{-2it(x-t)\cot\alpha} - f(x-t)e^{-2it(x-t)\cot\alpha}| + |f(x-t)e^{-2it(x-t)\cot\alpha} - f(x)|\right)\delta_n(t)dt \to 0$$

as $n \to \infty$ and $t \to 0$.
We have

$$f_n *_\alpha \delta_n \to f \text{ uniformly on compact sets.}$$

Similarly,

$$D^m(f_n *_\alpha \delta_n) \to D^m(f) \text{ uniformly on compact sets.}$$

Hence

$$f_n *_\alpha \delta_n \to f \text{ as } n \to \infty \text{ in } \mathcal{C}^\infty.$$

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Lemma 3.11. *If $f_n \to f$ as $n \to \infty$ in $\mathcal{L}^1$ then for every $\delta \in S$, $f_n *_\alpha \delta \to f *_\alpha \delta$ as $n \to \infty$ in $\mathcal{C}^\infty$.*

Proof: To show the convergence in $\mathcal{C}^\infty$, we assume that x vary over a compact set K.

$$\begin{aligned}
|(f_n *_\alpha \delta - f *_\alpha \delta)(x)| &= |((f_n * - f) *_\alpha \delta)(x)| \\
&= \left| \int_{-\infty}^{\infty} (f_n - f)(x-t)\delta(t) e^{-it(x-t)\cot\alpha} dt \right| \\
&\leq \int_{-\infty}^{\infty} |(f_n - f)(x-t)||\delta(t)| dt \\
&\leq \|f_n - f\|_1 \|\delta\|_\infty.
\end{aligned}$$

Since $f_n \to f$ in$\mathcal{L}^1$ and $\delta \in S$ has a compact support, therefore $x - t$ vary over a compact set also $|(f_n *_\alpha \delta - f *_\alpha \delta)(x)| \to 0$ as $n \to \infty$ on compact sets.
Similarly, we have

$$|D^m[(f_n *_\alpha \delta - f *_\alpha \delta)](x)| \leq \|f_n - f\|_1 \|D^m \delta\|_\infty.$$

Thus,

$$D^m(f_n *_\alpha \delta) \to D^m(f *_\alpha \delta) \text{ on compact sets.}$$

Hence,

$$f_n *_\alpha \delta \to f *_\alpha \delta \text{ as } n \to \infty \text{ in } \mathcal{C}^\infty.$$

3.3. **Fractional Hilbert Transform on Boehmians.** The following result is very important in the aftermath; The proof of the following theorem is similar to the proof of convolution theorem for FHT as in Theorem 2.3,we prefer to leave out the details.

Theorem 3.12. *If $f \in \mathcal{L}^1$ and $\delta \in \Delta$ then $H_\alpha[f *_\alpha \delta] = H_\alpha[f] *_\alpha \delta$.*

Definition 3.13. *The fractional Hilbert transform $\mathcal{H}_\alpha : \mathcal{B}_1 \to \mathcal{B}_2$ on Boehmians is defined by*

$$\mathcal{H}_\alpha \left[\frac{f_n}{\phi_n} \right] = \left[\frac{\mathcal{H}_\alpha f_n}{\phi_n} \right],$$

where $\frac{f_n}{\phi_n}$ is an arbitrary representative of any given Boehmian $B \in \mathcal{B}_1$. Since

$$f_n *_\alpha \phi_m = f_m *_\alpha \phi_n \quad \forall m, n \in \mathbb{N}.$$

By Theorem 3.12, we can write

$$\mathcal{H}_\alpha[f_n] *_\alpha \phi_m = \mathcal{H}_\alpha[f_m] *_\alpha \phi_n \quad \forall m, n \in \mathbb{N}.$$

Therefore, $\frac{\mathcal{H}_\alpha[f_n]}{\phi_n}$ represents a Boehmian in $\mathcal{B}_2$. In the similar manner, let $\frac{g_n}{\psi_n}$ be another representative of B then again by an application of Theorem 3.12,

$$\frac{\mathcal{H}_\alpha[f_n]}{\phi_n} \sim \frac{\mathcal{H}_\alpha[g_n]}{\psi_n},$$

thus the extended FHT on Boehmians $\mathcal{H}_\alpha : \mathcal{B}_1 \to \mathcal{B}_2$ is well defined.
The proof of the following theorem is similar to the proof of Hilbert transform on Boehmians, we prefer to omit the details. For details the reader is referred to (Karunakaran and Kalpakam, 2000).

Theorem 3.14. *Let $\mathcal{H}_\alpha : \mathcal{B}_1 \to \mathcal{B}_2$ be the extended FHT then*

(i) *If* $\frac{f_n}{\phi_n} \in \mathcal{B}_1$ *then* $\frac{\mathcal{H}_\alpha f_n}{\phi_n} \in \mathcal{B}_2$.
(ii) $\mathcal{H}_\alpha$ *is well defined.*
(iii) $\mathcal{H}_\alpha$ *is a continuous linear map.*
(iv) $\mathcal{H}_\alpha$ *is an injective map.*

Proof: The proof of the following theorem is similar to the those of Hilbert transform on Boehmians, we prefer to leave out the details. For details the reader is referred to (Karunakaran and Kalpakam, 2000).

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