Review

The Uncertainty Principle and the Minimal Space-Time Length Element

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Abstract: Quantum gravity theories rely on a minimal measurable length for their formulations, which clashes with the classical formulation of the uncertainty principle and with Lorentz invariance from general relativity. These incompatibilities led to the development of the generalized uncertainty principle (GUP) from string theories and its various modifications. GUP and covariant formulations of the uncertainty principle are discussed, together with implications for space-time quantization.

Keywords: General relativity; uncertainty principle; geodesics; black hole singularity; quantum gravity; Planck star; Lorentz invariance violations.

1. Introduction: general relativity, quantum mechanics and the problem of a minimal length

General Relativity (GR) and Quantum Mechanics (QM) constitute the two major pillars of modern physics. So far, these two theories in their various formulations have survived all experimental testing, which supports their role as fundamental theories of nature. While classical GR is a geometric theory for gravitation, classical QM describes phenomena other than gravitation at "Planck scales" by probability theory of states in Hilbert space. Owing to their fundamentality, one would expect that these two theories could be combined in a single, unified theory for quantum gravity. However, these two theories have major incompatibilities starting from their different frameworks, formulations and principles, which make their merging a daunting task. Nevertheless, attempts to unify these two fundamental theories have given rise to well-developed quantum gravity theories such as string theory and loop quantum gravity (LQG) [1,2].

GR is a Lorentz covariant geometric theory for gravitation put forward by Albert Einstein in 1916 [3], in which a radical conceptual change was introduced to classical gravitation. In GR the concept of classical gravitational force disappears and is substituted by a dynamical space–time geometry given by a pseudo-Riemannian manifold consisting of three spatial dimensions and a time dimension. The space-time manifold in GR presents a Lorentzian (- + + +) signature and it is shaped by energy-momentum densities from an energy-momentum tensor in Einstein´s field equations [3]. GR is also a background independent theory in which the space–time metric is the dynamical variable [4]. Space-time geometries are determined by mass, energy and momentum densities, and particles follow geodesic trajectories in the space-time manifolds, for which position and momentum are defined with absolute certainty. This is simply not allowed in QM.

QM was developed through a process of tackling several inconsistencies mainly in particle physics and thermodynamics which could not be solved by classical principles of physics. Its foundation as a consistent theory rested on a collection of postulates not truly derived from first principles [5,6], and on three fundamental pillars: Energy quantization, the concept and interpretation of the wave function and the uncertainty principle. For the uncertainty principle, classical QM states that the position and momentum of a particle in a trajectory cannot be defined with absolute certainty, which is in direct contradiction with GR. This principle is further completed by a similar statement on energy-time uncertainty.

Classical QM evolved into quantum field theory during the 1930s, and with it the problem of ultraviolet divergences. These divergences were later taken care of by the development of renormalization mathematical techniques [7,8]. But before that, in this context, the idea of a minimal measurable discrete length was put forward with Heisenberg being one of the main advocates [9]. His main argument was the necessity for a discreet length to overcome the divergences in quantum field theories, and also for the description of the range of elementary known particles. The proposals for a minimal measureable length were met with scepticism, because this concept was in direct contradiction with Lorenz invariance and general relativity. A minimal discrete length would imply the need of privileged reference frames. Snyder was the first to show that the two ideas, a minimal length and Lorentz invariance, could be combined by modifying the canonical commutators of position/momentum operators [10]. It was also realized relatively early that quantum uncertainties would affect the background space-time, leading to the necessity of its quantization in a quantum theory of gravity [11,12]. The proposal by Mead that Planck length constituted such a fundamental minimal length [13] was initially not taken seriously.

The classical uncertainty principle, one of the pillars of quantum mechanics, is not restricted to a minimal length or a minimal momentum if these are interpreted as uncertainties. Hence, the uncertainty in position or momentum can be arbitrarily small, leading to troublesome divergencies. Then string theory came in the 1980s by deriving a generalized uncertainty principle which stated the impossibility of measuring an arbitrarily small length [14-16]. In the 1990s a modification of the position/momentum commutator relations of space-time to a Hopf algebra was introduced [17], and Kempf modified the commutator relations to accommodate a minimal length in quantum field theories [18-21]. The generalized uncertainty principle could be derived from these modifications [18]. This generalized uncertainty principle with Planck length as a minimal measureable length was proposed as a solution to ultraviolet divergencies in quantum gravity at Planck energies. But another drawback appeared when GR was found to be apparently non-renormalizable when formulated as a quantum field theory. The introduction of a Lorentz covariant minimal length could be a way forward to tackle this issue [8].

Here we review the uncertainty principle and its main modifications for adaptation to a minimal length element and to Lorentz covariance.

2. The uncertainty principle

The uncertainty principle originally proposed by Heisenberg is a general property of wave systems, and as such it is considered a fundamental law of nature. Heisenberg put forward this principle for the canonical conjugated variables of momentum and position in 1927 [22], which was later generalized as an inequality by Kennard for any arbitrary wave function [23]. In 1945 Mandelshtam and Tamm derived a similar non-relativistic uncertainty principle between energy and time in the form of the Madelshtam-Tamm inequality [24]. In this latter inequality, time still remains as an independent privileged variable. The current classical uncertainty principle thus consists of two inequalities:

$$|\Delta \boldsymbol{p}||\Delta \boldsymbol{x}| \geq \frac{\hbar}{\mathbf{2}} \quad , \qquad |\Delta \boldsymbol{E}||\Delta \boldsymbol{t}| \geq \frac{\hbar}{\mathbf{2}} . \tag{1}$$

where, $\Delta \boldsymbol{p}$ represents the change in magnitude of momentum parametrized by coordinate time; $\Delta \boldsymbol{x}$ is the change in magnitude of the position vector; $\Delta \boldsymbol{E}$ and $\Delta \boldsymbol{t}$ represent the change in magnitude of energy and time, respectively; $\hbar$ is the reduced Plank constant.

These two uncertainty relations are considered a fundamental principle in nature behind many quantum phenomena [25-27]. And although Heisenberg utilized the "observer effect" as an intuitive interpretation, this principle is fundamentally intrinsic to any wave system [27-29]. The momentum/position classical uncertainty principle is conveniently represented by the Heisenberg commutator algebra, which is a reflection of the non-commutability of momentum and position operators:

$$[\hat{\boldsymbol{p}}^{\boldsymbol{i}}, \hat{\boldsymbol{x}}^{\boldsymbol{j}}] = -\boldsymbol{i}\hbar\boldsymbol{\delta}^{\boldsymbol{ij}}. \quad (2)$$

where the indices, denoted by latin letters, take on the values 1, 2 and 3; $\hat{\boldsymbol{p}}^{\boldsymbol{i}}, \hat{\boldsymbol{x}}^{\boldsymbol{j}}$ represent momentum and position operators, and $\boldsymbol{\delta}^{\boldsymbol{ij}}$, the Kronecker delta function.

The momentum/position commutator and the classical inequalities of the uncertainty principle were reinterpreted as standard deviations in momentum and position ($\boldsymbol{\sigma_p}, \boldsymbol{\sigma_x}$) by Kennard for any wave function [23,30]:

$$\boldsymbol{\sigma_p}\boldsymbol{\sigma_x} \geq \frac{\hbar}{\boldsymbol{2}}. \quad (3)$$

One key consequence of the uncertainty relations in QM is that momentum-position phase space is quantized. However, this does not directly imply the existence of a minimal length because in inequalities (1) and (3) the actual uncertainty in position is unrestricted (Figure 1A). Uncertainty in position can be arbitrarily small, leading also to divergence in momentum, which is highly problematic. This was soon shown to be in conflict with quantum gravity theories such as string theories [31] and LQG [1,32]. Their formulations require a minimal length proportional to Planck length (ℓ_p) [33-35]:

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}}\,. \quad (4)$$

where G and c represent the universal gravitational constant and the speed of light, respectively.

For string theories, $\boldsymbol{\ell_p}$ is already a fundamental length element for strings-particles [2,14,36]. LQG is a quantum theory for gravitation that starts from classical GR in its ADM formulism, in which space-time is foliated and then space lattice quantization is introduced [37]. As a consequence, this lattice quantization leads to a minimum length, and for example, LQG area and volume operators are quantized and proportional to $\boldsymbol{\ell_p}^{\boldsymbol{2}}$and $\boldsymbol{\ell_p}^{\boldsymbol{3}}$, respectively. But this concept of a fixed, measurable minimal length not only clashed with the original formulation of Heisenberg´s uncertainty principle, but also with Lorentz covariance. Nevertheless, the uncertainty principle provided a means to introduce a minimal length in relativity. As the gravitational field in classical GR depends on energy and momentum densities, the uncertainty principle would be expected to alter the background space-time geometry and introduce constraints to the classical space-time metric. Indeed, these constraints could be identified with a minimal length in quantum gravity. The starting point constitutes the extension of the position/momentum commutator relation from inequality (2) to the background Minkowski space-time. These modified commutator relations introduce a Lorentzian signature in the commutator, and are valid as a local projection of momentum and position operators on asymptotically non-curved tangent space [38]:

$$[\hat{\boldsymbol{p}}^{\mu}, \hat{\boldsymbol{x}}^{\nu}] = -\boldsymbol{i}\hbar\eta^{\mu\nu}. \quad (5)$$

Where the indices denoted by Greek letters take on the values 0 (time), 1, 2 and 3 (space) following standard tensor notation; $\eta^{\mu\nu}$ represents Minkowski space-time metric.

Hence, one of the first issues was to reconcile the classical uncertainty principle with the necessity for a measurable minimal length in quantum gravity theories. This gave rise to the generalized uncertainty principle and its variants.

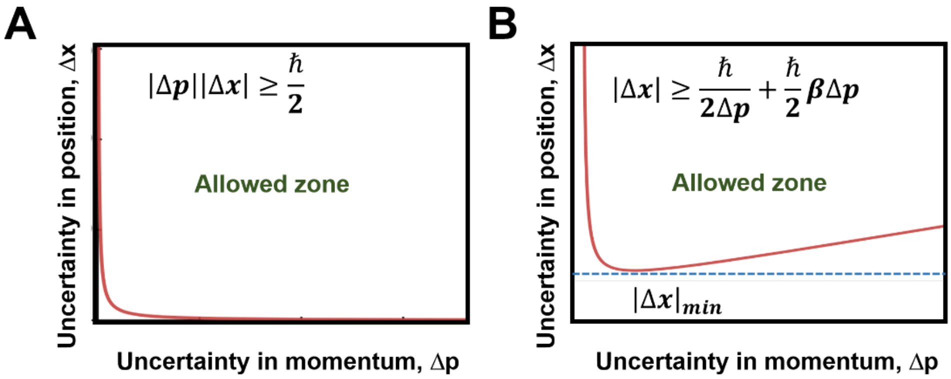

Figure 1. Classical uncertainty principle and GUP. (A) Plot of the classical uncertainty momentum-position inequality as shown on top, indicating the allowed region. Uncertainties in position and momentum diverge to infinity. (B) Graph plot of a GUP representation of the uncertainty principle as shown on top. A minimum in the function is reached representing a minimal measurable length, $|\Delta x|_{min}$. The allowed region by the inequality is shown. Plots are represented in relative units.

3. Generalized uncertainty principle (GUP) and its modifications

The uncertainty principle inequalities as originally formulated (inequality 1) imply a quantized momentum-position phase space, and subsequently, a quantized space as discussed above. However, the momentum-position uncertainty relation as shown in inequality (1) is not constrained to a minimal length (if considered as a non-zero uncertainty in position) and thus subject to ultraviolet divergences (Figure 1A). In this classical formulation, the space length represented as the uncertainty in position can asymptotically approach zero, making momentum diverge to infinity. This uncertainty relation is therefore unbound both in position and momentum uncertainties. This is in direct contrast with the need for a minimal length element, which is a common feature of gravity theories including string theory [1,14,31,36,39], LQG [2,32] and doubly special relativity [40].

Collisions of strings at Planckian energies also required a minimal length leading to a modification of the classical uncertainty inequalities into what is known as the generalized uncertainty principle (GUP)[14-16,41-44]. GUP formulations included boundaries to both momentum and position [14,44]. But the simplest forms of GUP led to corrections in inequality (1) that bounded only uncertainties in position by adding quadratic forms of momentum [19,45] :

$$|\Delta p||\Delta x| \geq \frac{\hbar}{2} + \frac{\hbar}{2}\,\beta\Delta p^2 + \frac{\hbar}{2}\gamma. \tag{6}$$

where β and γ represent functions dependent on the expectation value of momentum and position [18]. This re-formulation of the uncertainty principle presents a minimum of uncertainty in position, below which the uncertainty relation is not allowed (Figure 1B). By modelling string collisions at Planck energies, an explicit p-quadratic GUP

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formulation arises with expressions dependent on a fundamental quadratic length on Planck scale ($\boldsymbol{\delta\ell_p}^2$) [14,41-43,46]:

$$|\Delta \boldsymbol{x}| \geq \frac{\hbar}{2\Delta \boldsymbol{p}} + \frac{\boldsymbol{\alpha G}}{\boldsymbol{c}^3}\Delta \boldsymbol{p} \ , \ |\Delta \boldsymbol{x}| \geq \frac{\hbar}{2\Delta \boldsymbol{p}} + \boldsymbol{\delta\ell_p}^2 \Delta \boldsymbol{p} \ . \quad (7)$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ represent constants.

An uncertainty relation in the framework of quantum geometry theory can be derived for any accelerating particle in the absence of a gravitational field. The uncertainty relation perturbs the background Minkowski space-time through acceleration, and the particle experiences gravitation via a perturbation over the background Minkowski metric [38]. This perturbation can be reflected by local quantum deviations from the background flat space at high energy collisions, for example:

$$\boldsymbol{g}_{\mu\nu} = \ \eta_{\mu\nu} + \boldsymbol{h}_{\mu\nu} = \left(\boldsymbol{1} + \boldsymbol{c}^4 \frac{\ddot{\boldsymbol{x}}^{\alpha}\ddot{\boldsymbol{x}}_{\alpha}}{\boldsymbol{A}^2}\right) . \quad (8)$$

where $\boldsymbol{g}_{\mu\nu}, \eta_{\mu\nu}, \boldsymbol{h}_{\mu\nu}$ represent the co-variant pseudo-Riemannian metric tensor, Minkowski metric tensor and a metric perturbation, respectively; $\ddot{\boldsymbol{x}}^{\alpha}$, $\ddot{\boldsymbol{x}}_{\alpha}$ represent contravariant and covariant components of acceleration; and A represents maximal acceleration. By incorporating the perturbed metric from equation (8) into the canonical position-momentum commutator in Minskowski space, GUP in the p-quadratic form is recovered as a function of the particle mass, m, the maximal proper acceleration, A , and the quadratic form of a space-time length element, δs [38]:

$$|\Delta \boldsymbol{x}| \geq \frac{\hbar}{2\Delta \boldsymbol{p}} + \frac{\hbar \boldsymbol{c}^2}{\boldsymbol{m}^2 \boldsymbol{A}^2 \boldsymbol{\delta s}^2}\Delta \boldsymbol{p} \ . \quad (9)$$

This re-formulation of the uncertainty principle is equivalent to GUP as shown by inequality (7) by equating $\boldsymbol{\delta s}$ to the particle´s Compton length [38].

The inequality formulations for GUP can be expressed as commutator relationships between momentum and position operators by introducing functions of quadratic momentum $\boldsymbol{f}(\vec{\boldsymbol{p}}\,)^2$as follows:

$$[\hat{\boldsymbol{p}}^i, \hat{\boldsymbol{x}}^j] = -\boldsymbol{i}\hbar\boldsymbol{\delta}^{ij}(\boldsymbol{1} + \boldsymbol{f}(\vec{\boldsymbol{p}}\,)^2). \quad (10)$$

If we consider as an example the following commutator where the quadratic momentum is multiplied by a function, β, then the smallest uncertainty in position that could be related to a minimum length ($\Delta \boldsymbol{x}_{\boldsymbol{min}}$) would be given by [18,19]:

$$[\hat{\boldsymbol{p}}^i, \hat{\boldsymbol{x}}^j] = -\boldsymbol{i}\hbar\boldsymbol{\delta}^{ij}(\boldsymbol{1} + \boldsymbol{\beta}(\vec{\boldsymbol{p}}\,)^2) \ , \ \Delta \boldsymbol{x}_{\boldsymbol{min}} = \hbar\sqrt{\boldsymbol{\beta}} \quad (11)$$

This minimal length can then be related to quadratic length elements on the order of Planck length as shown in inequality (7).

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4. Relativistic formulations of GUP

The second main issue to be solved was the apparent incompatibility between a minimal measurable length and Lorentz co-variance. However, it had already been shown by Snyder that quantizing space-time does not necessarily imply the breaking of Lorentz co-variance [10].

One way to obtain relativistic, Lorentz covariant formulations implies modifications of the commutator relations in Minkowski space-time (equation 5). One first step is its generalization to curved space through a differential local perturbation over the Minkowski metric [38]:

$$[\hat{\boldsymbol{p}}^{\mu}, \hat{\boldsymbol{x}}^{\nu}] = -\boldsymbol{i}\hbar \boldsymbol{g}^{\mu\nu} \quad , \qquad \boldsymbol{g}^{\mu\nu} = \eta^{\mu\nu} + \boldsymbol{h}^{\mu\nu}. \qquad (\mathbf{12})$$

Such perturbation approaches have been used in semi-classical quantum gravity. For example by defining a metric tensor operator decomposed into a pseudo-Riemannian metric tensor plus a fluctuating tensor operator of quantum origin ($\delta\hat{g}_{\mu\nu}$) that can be identified with a classical energy-momentum tensor ($T_{\mu\nu}$) [47]:

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \delta\hat{g}_{\mu\nu} \quad , \quad \langle\delta\hat{g}_{\mu\nu}\rangle \equiv T_{\mu\nu} \, . \qquad (13)$$

The necessity for a fixed, measurable minimal space–time length in quantum gravity theories clashes with Lorentz-covariance, an inherent property of relativity [48,49]. Quantum gravity theories thus operate under a privileged frame of reference, which have restricted the application of GUP mainly to non-relativistic problems. While in some instances, the minimal length in LQG can be considered a free parameter subject to Lorentz covariance [50,51], the need for a covariant formulation for GUP has led to correcting its canonical commutator for Minkowski space [51]. For example, Quesne and Tkachuk generalized Kempf´s deformed commutator algebra in D-dimensions [18,52] to make it Lorentz covariant [53]. In this procedure, the quadratic forms of momentum and products of momentum and position were replaced by their contracted tensor formulations. The resulting commutator algebra is invariant under classical Lorentz transformations, and used to solve the relativistic Dirac oscillator [53,54]:

$$[\hat{p}^{\mu}, \hat{x}^{\nu}] = i\hbar[(1 - \beta p^{\alpha}p_{\alpha})g^{\mu\nu} - \beta' p^{\mu}p^{\nu}] \quad ;$$
$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\hbar(p^{\mu}p^{\nu} - p^{\nu}p^{\mu})\frac{2\beta - \beta' - (2\beta + \beta')\beta p^{\alpha}p_{\alpha}}{1 - \beta p^{\alpha}p_{\alpha}} \quad ;$$
$$[\hat{p}^{\mu}, \hat{p}^{\nu}] = 0. \qquad (14)$$

Where in the context of these equations, β, β' correspond to non-negative deforming parameters. In this modified relativistic GUP the smallest uncertainty in position is given by:

$$(\Delta x^{i})_{min} = \hbar\sqrt{(D\beta + \beta')[1 - \beta\langle(P^{0})^{2}\rangle]}. \qquad (15)$$

Where D corresponds to the number of dimensions.

A similar strategy was undertaken by Todorinov et al to comply with Lorentz covariance in Minkowski space-time [25,51]:

$$[\hat{\boldsymbol{p}}^{\mu}, \hat{\boldsymbol{x}}^{\nu}] = -\boldsymbol{i}\hbar\left(\mathbf{1} + (\boldsymbol{\varepsilon} - \boldsymbol{\alpha})\boldsymbol{\lambda}^{2}\boldsymbol{p}^{\rho}\boldsymbol{p}_{\rho}\right)\boldsymbol{\eta}^{\mu\nu} - \boldsymbol{i}\hbar(\boldsymbol{\beta} + 2\boldsymbol{\xi})\boldsymbol{\lambda}^{2}\boldsymbol{p}^{\mu}\boldsymbol{p}^{\nu}. \qquad (\mathbf{16})$$

Where in the context of this equation α, β, ε and ξ are dimensionless parameters to be adjusted to the specific problem, and λ a parameter with dimensions of inverse momentum. This formulation was applied to three relativistic systems: the Klein-Gordon equation for the hydrogen atom, the Schrödinger equation for a particle in a box and a linear harmonic oscillator, and the Dirac equation [51]. For these examples, GUP corrections were obtained only for the Schrödinger equation.

Recently, an approximation towards a GUP formulation in pseudo-Riemannian curved spaces has recently been proposed, using normal coordinates defined in tangent space as follows [55]:

$$[x^a, \hat{p}_b] = i\hbar(\alpha K^a{}_b - u^a u_b). \tag{17}$$

Where x^a corresponds to normal coordinates; α is a constant; $K^a{}_b$ represents components of the extrinsic curvature tensor associated with the equi-geodesics; u^a, u_b represent contravariant and covariant components of the proper velocity 4-vector.

5. Covariant reformulation of the classical uncertainty principle.

To make the uncertainty principle compatible with GR, we recently tried a generalization of the classical uncertainty principle inequalities strictly from covariant tensor formulations. We assumed that the following (or modified) statement could be a starting point:

$$\left|\Delta P^\mu \Delta x_\mu\right| \geq f(\hbar). \tag{18}$$

Where $f(\hbar)$ represents a function of the reduced Planck constant. Such a formulation would introduce a Lorentz covariant constraint through a contraction of the change in relativistic momentum and position 4-vectors. However, it turned out that such formulation did not recover the two classical inequalities. Hence, we decided to re-express the classical inequalities in a covariant form, allowing its application as a mathematical constraint over GR geodesics [56,57]. This formulation extended the uncertainty inequality to a differential length of relativistic proper space-time line element $(d\tau^2)$ as a function of Planck length, ℓ_p, and a geodesic-related scalar (G_{geo}) as follows:

$$\left|G_{\text{geo}}\, d\tau^2\right| \geq (1+\gamma)\, \ell_p^2\,. \tag{19}$$

where the gamma factor γ and G_{geo} are defined in terms of the total energy of the particle (E), its mass (m) and Christoffel connectors ($\Gamma^{\ \mu}_{\alpha\beta}$) in units of c set to 1:

$$\gamma = \frac{dt}{d\tau} \equiv \frac{E}{m}\ ,$$
$$G_{\text{geo}} \equiv 2Gm\ \left|u_0 \Gamma^{\ 0}_{\alpha\beta}\, u^\alpha u^\beta\right| + 2Gm \left|u_j \Gamma^{\ j}_{\alpha\beta}\, u^\alpha u^\beta\right|. \tag{20}$$

This covariant reformulation of the classical uncertainty principle sets a length limit for the quadratic proper space-time line element. Its application as a constraint to Minkowski space, required the introduction of a time-dependent differential perturbation (ε) to the g_{00} component of the metric [56,57]:

$$g_{00} = \eta_{00} + h_{00} = -1 - \varepsilon(t)\,. \tag{21}$$

This correction to the metric established a limit to the space-time quadratic distance in terms of energy fluctuations ($\dot{\boldsymbol{E}} = \frac{\boldsymbol{dE}}{\boldsymbol{dt}}$) arising from the uncertainty principle as follows:

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$$|d\tau^2| \geq \frac{2c^5\ell_p^2}{G|\dot{E}|}. \quad (22)$$

When applied to the metric of an expanding universe, as represented by the FRW metric [58-60], the quadratic space-time line element was calculated in terms of two functions [57]. The first one derived from energy fluctuations from the uncertainty principle (E_{un}) and the second from the expansion rate of the universe (H_{ex}):

$$E_{un} \equiv u^0 u_o \, \dot{\varepsilon} \quad , \qquad H_{ex} \equiv 6u^1 u_1 H \quad ,$$

$$|d\tau^2| \geq (1+\gamma)\frac{\ell_p^2}{Gp^0|H_{ex} - E_{un}|}. \quad (23)$$

where $\dot{\boldsymbol{\varepsilon}}$ represents energy fluctuations of quantum origin, and the dot indicating derivation by coordinate time. This covariant expression was applied to other geometries, such as that of Schwarzschild´s metric for a point mass. This metric was chosen as it contains a singularity at radial position 0 [59,61-63]. The imposition of the covariant formulation of the classical uncertainty principle defined an exclusion zone around the singularity at R=0 below which no GR geodesic is allowed [56]. This condition ensured a minimal non-zero uncertainty in radial distance right at the singularity which corresponded to:

$$dR^2 = \frac{2Mc}{mu^1}\ell_p^2 \, . \quad (24)$$

where M corresponds to the mass of the black hole. This minimal uncertainty avoids the singularity of the classical Schwarzschild solution to GR field equations. If interpreted as a standard deviation, the average R coordinate position of a particle at the singularity will still be 0, but allowing a radius of uncertainty that would counteract the information paradox at the singularity [31,64]. A calculation of this minimal uncertainty of dR for a stellar mass black hole provides a value within the range of 10^{-15} to 10^{-16} cm. The uncertainty principle had been previously proposed to be the source of a repulsion force that prevents particles from reaching the singularity within the framework of LGQ and string theory. The matter contained within a black hole would form a "fuzzball" [65] or a "Planck star" [66]. The repulsion force by the uncertainty principle as described by LQG would occur when reaching Planck density [67], and this leads the radius of a Planck star to be:

$$R \sim \left(\frac{M}{m_p}\right)^n \ell_p \, . \quad (25)$$

where in the context of this equation, M corresponding to the mass of the Planck star and $\boldsymbol{m_p}$ to Planck mass. Considering scenarios where n=1/3 or 1, the radius of a Planck star would be comprised between 10^{-10} to 10^{-14} cm [66], close to our calculations for a minimal radial uncertainty at the singularity [56].

6. Lorentz invariance violations (LIV) and space-time quantization

Current gravity theories such as string theory and LQG predict Lorentz invariance violations due to the discreet nature of space-time and a minimal measurable length. For example, *in vacuo* dispersion of photons and neutrinos, or deviations of polarization over astronomical distances [1,14] [68-73]. The experimental detection of LIV and the energy scales in which LIVs might be detected could help to either refine quantum gravity theories by setting up appropriate mathematical constraints, or at least discard scenarios incompatible with the experimental data [68]. Detection (or not) of LIVs could help with lattice quantization in LQG, the time problem and the choice for privileged reference frames [8,32,74,75]. If proven, LIVs could demonstrate space-time quantization, and set up the proper length scales and energies for quantum gravity [73].

However, the experimental detection of LIV is controversial. Several studies have attempted to quantify upper limits to LIV constraints [69-72]. Measurement of energy and helicity-dependent photon propagation velocities over astronomical distances could uncover quantum gravity effects such as space quantization [66]. By measuring deviations of GRB 041219A gamma ray burst photons, an upper limit on the vacuum birefringence of 1.1 10^{-14} was estimated which would correspond to spatial volume units of less than 10^{-42} m^3 [71]. While a number of recent studies are reporting LIV violations at different energy orders, other studies estimate very stringent constraints, or even fail to detect LIVs [68,70,71,76,77].

Nevertheless, space-time quantization can be compatible with a minimal, Lorentz-covariant length element, as shown by GUP and other covariant formulations including ours [56,57]. Hence, it is yet unclear whether LIVs could be definitely detected within our current energy scales.

7. Conclusion

The concept of a fixed, measurable minimal length in quantum mechanics was originally brought up as a possible way to counteract troublesome ultraviolet divergences appearing in quantum field theories. However, the implementation of a Lorentz covariant minimal length element that can be applied to curved spaces might be a way forward to unify quantum mechanics and general relativity.

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