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On the Total Edge Irregularity Strength of Odd and Even Staircase Graphs

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Abstract

A total m-labelling α on graph Γ is called an edge irregular total m-labelling if for any two different edges of Γ, their weights respect to α are distinct, where the weight of any edge is defined as the sum of its label and the labels of its end vertices. We determine the minimum m such that Γ can be labelled by an edge irregular total m-labelling whenever Γ is an odd staircase graph or an even staircase graph.

Keywords: total edge irregularity strength, odd staircase graphs, even staircase graphs

Mathematics Subject Classification 2010 : 05C78, 05C85

1 Introduction

Graph labelling plays an important role in the development of graph theory nowadays. A labelling of a graph, also called as valuation, is a function that assigns usually positive or non-negative integers to the graph elements subject to a certain condition. The set of all vertices alone, the set of all edges alone and the set of all vertices and

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altogether all edges are the most common sets taken as domain of labelling. Whenever a labelling has the set of all vertices and all edges as its domain, then the labelling is said to be total. From the most complete recent survey on labelling by Gallian [2] we know that there are various kinds of labelling on graphs. One of well-known labellings is edge irregular total labelling proposed by Bača et.all [1] as follows. Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a connected simple undirected graph with non empty vertex set V_Γ and edge set E_Γ. Bača et al. [1] investigated the weight $wt_\alpha(ab)$ of edge ab under a total m-labelling $\alpha : V_\Gamma \cup E_\Gamma \to \{1, 2, ..., m\}$ defined by

$$wt_\alpha(ab) = \alpha(a) + \alpha(ab) + \alpha(b)$$

for each $ab \in E_\Gamma$. The total labelling α is called an edge irregular total m-labelling if for any two different edges ab and $a'b'$, the weights $wt_\alpha(ab)$ and $wt_\alpha(a'b')$ are not the same. The minimum m such that Γ can be labelled by an edge irregular total m-labelling, denoted by $tes(\Gamma)$, is called the total edge irregularity strength of graph Γ. In general, the total edge irregularity strength of a given graph is not easy to obtain. The following result gives a helpful hint on the lower bound of the total edge irregularity strength of arbitrary graph.

Theorem 1.1 [1] *Let Γ be any graph of $|E_\Gamma|$ edges. If the maximum vertex degree of Γ is Δ_Γ, then*

$$tes(\Gamma) \geq max\left\{\left\lceil\frac{|E_\Gamma|+2}{3}\right\rceil, \left\lceil\frac{\Delta_\Gamma+1}{2}\right\rceil\right\}.$$

Apart from the above theorem, the following conjecture on the exact value of $tes(\Gamma)$ for any graph Γ was presented by Ivanco and Jendrol in [3]

Conjecture 1.2 [3] *For arbitrary graph Γ of $|E_\Gamma|$ edges and of maximum vertex degree Δ_Γ, it follows that*

$$tes(\Gamma) = max\left\{\left\lceil\frac{|E_\Gamma|+2}{3}\right\rceil, \left\lceil\frac{\Delta_\Gamma+1}{2}\right\rceil\right\}.$$

For some classes of graphs, including complete graphs, complete bipartite graphs and trees, it has been proved that the conjecture is true. The total edge irregularity strength of any tree was given by Ivančo and Jendrol ([3]) while the for complete graphs and complete bipartite graphs were presented by Jendrol, et al. [4]. Some other works also presented. Some are given in [6] and [7], where Putra and Susanti presented the tes of centralized uniform theta graphs and the tes of uniform theta graphs, respectively. In [8], Ratnasari and Susanti provided the exact value of total

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edge irregularity strength of some ladder related graphs. In [9], it is given the exact value of the *tes* of some arithmetic book graphs. Moreover, [10] it is also provided the total edge irregularity strength of q-tuplet book graphs. Meanwhile in [5] it is given the *tes* for some double fan related graphs. In [11] it is given the *tes* of some staircase graphs including mirror-staircase and double staircase graphs. In this paper, we consider particular arithmetic staircase graphs, namely odd staircase graphs and even staircase graphs and give the exact value of their total edge irregularity strengths.

2 Main Results

In this section we present the total edge irregularity strength of several classes of graphs. For the first discussion we consider odd staircase graphs. Let us denote the odd staircase graph of level $s \geq 1$ by OSC_s (see Figure 1). For this graph, we have

$$V_{OSC_s} = \{a_{p,q} | p = 0, 1, 2, \ldots, 2s-1, q = \lfloor \frac{p}{2} \rfloor, \ldots, s\}$$

as the vertex set and E_{OSC_s} which consists all edges given on the table below

edges	p	q
$a_{p,q}a_{p+1,q}$	0	$0, 1, 2, \ldots, s$
$a_{p,q}a_{p+1,q}$	$1, \ldots, 2s-2$	$\lceil \frac{p}{2} \rceil, \ldots, s$
$a_{p,q}a_{p,q+1}$	$0, 1$	$0, 1, 2, \ldots, s-1$
$a_{p,q}a_{p,q+1}$	$2, \ldots, 2s-1$	$\lfloor \frac{p}{2} \rfloor, \ldots, s-1.$

It is routine that $|V_{OSC_s}| = s^2 + 3s$ and $|E_{OSC_s}| = 2s^2 + 3s - 1$. The following theorem gives the exact value of *tes* of OSC_s.

Theorem 2.1 *Let OSC_s be the odd staircase graph of $s \geq 1$ level. Then the total edge irregularity strength of OSC_s is*

$$tes(OSC_s) = \left\lceil \frac{2s^2 + 3s + 1}{3} \right\rceil.$$

Proof: Obviously, the maximum degree of the odd staircase graph is 2 for $s = 1$ and 4 for otherwise. Thus, by Theorem (1.1), we have

$$tes(OSC_s) \geq \left\lceil \frac{2s^2 + 3s + 1}{3} \right\rceil.$$

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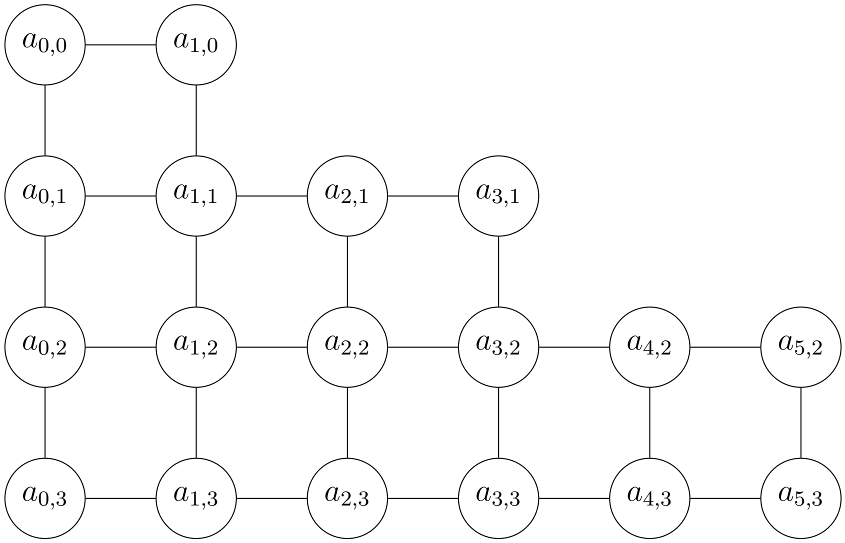

Figure 1: Odd Staircase Graph OSC_3

To prove the upper bound, i.e. $tes(OSC_s) \leq \left\lceil \frac{2s^2+3s+1}{3} \right\rceil$, we are constructing a total edge irregularity m-labelling with $m = \left\lceil \frac{2s^2+3s+1}{3} \right\rceil$. Before we give the labelling, we determine the largest positive integer t such that

$$t^2 \leq \left\lceil \frac{2s^2+3s+1}{3} \right\rceil - 1.$$

(On Table 1 it is given several s's and t's.) Now we define a labelling

$$\alpha_1 : V_{OSC_s} \cup E_{OSC_s} \to \left\{1, 2, \ldots, \left\lceil \frac{2s^2+3s+1}{3} \right\rceil \right\}$$

as follows

label of edges and vertices	p and q
$\alpha_1(a_{p,q}) = q^2 + 1$	$p = 0, 1, \ldots, 2q+1$ $q = 0, 1, 2, \ldots, t$
$\alpha_1(a_{p,q}a_{p+1,q}) = p + q + 1$	$p = 0, 1, \ldots, 2q$ $q = 0, \ldots, t-1$
$\alpha_1(a_{p,q}a_{p+1,q}) = p + q + 1$	$p = 0, 1, \ldots, 2t-2$ $q = t$
$\alpha_1(a_{p,q}a_{p,q+1}) = p + q + 1$	$p = 0, 1, \ldots, 2q+1$ $q = 0, \ldots, t-1.$

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If $t = s$, then the labelling is done. If $t < s$, then we continue to assign labels as the following

label of edges and vertices	p and q
$\alpha_1(a_{p,q}a_{p+1,q}) = p + q + 1$	$p = 2t-1, 2t$ $q = t$
$\alpha_1(a_{p,q}) = \lceil \frac{2s^2+3s+1}{3} \rceil$	$p = 0, \ldots, 2q+1$ $q = t+1, \ldots, s-1$
$\alpha_1(a_{p,q}) = \lceil \frac{2s^2+3s+1}{3} \rceil$	$p = 0, \ldots, 2s-1$ $q = s$
$\alpha_1(a_{p,q}a_{p,q+1}) = p + t^2 + 3t + 3 - \lceil \frac{2s^2+3s+1}{3} \rceil$	$p = 0, \ldots, 2t+1$ $q = t$
$\alpha_1(a_{p,q}a_{p+1,q}) = p + (t+k-1)(2t+2k+3) + 6 - 2\lceil \frac{2s^2+3s+1}{3} \rceil$	$p = 0, \ldots, 2q+1$ $q = t+k$ $k = 1, \ldots, s-t-1$
$\alpha_1(a_{p,q}a_{p,q+1}) = p + (t+k)(2t+2k+3) + 4 - 2\lceil \frac{2s^2+3s+1}{3} \rceil$	$p = 0, 1, \ldots, 2(t+k)+1$ $q = t+k$ $k = 1, \ldots, s-t-1$
$\alpha_1(a_{p,q}a_{p+1,q}) = p + 2s^2 + s + 3 - 2\lceil \frac{2s^2+3s+1}{3} \rceil$	$q = s$ $p = 0, 1, \ldots, 2s-2.$

From the above assignment, we obtain the following edge weights:

weights	p and q
$wt_{\alpha_1}(a_{p,q}a_{p+1,q}) = p + 2q^2 + q + 3$	$p = 0, 1, \ldots, 2q$ $q = 0, 1, 2, \ldots, t$
$wt_{\alpha_1}(a_{p,q}a_{p,q+1}) = p + 2q^2 + 3q + 4$	$p = 0, 1, \ldots, 2q+1$ $q = 1, 2, \ldots, t$
$wt_{\alpha_1}(a_{p,q}a_{p+1,q}) = p + (t+k-1)(2t+2k+1) + 4$	$p = 0, \ldots, 2q$ $q = t+k$ $k = 1, \ldots, s-t-1$
$wt_{\alpha_1}(a_{p,q}a_{p,q+1}) = p + (t+k)(2t+2k+3) + 4$	$p = 0, 1, \ldots, 2(t+k)+1$ $q = t+k$ $k = 1, \ldots, s-t-1$
$wt_{\alpha_1}(a_{p,q}a_{p+1,q}) = p + 2s^2 + s + 3$	$q = s$ $p = 0, 1, \ldots, 2s-2.$

The fact that the weights are all different can be verified as a routine. ■

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Table 1 Several s's and t's such that $t^2 \leq \lceil \frac{2s^2+3s+1}{3} \rceil - 1$

s	t		s	t		s	t		s	t		s	t		s	t
1	1		6	5		11	9		16	13		21	17		26	21
2	2		7	6		12	10		17	14		22	18		27	22
3	3		8	7		13	11		18	15		23	19		28	23
4	3		9	7		14	12		19	16		24	20		29	24
5	4		10	8		15	12		20	16		25	21		30	25

Below we give the tes for the second graph, namely the even staircase graphs ESC_s of level $s \geq 1$. Let ESC_s be the even staircase graph of level $s \geq 1$ (see Figure 2). Let the vertex set be

$$V_{ESC_s} = \{a_{p,q} | p = 0, 1, 2, \ldots, 2s \text{ and } q = \lfloor \frac{p-1}{2} \rfloor, \ldots, s\}.$$

We have that E_{ESC_s} consists of all edges as shown below

edges	p	q
$a_{p,q}a_{p+1,q}$	$0, 1, \ldots, 2s-1$	$\lfloor \frac{p}{2} \rfloor, \ldots, s$
$a_{p,q}a_{p,q+1}$	0	$0, 1, 2, \ldots, s-1$
$a_{p,q}a_{p,q+1}$	$1, 2, \ldots, 2s$	$\lfloor \frac{p-1}{2} \rfloor, \ldots, s-1.$

By a simple counting it can be shown that $|V_{ESC_s}| = s^2+4s+1$ and $|E_{ESC_s}| = 2s^2+5s$. In the following theorem the exact value of $tes(ESC_s)$ is given.

Theorem 2.2 *For any $s \geq 1$, let ESC_s be the odd staircase graph of s level. Then the total edge irregularity strength of ESC_s is*

$$tes(ESC_s) = \left\lceil \frac{2s^2+5s+2}{3} \right\rceil.$$

Proof: Obviously, the maximum degree of ESC_s is 2 for $s = 1$ and is equal to 4 otherwise. Thus we have

$$tes(ESC_s) \geq \left\lceil \frac{2s^2+5s+2}{3} \right\rceil$$

by Theorem (1.1). For the upper bound, we prove that $tes(ESC_s) \leq \left\lceil \frac{2s^2+5s+2}{3} \right\rceil$, by showing that there exists a total edge irregularity m-labelling with $m = \left\lceil \frac{2s^2+5s+2}{3} \right\rceil$. For the first step, we determine the biggest positive integer t such that

$$t(t+1) \leq \left\lceil \frac{2s^2+5s+2}{3} \right\rceil - 1.$$

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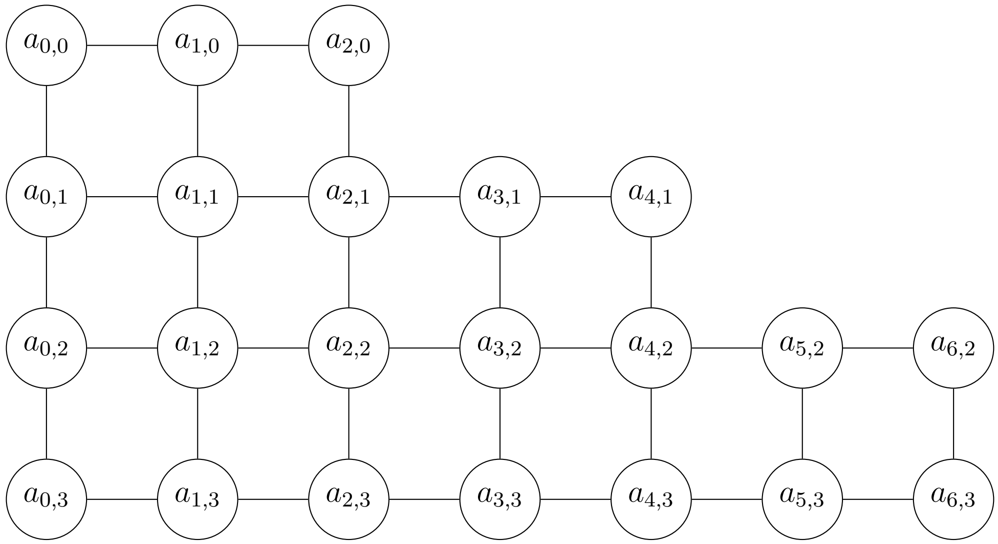

Figure 2: Even Staircase Graph ESC_3

(Several s's and t's are listed on Table 2.) We then construct a total m-labelling

$$\alpha_2 : V_{ESC_s} \cup E_{ESC_s} \to \left\{1, 2, \ldots, \left\lceil \frac{2s^2+5s+2}{3} \right\rceil \right\}$$

with $m = \left\lceil \frac{2s^2+5s+2}{3} \right\rceil$ in the following way:

edges and vertices label	p and q
$\alpha_2(a_{p,q}) = q^2 + q + 1$	$p = 0, 1, \ldots, 2q+2$ $q = 0, 1, \ldots, t$
$\alpha_2(a_{p,q}a_{p+1,q}) = p + q + 1$	$p = 0, 1, \ldots, 2q+1$ $q = 0, \ldots, t$
$\alpha_2(a_{p,q}a_{p,q+1}) = p + q + 1$	$p = 0, 1, \ldots, 2q+2$ $q = 0, \ldots, t-1.$

We stop the process whenever $t = s$. For the case $t < s$, we continue with the following

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assignment

edges and vertices label	p and q
$\alpha_2(a_{p,q}a_{p+1,q}) = p+q+1$	$p=2t, 2t+1$ $q=t$
$\alpha_2(a_{p,q}) = \lceil \frac{2s^2+5s+2}{3} \rceil$	$p=0,1,\ldots,2q+2$ $q=t+1,\ldots,s-1$
$\alpha_2(a_{p,q}) = \lceil \frac{2s^2+5s+2}{3} \rceil$	$p=0,1\ldots,2$ $q=s$
$\alpha_2(a_{p,q}a_{p,q+1}) = p+t^2+4t+4-\lceil \frac{2s^2+5s+2}{3} \rceil$	$p=0,1,\ldots,2t+2$ $q=t$
$\alpha_2(a_{p,q}a_{p+1,q}) = p+(t+k-1)(2t+2k+5)+8-2\lceil \frac{2s^2+5s+2}{3} \rceil$	$p=0,\ldots,2q+1$ $q=t+k$ $k=1,\ldots,s-t-1$
$\alpha_2(a_{p,q}a_{p,q+1}) = p+(t+k)(2t+2k+5)+5-2\lceil \frac{2s^2+5s+2}{3} \rceil$	$p=0,1,\ldots,2(t+k+1)$ $q=t+k$ $k=1,\ldots,s-t-1$
$\alpha_2(a_{p,q}a_{p+1,q}) = p+(s-1)(2s+5)+8-2\lceil \frac{2s^2+5s+2}{3} \rceil$	$q=s$ $p=0,1,\ldots,2s-1.$

Table 2 Several s's and t's such that $t(t+1) \leq \lceil \frac{2s^2+5s+2}{3} \rceil - 1$

s	t		s	t		s	t		s	t		s	t		s	t
1	1		6	5		11	9		16	13		21	17		26	21
2	2		7	6		12	10		17	14		22	18		27	22
3	2		8	7		13	11		18	15		23	19		28	23
4	3		9	7		14	11		19	16		24	20		29	24
5	4		10	8		15	12		20	16		25	20		30	25

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We then have the weights of the edges as follows:

weights	p and q
$wt_{\alpha_2}(a_{p,q}a_{p+1,q}) = p + 2q^2 + 3q + 3$	$p = 0, 1, \ldots, 2q + 1$ $q = 0, 1, \ldots, t$
$wt_{\alpha_2}(a_{p,q}a_{p,q+1}) = p + 2q^2 + 5q + 5$	$p = 0, 1, \ldots, 2q + 2$ $q = 1, 2, \ldots, t$
$wt_{\alpha_2}(a_{p,q}a_{p+1,q}) = p + (t + k - 1)(2t + 2k + 5) + 8$	$p = 0, 1, \ldots, 2q + 1$ $q = t + k$ $k = 1, \ldots, s - t - 1$
$wt_{\alpha_2}(a_{p,q}a_{p,q+1}) = p + (t + k)(2t + 2k + 5) + 5$	$p = 0, 1, \ldots, 2(t + k) + 1$ $q = t + k$ $k = 1, \ldots, s - t - 1$
$wt_{\alpha_2}(a_{p,q}a_{p+1,q}) = p + 2s^2 + 3s + 3$	$q = s$ $p = 0, 1, \ldots, 2s - 1$

The weights constitute numbers from 3 up to $2s^2 + 5s + 2$ and all different. This completes the proof. ■

For the third observation, we consider the double odd staircase graph of level $s \geq 1$ denoted by $DOSC_s$ (see Figure 3). We have

$$\begin{aligned} V_{DOSC_s} =& \{l_{p,q} | p = 1, 2, \ldots, 2s - 1, q = \lceil \frac{p-1}{2} \rceil, \ldots, s\} \cup \{r_{p,q} | p = 1, 2, \ldots, 2s - 1, \\ & q = \lceil \frac{p-1}{2} \rceil, \ldots, s\} \end{aligned}$$

and E_{DOSC_s} which consists of edges as given below

edges	p	q
$l_{p,q}r_{p,q}$	1	$0, 1, 2, \ldots, s$
$l_{p+1,q}l_{p,q}$	$1, 2, \ldots, 2s - 2$	$p, \ldots, 2s - 1$
$r_{p,q}r_{p+1,q}$	$1, 2, \ldots, 2s - 2$	$p, \ldots, 2s - 1$
$l_{p,q}l_{p,q+1}$	$1, 2, \ldots, 2s - 1$	$\left\lceil \frac{p-1}{2} \right\rceil, \ldots, s - 1$
$r_{p,q}l_{p,q+1}$	$1, 2, \ldots, 2s - 1$	$\left\lceil \frac{p-1}{2} \right\rceil, \ldots, s - 1.$

By a routine counting we have that $|V_{DOSC_s}| = 2s^2 + 4s - 2$ and $|E_{DOSC_s}| = 4s^2 + 3s - 3$.

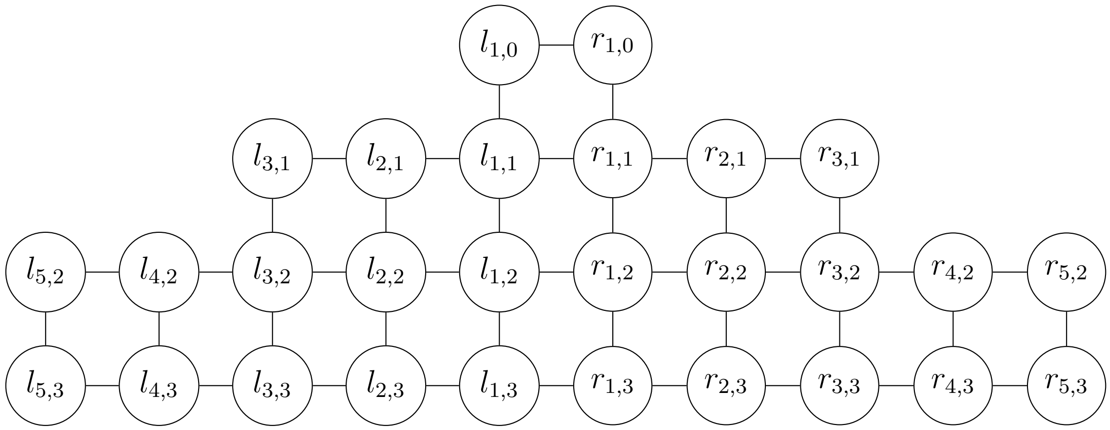

Figure 3: Double Odd Staircase Graph $DOSC_3$

In the following theorem we give the exact value of tes of $DOSC_s$ for any $s \geq 1$.

Theorem 2.3 *Let $DOSC_s$ be the double odd staircase graph of $s \geq 1$ level. Then the total edge irregularity strength of $DOSC_s$ is*

$$tes(DOSC_s) = \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil.$$

Proof: It is easy to observe that the maximum degree of the double odd staircase graph is 2 or 4, for $s = 1$ an $s \geq 2$, respectively. Therefore, by Theorem 1.1, we have

$$tes(DOSC_s) \geq \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil.$$

To complete the prove it is sufficient to show that $tes(DOSC_s) \leq \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil$ by defining a total edge irregularity m-labelling with $m = \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil$. Prior, we determine the largest positive integer t such that

$$2t^2 - t + 1 \leq \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil.$$

Table 3 Several s's and t's such that $2t^2 - t + 1 \leq \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil$

s	t		s	t		s	t		s	t		s	t
1	1		6	5		11	9		16	13		21	17
2	2		7	6		12	10		17	14		22	18
3	2		8	7		13	11		18	15		23	19
4	3		9	7		14	11		19	16		24	20
5	4		10	8		15	12		20	16		25	20

We define a total m-labelling

$$\alpha_3 : V_{DOSC_s} \cup E_{DOSC_s} \rightarrow \left\{1, 2, \ldots, \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil \right\}$$

with $m = \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil$ by the following definition

edges and vertices label	p and q
$\alpha_3(l_{p,q}r_{p,q}) = 1$	$p = 1$ $q = 0$
$\alpha_3(l_{1,q}r_{1,q}) = 3q + 1$	$1 \leq q \leq s$
$\alpha_3(l_{p,q}) = 2q^2 - q + 1$	$p = 1, \ldots, 2q + 1$ $q = 1, 2, \ldots, t$
$\alpha_3(r_{p,q}) = 2q^2 - q + 1$	$p = 1, \ldots, 2q + 1$ $q = 1, 2, \ldots, t$
$\alpha_3(l_{p,q}l_{p+1,q}) = -p + 3q + 1$	$p = 1, \ldots, 2q$ $q = 1, 2, \ldots, t - 1$
$\alpha_3(l_{p,q}l_{p+1,q}) = -p + 3q + 1$	$p = 1, \ldots, 2t - 2$ $q = t$
$\alpha_3(l_{p,q}l_{p,q+1}) = -p + 3q + 2$	$p = 1, \ldots, 2q + 1$ $q = 1, 2, \ldots, t - 1$
$\alpha_3(r_{p,q}r_{p+1,q}) = p + 3q + 1$	$p = 1, \ldots, 2q$ $q = 1, 2, \ldots, t - 1$
$\alpha_3(r_{p,q}r_{p+1,q}) = p + 3q + 1$	$p = 1, \ldots, 2t - 2$ $q = t$
$\alpha_3(r_{p,q}r_{p,q+1}) = p + 3q + 1$	$p = 1, \ldots, 2q + 1$ $q = 1, 2, \ldots, t - 1.$

The labelling is complete in the case $t = s$. If $t < s$, we continue the labelling as follows

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edges and vertices label	p and q
$\alpha_3(l_{p,q}l_{p+1,q}) = -p+3q+1$	$p=2t-1, 2t$ $q=t$
$\alpha_3(r_{p,q}r_{p+1,q}) = p+3q+1$	$p=2t-1, 2t$ $q=t$
$\alpha_3(l_{p,q}) = \left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p=1,2,\ldots,2q+1$ $q=t+1,\ldots,s$
$\alpha_3(r_{p,q}) = \left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p=1,2,\ldots,2q+1$ $q=t+1,\ldots,s$
$\alpha_3(l_{p,q}l_{p,q+1}) = 2t^2+6t+4-i-\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p=1,\ldots,2p+1$ $q=t$
$\alpha_3(l_{p,q}l_{p+1,q}) = 4(t+k)^2+(t+k)-p+3-2\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p=1,\ldots,2q$ $q=t+k$ $k=1,2,\ldots,s-t-1$
$\alpha_3(l_{p,q}l_{p,q+1}) = 4(t+k)^2+5(t+k)-p+5-2\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p=1,\ldots,2q+1$ $q=t+k$ $k=1,2,\ldots,s-t-1$
$\alpha_3(l_{p,q}l_{p+1,q}) = 4s^2+s-p+1-2\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p=1,\ldots,2q-2$ $q=s$
$\alpha_3(r_{p,q}r_{p,q+1}) = 2t^2+2t+2+p-\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p=1,\ldots,2q+1$ $q=t$
$\alpha_3(r_{p,q}r_{p+1,q}) = 4(t+k)^2+(t+k)+p+3-2\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p=-(2q+1),\ldots,2q$ $q=t+k$ $k=1,2,\ldots,s-t-1$
$\alpha_3(r_{p,q}r_{p,q+1}) = 4(t+k)^2+5(t+k)+p+4-2\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p=1,\ldots,2q+1$ $q=t+k$ $k=1,2,\ldots,s-t-1$
$\alpha_3(r_{p,q}r_{p+1,q}) = 4s^2+s+p+1-2\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p=1,\ldots,2q-2$ $q=s.$

We obtain edge weights as follows:

weight	p and q
$wt_{\alpha_3}(l_{p,q}r_{p,q}) = 3$	$p = 1$ $q = 0$
$wt_{\alpha_3}(l_{p,q}r_{1,q}) = 4q^2 + q + 3$	$1 \leq q \leq s$
$wt_{\alpha_3}(l_{p,q}l_{p+1,q}) = 4q^2 + q - p + 3$	$p = 1, 2, \ldots, 2q$ $q = 1, 2, \ldots, t$
$wt_{\alpha_3}(l_{p,q}l_{p,q+1}) = 4q^2 + 5q - p + 5$	$p = 1, 2, \ldots, 2q + 1$ $q = 1, 2, \ldots, t$
$wt_{\alpha_3}(l_{p,q}l_{p+1,q}) = 4(m + k)^2 + (m + k) - p + 3$	$p = 1, \ldots, 2q$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$wt_{\alpha_3}(l_{p,q}l_{p,q+1}) = 4(t + k)^2 + 5(t + k) - p + 5$	$p = 1, \ldots, 2q + 1$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$wt_{\alpha_3}(l_{p,q}l_{p+1,q}) = 4s^2 + s - p + 1$	$p = 1, \ldots, 2q - 2$ $q = s$
$wt_{\alpha_3}(r_{p,q}r_{p+1,q}) = 4q^2 + q + p + 3$	$p = 1, 2, \ldots, 2q$ $q = 1, 2, \ldots, t$
$wt_{\alpha_3}(r_{p,q}r_{p,q+1}) = 4q^2 + 5q + p + 4$	$p = 1, 2, \ldots, 2q$ $q = 1, 2, \ldots, t$
$wt_{\alpha_3}(r_{p,q}r_{p+1,q}) = 4(t + k)^2 + (t + k) + p + 3$	$p = 1, \ldots, 2q$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$wt_{\alpha_3}(r_{p,q}r_{p,q+1}) = 4(t + k)^2 + 5(t + k) + p + 4$	$p = 1, \ldots, 2q + 1$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$ $k \geq 1$
$wt_{\alpha_3}(r_{p,q}r_{p+1,q}) = 4s^2 + s + p + 1$	$p = 1, \ldots, 2q - 2$ $q = s.$

It is a routine to verify that all weights are distinct. Therefore the theorem is confirmed to be true. ■

We now come to the last graph to observe, i.e. the mirror odd staircase graph of level

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$s \geq 1$ denoted by $MOSC_s$ (see Figure 4). We have

$$V_{MOSC_s} = \{a_{p,q} | p = -1, 0, 1, q = 0, 1, 2, \dots, s\} \cup \{a_{p,q} | p = 2, \dots, 2s-1, q = \left\lceil \frac{p-1}{2} \right\rceil, \dots, s\}$$
$$\cup \{a_{p,q} | p = -2, \dots, -(2s-1), q = -\left\lceil \frac{p-1}{2} \right\rceil, \dots, s\}$$

and E_{MOSC_s} which consists of edges as given below

edges	p	q
$a_{p,q}a_{p+1,q}$	$-1, 0$	$0, 1, 2, \dots, s$
$a_{p,q}a_{p+1,q}$	$1, 2, \dots, 2s-2$	$\left\lceil \frac{p}{2} \right\rceil, \dots, s$
$a_{p,q}a_{p+1,q}$	$-(2s-1), \dots, -2$	$-\left\lceil \frac{p}{2} \right\rceil, \dots, s$
$a_{p,q}a_{p,q+1}$	$-1, 0, 1$	$0, 1, \dots, s-1$
$a_{p,q}a_{p,q+1}$	$2, \dots, 2s-1$	$\left\lceil \frac{p-1}{2} \right\rceil, \dots, s-1$
$a_{p,q}a_{p,q+1}$	$-(2s-1), \dots, -2$	$-\left\lceil \frac{p-1}{2} \right\rceil, \dots, s-1.$

It is easy to check that $|V_{MOSC_s}| = 2s^2 + 5s - 1$ and $|E_{MOSC_s}| = 4s^2 + 5s - 2$.

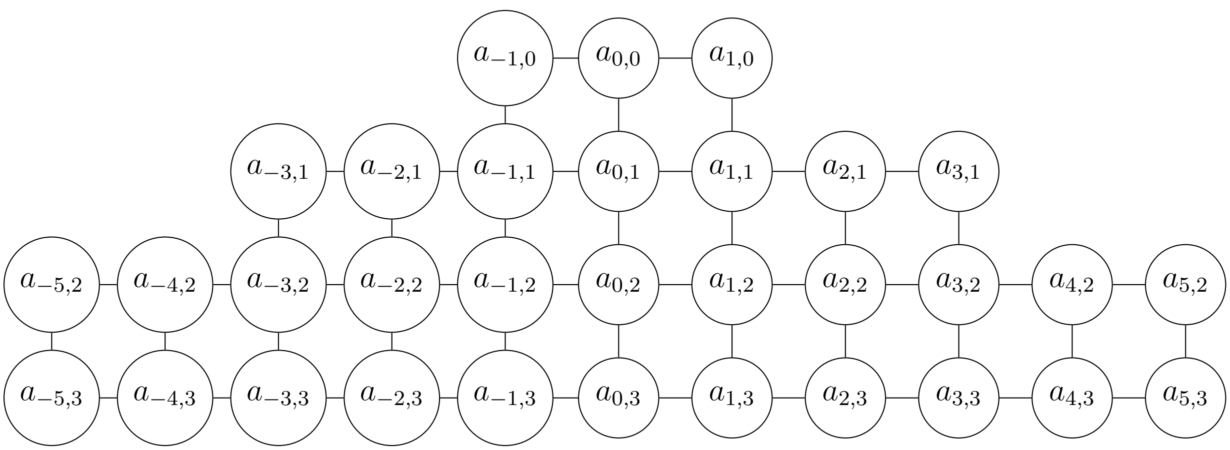

Figure 4: Mirror Odd Staircase Graph $MOSC_3$

Theorem 2.4 *Let for any $s \geq 1$, $MOSC_s$ be the mirror odd staircase graph of s level. Then the total edge irregularity strength of $MOSC_s$ is*

$$tes(MOSC_s) = \left\lceil \frac{4s^2 + 5s}{3} \right\rceil.$$

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Proof: It is clear that the maximum degree of $MOSC_3$ is 3 for $s=1$ and 4 for $s\neq 1$. Thus, we obtain

$$tes(MOSC_s) \geq \left\lceil \frac{4s^2+5s}{3} \right\rceil.$$

For completing the proof we show that $tes(MOSC_s) \leq \left\lceil \frac{4s^2+5s}{3} \right\rceil$ by constructing a total edge irregularity m-labelling with $m = \left\lceil \frac{4s^2+5s}{3} \right\rceil$. Similarly to the previous graphs, before we define the labelling, we determine the largest positive integer t such that

$$2t^2+1 \leq \left\lceil \frac{4s^2+5s}{3} \right\rceil.$$

Table 4 Several s's and t's such that $2t^2+1 \leq \left\lceil \frac{4s^2+5s}{3} \right\rceil$

s	t		s	t		s	t		s	t		s	t
1	1		6	5		11	9		16	13		21	17
2	2		7	6		12	10		17	14		22	18
3	2		8	7		13	11		18	15		23	19
4	3		9	7		14	11		19	16		24	20
5	4		10	8		15	12		20	16		25	20

We define a total m-labelling

$$\alpha_4 : V_{MOSC_s} \cup E_{MOSC_s} \rightarrow \left\{1, 2, \ldots, \left\lceil \frac{4s^2+5s}{3} \right\rceil \right\}$$

with $m = \left\lceil \frac{4s^2+5s}{3} \right\rceil$ in the following manner:

edges and vertices label	p and q
$\alpha_4(a_{p,0}) = 1$	$q = -1, 0, 1$
$\alpha_4(a_{p,q}) = 2q^2+1$	$p = -(2q+1), \ldots, 2q+1$ $q = 1, 2, \ldots, t$
$\alpha_4(a_{p,q}a_{p+1,q}) = p+3q+2$	$p = -(2q+1), \ldots, 2q$ $q = 0, \ldots, t-1$
$\alpha_4(a_{p,q}a_{p+1,q}) = p+3q+2$	$p = -(2q+1), \ldots, 2q-2$ $q = t$
$\alpha_4(a_{p,q}a_{p,q+1}) = p+3q+2$	$p = -(2q+1), \ldots, 2q+1$ $q = 0, \ldots, t-1.$

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We stop the labelling whenever $t = s$. For the case $t < s$, then we continue with the following labels

edges and vertices label	p and q
$\alpha_4(a_{p,q}a_{p+1,q}) = p + 3q + 2$	$p = 2t - 1, 2t$ $q = t$
$\alpha_4(a_{p,q}) = \left\lceil \frac{4s^2 + 5s}{3} \right\rceil$	$p = -(2q + 1), \ldots, 2q + 1$ $q = t + 1, \ldots, s$
$\alpha_4(a_{p,q}a_{p,q+1}) = p + (2t + 3)(t + 2) - 1 - \left\lceil \frac{4s^2 + 5s}{3} \right\rceil$	$p = -(2q + 1), \ldots, 2q + 1$ $q = t$
$\alpha_4(a_{p,q}a_{p+1,q}) = p + 4(t + k)^2 + 3(t + k) + 4 - 2\left\lceil \frac{4s^2 + 5s}{3} \right\rceil$	$p = -(2q + 1), \ldots, 2q$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$\alpha_4(a_{p,q}a_{p,q+1}) = p + 4(t + k)^2 + 7(t + k) + 6 - 2\left\lceil \frac{4s^2 + 5s}{3} \right\rceil$	$p = -(2q + 1), \ldots, 2q + 1$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$\alpha_4(a_{p,q}a_{p+1,q}) = p + 4s^2 + 3s + 2 - 2\left\lceil \frac{4s^2 + 5s}{3} \right\rceil$	$p = -(2q - 1), \ldots, 2q - 2$ $q = s.$

From the definition of α_4 we obtain the weight of all edges as follows:

weight	p and q
$wt_{\alpha_4}(a_{p,q}a_{p+1,q}) = p + 4$	$p = -1, 0$ $q = 0$
$wt_{\alpha_4}(a_{p,q}a_{p+1,q}) = p + 4q^2 + 3q + 4$	$p = -(2q+1), \ldots, 2q$ $q = 1, 2, \ldots, t$
$wt_{\alpha_4}(a_{p,q}a_{p+1,q}) = p + 4(t+k)^2 + 3(t+k) + 4$	$p = -(2q+1), \ldots, 2q$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$wt_{\alpha_4}(a_{p,q}a_{p,q+1}) = p + 6$	$p = -1, 0, 1$ $q = 0$
$wt_{\alpha_4}(a_{p,q}a_{p,q+1}) = p + 4q^2 + 7q + 6$	$p = -(2q+1), \ldots, 2q + 1$ $q = 1, 2, \ldots, t$
$wt_{\alpha_4}(a_{p,q}a_{p,q+1}) = p + 4(t+k)^2 + 7(t+k) + 6$	$p = -(2q+1), \ldots, 2q + 1$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$wt_{\alpha_4}(a_{p,q}a_{p+1,q}) = p + 4s^2 + 3s + 2$	$p = -(2q-1), \ldots, 2s - 2$ $q = s.$

It can be verified in a routine way that the weights of all edges in $E(MOSC_s)$ are all different. Hence the theorem is proved. ■

3 Conslusion

From Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4. we conclude that the $tes(\Gamma)$ for $\Gamma = OSC_s, ESC_S, DOSC_s, MOSC_s$, is equal to $\left\lceil \frac{|E_\Gamma|+2}{3} \right\rceil$. These results obviously support the conjecture of Ivanco and Jendrol [3].

References

[1] Bača, M., Jendrol, S., Miller, M., Ryan, J. On irregular total labelling. *Discrete Math*, **307**, 1378-1388 (2007).

[2] Gallian, J.A. A dynamic survey of graph labelling. *The Electronic Journal of Combinatorics*, **18**, 247-252 (2015).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 August 2022 doi:10.20944/preprints202208.0259.v1

[3] Ivančo, J., Jendrol, S., The total edge irregularity strength of trees, *Discuss. Math. Graph Theory*, 26 pp. 449-456 (2006).

[4] Jendrol, S., Miskuf, J., Sotak, R., Total edge irregularity strength of complete graphs and complete bipartite graphs, *Discrete Mathematics*, Volume 310, Issue 3, Pages 400-407 (2010).

[5] Khatimah, H., Susanti, Y., On Total Edge Irregularity Strength of Some Graphs Related to Double Fan Graphs, *Journal of the Indian Mathematical Society*, Vol. 87(1-2), (2020).

[6] Putra, R.W., Susanti, Y., On total edge irregularity strength of centralized uniform theta graphs, *AKCE International Journal of Graphs and Combinatorics*, volume 15 issue 1 page 7-13. (2018)

[7] Putra, R.W., Susanti, Y., The total edge irregularity strength of uniform theta graphs, *Journal of Physics: Conference Series*, **1097** 012069 (2018).

[8] Ratnasari L., Susanti, Y., Total edge irregularity strength of ladder related graphs, *Asian-European Journal of Mathematics*, Vol. 13 Issue 04 (June 2020).

[9] Ratnasari, L., Wahyuni, S., Susanti, Y., Junia Eksi Palupi, D., Surodjo, B., Total Edge Irregularity Strength of Arithmetic Book Graphs, *Journal of Physics: Conference Series* **1306**(1), 012032 (2019).

[10] Ratnasari, L., Wahyuni, S., Susanti,. Y., Palupi, D.J.E., 2021, Total Edge Irregularity Strength of q Tuple Book Graphs, Journal of Mathematics Research, Canadian Center of Science and Education, vol. 13(1), pages 1-16

[11] Susanti, Y., Puspitasari, Y.I., Khotimah, H., On Edge Irregularity Strength of Staircase Graphs and Related Graphs, *Iranian Journal of Mathematical Sciences and Informatics*, Volume 15, Issue 1 (4-2020)