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Almost Global Pullback Attraction In Non-Autonomous Systems

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Abstract

This short report contains a result that characterizes almost global pullback attractor of discrete-time non-autonomous systems. Analogously to the multiple Lyapunov functions approach for switched systems, we show here that existence of multiple Lyapunov densities implies pullback convergence of almost all initial states to the origin for a discrete-time non-autonomous system.

1 Introduction

We claim that Lyapunov density functions are appropriate certification tools for the pullback attraction in non-autonomous systems. They have been used for almost global stability of switched systems in [1]. In the following we show that existence of multiple Lyapunov densities implies pullback convergence of a discrete-time non-autonomous system.

2 Lyapunov Density Theorem for a Discrete-Time Dynamical System

Let us consider a discrete-time semi-dynamical system on $\mathbb{R}^N$[1] given by

$$x(t+1) = F(x(t)), \quad F : \mathbb{R}^N \to \mathbb{R}^N. \tag{1}$$

We assume that $F(0) = 0$, and that F is nonsingular on $\mathbb{R}^N - \{0\}$, i.e., $V \subset \mathbb{R}^N - \{0\}$ and $\mathrm{Leb}(V) = 0 \implies \mathrm{Leb}(F^{-1}(V)) = 0$. An initial point $x_0 \in \mathbb{R}^N - \{0\}$ is said to be attracted to zero if

$$\lim_{t\to\infty} \|F^t(x_0)\| = 0. \tag{2}$$

[1]Here, we choose $\mathbb{R}^N$ for simplicity. All results of this paper can be extended to the case where the state space is a manifold and a Riemannian measure is considered instead of the Lebesgue measure.

The Frobenius-Perron operator $\mathbb{P}$ of the function F can be defined over $\mathcal{M}(\mathbb{R}^N - \{0\})$, i.e., the set of measurable functions from $\mathbb{R}^N - \{0\}$ to $\mathbb{R}$, via

$$\int_V \mathbb{P}\rho \, dx = \int_{F^{-1}(V)} \rho \, dx, \quad V \subset \mathbb{R}^N - \{0\} \tag{3}$$

uniquely, thanks to the Radon-Nikodym theorem and the backward invariance of $\mathbb{R}^N - \{0\}$. Note that we allow both sides of (3) to be infinite for some V's. The following theorem is stated and proved in [2], which also follows from Lemma 2 in [5]. We will use the positivity of the Frobenius-Perron operator: $\rho \geq 0 \implies \mathbb{P}\rho \geq 0$. When F is differentiable and invertible, the Frobenius-Perron operator has the following explicit expression [4]:

$$\mathbb{P}\rho(x) = \rho(F^{-1}(x)) \cdot \left| \frac{d}{dx} F^{-1}(x) \right| \tag{4}$$

Similarly, the Koopman operator $\mathbb{U} : \mathcal{M}(\mathbb{R}^N - \{0\}) \to \mathcal{M}(\mathbb{R}^N - \{0\})$ is defined by

$$\mathbb{U}f(x) = f\left(F(x)\right). \tag{5}$$

In particular, when f is the characteristic function 1_V of a set $V \subset \mathbb{R}^N - \{0\}$, we have $\mathbb{U}1_V(x) = 1_V\left(F(x)\right) = 1_{F^{-1}(V)(x)}$ and

$$\int \mathbb{P}\rho \cdot 1_V \, dx = \int \rho \cdot \mathbb{U}1_V \, dx \tag{6}$$

by the definition of $\mathbb{P}$ in (3). This can be generalized to measurable functions f as

$$\int \mathbb{P}\rho \cdot f \, dx = \int \rho \cdot \mathbb{U}f \, dx. \tag{7}$$

Note that both sides of (7) can simultaneously be infinite and we do not make any claim about the duality of spaces.

Theorem 1 ([2])**.** *Almost all initial solutions of* (1) *converge to zero if there exists a measurable function* $\rho^* : \mathbb{R}^N - \{0\} \to \mathbb{R}$ *such that*

- $\rho^*(x) > 0$ *for almost every* $x \in \mathbb{R}^N - \{0\}$,
- $\mathbb{P}\rho^*(x) < \rho^*(x)$ *for almost every* $x \in \mathbb{R}^N - \{0\}$,
- ρ^* *is integrable on the set* $\{x \in \mathbb{R}^N : \|x\| > \varepsilon\}$ *for any* $\varepsilon > 0$.

3 Multiple Lyapunov Density Theorem for a Discrete-Time Non-Autonomous System

Let us consider a discrete-time non-autonomous system given by

$$x(t+1) = F_t(x(t)), \quad F_t : \mathbb{R}^N \to \mathbb{R}^N, \quad t \in \mathbb{Z}, \tag{8}$$

where each F_t is nonsingular. We use the notation $F_{t,t_0} = F_{t-1} \circ \cdots \circ F_{t_0}$. Hence a solution $x(t)$ of (8) satisfies $x(t) = F_{t,t_0}(x(t_0))$. We assume that $F_t(0) = 0$, and $F_t^{-1}(\{0\}) = \{0\}$ for all t. For the nonautonomous system (8), two different types of attraction can be defined [3]. A point $x_0 \in \mathbb{R}^N - \{0\}$ is said to be forward attracted to zero at time t_0 if

$$\lim_{t\to\infty} \|F_{t,t_0}(x_0)\| = 0, \tag{9}$$

whereas a point $x_0 \in \mathbb{R}^N - \{0\}$ is said to be pullback attracted to zero at time t if

$$\lim_{t_0\to-\infty} \|F_{t,t_0}(x_0)\| = 0. \tag{10}$$

For each $t \in \mathbb{Z}$, we define the Frobenius-Perron operator $\mathbb{P}_t$ and the Koopman operator $\mathbb{U}_t$ for the map F_t as above. Namely, $\mathbb{P}_t : \mathcal{M}(\mathbb{R}^N - \{0\}) \to \mathcal{M}(\mathbb{R}^N - \{0\})$ using

$$\int_V \mathbb{P}_t\rho \, dx = \int_{F_t^{-1}(V)} \rho \, dx, \quad V \subset \mathbb{R}^N - \{0\}, \tag{11}$$

and

$$\mathbb{U}_t f(x) = f\left(F(x)\right). \tag{12}$$

Note that, similar to (7) we have

$$\int \mathbb{P}_t\rho \cdot f \, dx = \int \rho \cdot \mathbb{U}_t f \, dx. \tag{13}$$

The following result is a generalization of Theorem 1 to non-autonomous systems.

Theorem 2. *For any $t \in \mathbb{Z}$, almost every state in $\mathbb{R}^N - \{0\}$ is pullback attracted to zero at time t, if there exists a family of measurable functions $\{\rho_t^* : \mathbb{R}^N - \{0\} \to \mathbb{R}\}_{t\in\mathbb{Z}}$ such that for all $t \in \mathbb{Z}$*

- *$\rho_t^*(x) > 0$ for almost every $x \in \mathbb{R}^N - \{0\}$,*
- *ρ_t^* is integrable on the set $\{x \in \mathbb{R}^N : \|x\| > \varepsilon\}$ for any $\varepsilon > 0$,*

and

- $\inf_{t\in\mathbb{Z}} [\rho_{t+1}^* - \mathbb{P}_t\rho_t^*](x) > 0.$

Example 1. *Consider the system*

$$x(t+1) = a^t \cdot x(t), \quad t \in \mathbb{Z}, \tag{14}$$

where $a > 1$. The zero solution of this system is not forward attracting but is pullback attracting which can be shown using the Lyapunov densities

$$\rho_t^*(x) = \frac{a^{\bar{t}}}{x^2}, \tag{15}$$

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where $\bar{t}$ denotes the sum of the integers from 0 to t if t is nonnegative, and the absolute value of the sum of integers from $t+1$ to 0 if t is negative. Note that $\bar{t} \geq 0$ and

$$\bar{t} + t + 1 = \overline{t+1}$$

for all $t \in \mathbb{Z}$. The Frobenius-Perron operator of the system (14) at time t can be obtained using (4) as

$$\mathbb{P}_t \rho(x) = \rho(a^{-t}x) \cdot a^{-t} = \frac{a^{\bar{t}+t}}{x^2} = \frac{a^{\overline{t+1}-1}}{x^2}, \tag{16}$$

and therefore

$$\inf_t [\rho^*_{t+1} - \mathbb{P}_t \rho^*_t](x) = \inf_t \frac{a^{\overline{t+1}} - a^{\overline{t+1}-1}}{x^2} = \inf_t \frac{a^{\overline{t+1}} \cdot (1 - a^{-1})}{x^2} = \frac{(1 - a^{-1})}{x^2}. \tag{17}$$

Hence, the conditions of Theorem 2 are satisfied and therefore the zero solution of (14) is pullback attracting almost all states.

Let us define the Frobenius-Perron operator from time t_0 to time t as

$$\mathbb{P}_{t,t_0} := \mathbb{P}_{t-1} \circ \cdots \circ \mathbb{P}_{t_0} \tag{18}$$

and the Koopman operator from time t_0 to time t as

$$\mathbb{U}_{t,t_0} := \mathbb{U}_{t_0} \circ \cdots \circ \mathbb{U}_{t-1}. \tag{19}$$

Consecutive applications of (13) yields to

$$\int \mathbb{P}_{t,t_0} \rho \cdot f \, dx = \int \rho \cdot \mathbb{U}_{t,t_0} f \, dx. \tag{20}$$

We will use the following characterization of attraction using the Koopman operators:

Lemma 1. *x_0 is pullback attracted to zero at time t for the discrete-time non-autonomous system (8) if and only if $\sum_{t_0=-\infty}^{t} \mathbb{U}_{t,t_0} 1_{\|x\| \geq \varepsilon}(x_0) < \infty$ for any $\varepsilon > 0$.*

Proof. $\sum_{t_0=-\infty}^{t} \mathbb{U}_{t,t_0} 1_{\|x\| \geq \varepsilon}(x_0)$ is the number of visits of $F_{t,t_0}(x_0)$ to $\{x \in \mathbb{R}^N : \|x\| \geq \varepsilon\}$ as $t_0 \to -\infty$. Hence it is finite if and only if $F_{t,t_0}(x_0)$ is eventually trapped in $\{x \in \mathbb{R}^N : \|x\| < \varepsilon\}$ as $t_0 \to -\infty$, which is equivalent to (10). □

A sufficient condition for almost everywhere pullback attraction can be given as follows:

Lemma 2. *Almost all $x_0 \in \mathbb{R}^n$ is pullback attracted to zero at time t if there exists a measurable function $\bar{\rho} : \mathbb{R}^N - \{0\} \to \mathbb{R}$ such that*

- *$\bar{\rho}(x) > 0$ for almost every $x \in \mathbb{R}^N - \{0\}$,*

- $\lim_{t_0 \to -\infty} \sum_{s=t_0}^{t} \mathbb{P}_{t,s}\bar{\rho}$ *is well-defined and integrable on the set* $\{x \in \mathbb{R}^N : \|x\| > \varepsilon\}$ *for any* $\varepsilon > 0$.

Proof.

$$\begin{aligned}
\infty > \int_{\|x\|>\varepsilon} \lim_{t_0\to-\infty} \sum_{s=t_0}^{t} \mathbb{P}_{t,s}\bar{\rho}\, dx &= \int \left(\lim_{t_0\to-\infty} \sum_{s=t_0}^{t} \mathbb{P}_{t,s}\bar{\rho} \right) \cdot 1_{\|x\|>\varepsilon}\, dx \\
&= \lim_{t_0\to-\infty} \sum_{s=t_0}^{t} \int \mathbb{P}_{t,s}\bar{\rho} \cdot 1_{\|x\|>\varepsilon}\, dx \\
&= \lim_{t_0\to-\infty} \sum_{s=t_0}^{t} \int \bar{\rho} \cdot \mathbb{U}_{t,s} 1_{\|x\|>\varepsilon}\, dx \\
&= \int \bar{\rho} \cdot \left(\lim_{t_0\to-\infty} \sum_{s=t_0}^{t} \mathbb{U}_{t,s} 1_{\|x\|>\varepsilon} \right) dx,
\end{aligned}$$

where the second and forth line follows from Fubini Theorem and the third line follows from (20). The finiteness of the last integral and the almost everywhere positivity of $\bar{\rho}$ implies that

$$\lim_{t_0\to-\infty} \sum_{s=t_0}^{t} \mathbb{U}_{t,s} 1_{\|x\|>\varepsilon}(x_0) < \infty$$

for almost every x_0. Hence, the proof follows by Lemma 1. □

We can now give the proof of Theorem 2:

Proof of Theorem 2. Define $\bar{\rho}_t := \rho_t^* - \mathbb{P}_{t-1}\rho_{t-1}^*$ and $\bar{\rho} := \inf_t \bar{\rho}_t$. By assumption, $\bar{\rho} > 0$ almost everywhere, which implies that $\bar{\rho}_t > 0$ almost everywhere for each $t \in \mathbb{Z}$. Let us show that the conditions of Lemma 2 are satisfied for $\bar{\rho}$:

$$\begin{aligned}
\lim_{t_0\to-\infty} \sum_{s=t_0}^{t} \mathbb{P}_{t,s}\bar{\rho} \le \lim_{t_0\to-\infty} \sum_{s=t_0}^{t} \mathbb{P}_{t,s}\bar{\rho}_s &= \lim_{t_0\to-\infty} \sum_{s=t_0}^{t} \left(\mathbb{P}_{t,s} \left(\rho_s^* - \mathbb{P}_{s-1}\rho_{s-1}^* \right) \right) \\
= \lim_{t_0\to-\infty} \sum_{s=t_0}^{t} \left(\mathbb{P}_{t,s}\rho_s^* - \mathbb{P}_{t,s-1}\rho_{s-1}^* \right) &= \rho_t^* - \lim_{t_0\to-\infty} \mathbb{P}_{t,t_0}\rho_{t_0}^*.
\end{aligned}$$

The final limit above exists and is integrable on $\{\|x\| \ge \varepsilon\}$ by the dominated convergence theorem since $\bar{\rho}_t > 0$ and the positivity of the Frobenius-Perron operator implies that

$$\rho_t^* \ge \mathbb{P}_{t,t-1}\rho_{t-1}^* \ge \mathbb{P}_{t,t-2}\rho_{t-2}^* \ge \cdots > 0$$

Since ρ_t^* is also integrable on $\{\|x\| \ge \varepsilon\}$ and the final time t was arbitrary, the proof follows by Lemma 2. □

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4 Discussions

The result in this report suggests that Lyapunov densities are suitable tools for the certification of pullback attraction in non-autonomous systems. Extensions of this result to continuous-time non-autonomous system is an open problem. As another future problems, certification of almost sure convergence or in-probability convergence of random systems can be considered.

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