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Effect of the Trip-Length Distribution on Network-Level Traffic Dynamics: Exact and Statistical Results

Jorge A. Laval[a,*]

[a]*School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, United States*
Correspondence: jlaval3@ce.gatech.edu

Abstract

This paper presents additional results of the generalized bathtub model for urban networks, including a simpler derivation and exact solutions for uniformly distributed trip lengths. It is shown that in steady state this trip-based model is equivalent to the more parsimonious accumulation-based model, and that the trip-length distribution has merely a transient effect on traffic dynamics, which converge to the same point in the macroscopic fundamental diagram (MFD). To understand the statistical properties of the system, a queueing approximation method is proposed to compute the network accumulation variance. It is found that (i) the accumulation variance is much larger than predicted by traditional queueing models, due to the nonlinear dynamics imposed by the MFD, (ii) the trip-length distribution has no effect on the accumulation variance, indicating that the proposed formula for the variance might be universal, (iii) the system exhibits critical behavior near the capacity state where the variance diverges to infinity. This indicates that the tools from critical phenomena and phase transitions might be useful to understand congestion in cities.

Keywords: Bathtub model; Trip-length distribution; Macroscopic fundamental diagram

1. Introduction

The aggregated modeling of urban networks has a long history dating back to Godfrey (1969), and numerous theories and models have been published since; see Johari et al. (2021) for a recent review and reference therein. The starting point for all macroscopic urban models is the reservoir (or bathtub) model for cities, which simply states the conservation of $Q(t)$, the number of vehicles inside the network at time t:

$$Q'(t) = \lambda(t) - \mu(t) \tag{1}$$

where $\lambda(t), \mu(t)$ give the inflow and outflow of the network, respectively, in units of flow. The main assumption in these models is that the speed of vehicles inside the network at time t, $v(t)$, is identical for all vehicles and given by the speed macroscopic fundamental diagram (speed-MFD):

$$v(t) = V(Q(t)) \tag{2}$$

Unlike the inflow $\lambda(t)$, which can be easily determined, the network outflow $\mu(t)$ can be difficult to formalize due to its convoluted dependence on the probability distribution of trip lengths. Depending on the assumptions to formulate the outflow function $\mu(t)$, the literature can be divided into accumulation-based (Daganzo, 2007, Geroliminis, 2009, Yildirimoglu et al., 2015) and trip-based models (Arnott, 2013, Daganzo and Lehe, 2015, Leclercq et al., 2017, Lamotte and Geroliminis, 2018, Mariotte et al., 2017, Mariotte and Leclercq, 2019, Leclercq and Paipuri, 2019,

*Corresponding author
Email address: `jorge.laval@ce.gatech.edu` (Jorge A. Laval)
[1]790 Atlantic Dr NW, Atlanta, GA, 30313

Batista et al., 2019, Paipuri et al., 2019). Accumulation-based models assume that the outflow is given by the network production divided by the average trip length, ℓ:

$$\mu = Q\, V(Q)/\ell. \qquad \text{(accumulation-based model)} \tag{3}$$

This expression for the outflow was first presented in Daganzo (2007) assuming a constant trip length, but it turns out to have its origins in Vickrey (1991) although it was not published until Vickrey (2020) as described in Jin (2020). As pointed out in this last reference, Vickrey (1991) already shows that (3) follows from assuming exponentially distributed trip lengths, an important observation that surfaced in the literature not until Arnott (2013) who considers it an unrealistic assumption. Leclercq et al. (2015) shows that the accumulation-based models suffer from significant numerical viscosity even when the time step is small. Outflows may then overreact to sudden demand surge leading to inconsistent information propagation between opposite perimeter boundaries. They also point out that the assumption of constant trip length for all vehicles inside a reservoir is not consistent with what is observed when the local dynamics are taken into account. This has prompted several extensions of the accumulation-based model, including multi-reservoir partitions with explicit calculation of trip distances, possibly as a function of congestion levels (Batista et al., 2021, Yildirimoglu and Geroliminis, 2014, Batista et al., 2019, Batista and Leclercq, 2019, Yildirimoglu et al., 2018, Ramezani et al., 2015).

Trip-based reservoir models have been proposed that guarantee that all vehicles travel their own trip length, at the expense of mathematical tractability. Originally proposed by Arnott (2013) on a footnote, these models recognize that each commuter i has different trip lengths, L_i, which has to equal the distance traveled inside the network during their trip time $T_i(t)$ at the prevailing speeds:

$$L_i = \int_t^{t+T_i(t)} V(Q(s))\, ds. \qquad \text{(trip length)} \tag{4}$$

The resulting outflow function was proposed implicitly in Lamotte and Geroliminis (2018) without proof, arguing that it is a direct consequence of Eq. (4) when the system is initially empty:

$$\mu(t) = V(Q(t)) \int_0^t \lambda(s) f_s\left(\int_s^t V(Q(u))\, du\right) ds, \qquad \text{(trip-based model)} \tag{5}$$

where $f_s(x)$ is the probability density of trip lengths of vehicles entering the network at time s. Jin (2020) introduces one spatial dimension to the problem, x, the *remaining distance* to reach the destination, and shows that when the distribution of trip lengths f in Eq. (5) is exponential with mean ℓ, we obtain the accumulation-based model Eq. (3). All attempts in the literature to solve the trip-based model for other distributions have been numerical using discrete-event simulation (Mariotte et al., 2017, Lamotte and Geroliminis, 2018). To the best of our knowledge, with the exception of Jin (2020), no other attempts have been made to develop analytical solutions to the trip-based model.

To fill this gap, section 2 below uses the framework proposed in Jin (2020) to derive analytical solutions of the trip-based model and to characterize its steady state, which coincides with the accumulation-based model independently of the trip-length distribution. Perhaps a more important gap in the literature is that the stochastic nature of arrival flows has been completely neglected, at least when it comes to analytical formulations; see Johari et al. (2021). Section 3 fills this gap by drawing the analogy with the M/G/∞ queue and shows that accumulations exhibit a much larger variance than predicted by the M/G/∞ queue, due to the nonlinear dynamics imposed by the MFD. Finally, discussion and outlook are presented in section 4.

2. Jin's formulation: Alternative derivation and additional results

The framework proposed in Jin (2020) introduces one spatial dimension to the problem, x, such that a vehicle trajectory $x_i(t)$ gives the *remaining distance* to reach commuter i's destination, and $x_i'(t) = -v(t)$. By considering the probability density function of remaining trip distances, Jin (2020) derives the conservation law that unifies existing reservoir models. Alternatively, one can simply note that these vehicle trajectories must obey the conservation law partial differential equation (PDE):

$$\frac{\partial}{\partial t} k(t,x) + \frac{\partial}{\partial x} q(t,x) = \Lambda(t,x) \tag{6}$$

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where $k(t,x)$ is the density of vehicles in the region at time t whose remaining distance is x, $q(t,x)$ is the corresponding flow, and the source term $\Lambda(t,x)$ is the net inflow to the system in units of flow per km. As opposed to regular traffic, in our problem *(i)* there is no fundamental diagram relating flow to density and therefore Eq. (6) it is not a hyperbolic conservation law such as the kinematic wave model. Instead, given assumption Eq. (2) above, the flow is given by the fundamental traffic flow relationship:

$$q(t,x) = v(t)\cdot k(t,x), \tag{7}$$

which turns the conservation law Eq. (6) into the transport PDE:

$$\frac{\partial}{\partial t}\,k(t,x) - v(t)\frac{\partial}{\partial x}\,k(t,x) = \Lambda(t,x) \tag{8}$$

where the negative sign follows from $x_i'(t) = -v(t)$. Other important differences compared to regular traffic are that *(ii)* the spacing between vehicle trajectories is immaterial because these vehicles could be in different links of the network; *(iii)* $\Lambda(t,x), x > 0$ only includes arrivals to the region because all departures take place at $x = 0$. It follows from *(iii)* that the outflow from the region is simply the flow at $x = 0$, i.e. $\mu(t) = q(t,0)$, or equivalently using Eq. (7)[2]:

$$\mu(t) = v(t)\cdot k(t,0) \qquad \text{(generalized bathtub model)} \tag{10}$$

It follows that all we need to solve the model is $k(t,0)$ but unfortunately its analytical derivation quickly becomes intractable. For clarity, let

$$z(t) \equiv \int_0^t V(Q(u))\,du \tag{11}$$

and the distance traveled by a vehicle during time interval (s,t):

$$z_s^t \equiv z(t) - z(s) = \int_s^t V(Q(u))\,du. \tag{12}$$

The initial value problem (IVP) solution to the transport PDE Eq. (8) is well known in the literature. Given initial densities $k(0,x) = g(x)$, the solution can be expressed as:

$$k(t,x) = g(x + z(t)) + \int_0^t \lambda(s) f_s\left(x + z_s^t\right)\,ds, \tag{13a}$$

$$z'(t) = V(Q(t)), \tag{13b}$$

$$Q(t) = \int_0^\infty k(t,x)\,dx \tag{13c}$$

which cannot be solved analytically in our case. However, it can be used to show the following:

Proposition 2.1. *Lamotte and Geroliminis's (Lamotte and Geroliminis, 2018) conjecture in Eq. (5) can now be proven.*

Proof. From Eq. (13a) we see that for $x = 0$ and empty initial conditions:

$$k(t,0) = \int_0^t \lambda(s) f_s\left(z_s^t\right)\,ds, \tag{14}$$

which in combination with Eq. (10) gives the desired result. □

[2]This result can also be obtained by integrating the conservation law Eq. (8) for all trip distances, which gives the generalized bathtub model:

$$Q'(t) = \lambda(t) - v(t)\cdot k(t,0) \tag{9}$$

after noting that $\lambda(t) = \int_0^\infty \Lambda(t,x)\,dx$, $\int_0^\infty \partial k(t,x)/\partial x\;\,dx = k(t,\infty) - k(t,0)$ and that $k(t,\infty) = 0$.

2.1. Analytical solutions for $Q(t)$

Although Eq. (13) cannot be solved analytically for $k(t,x)$, we have been able to find explicit solutions for $Q(t) = \int_0^\infty k(t,x)\,dx$ for exponential and uniform trip length distributions under a time-independent conditions $\lambda(t) = \lambda, f_s(\cdot) = f(\cdot)$ and a Greenshields speed-accumulation relationship, $V(Q) = 1 - Q$, where we assume that both the free-flow speed and jam accumulation are 1, without loss of generality. Let $\hat{\mu}$ the maximum outflow of the accumulation-based model in this case:

$$\hat{\mu} \equiv \max_Q\{Q(1-Q)/\ell\} = \frac{1}{4\ell}, \tag{15}$$

and let the intensity be $\rho \equiv \lambda/\hat{\mu}$, or:

$$\rho \equiv 4\lambda\ell. \tag{16}$$

In this case, one needs to solve the following system of equations:

$$Q'(t) = \lambda(t) - v(t)\cdot k(t,0), \tag{17a}$$
$$k(t,0) = g(z(t)) + \int_0^t \lambda(s) f\left(z_s^t\right) ds, \tag{17b}$$
$$z'(t) = V(Q(t)), \tag{17c}$$

subject to appropriate boundary conditions. When the distribution of trip lengths f is exponential with mean ℓ, we obtain the accumulation-based model Eq. (3), whose analytical solution was presented in Laval et al. (2017). Alternatively, here we solve Eq. (17) with initial conditions $Q(0) = Q_0$ to obtain:

$$Q(t) = \frac{1}{2}\left(1 - c_1 \tan\left(\sec^{-1}(-c_2) - c_3 t\right)\right), \tag{18a}$$
$$z(t) = \frac{t}{2} + \ell \log\left(c_2 \cos\left(c_3 t + \sec^{-1} c_2\right)\right), \tag{18b}$$

where $c_1 \equiv \sqrt{\rho - 1}$, $c_2 = \sqrt{\rho - 4Q_0 V(Q_0)}/c_1$, and $c_3 = c_1/(2\ell)$ are constants. If we let the distribution of trip lengths f be uniform in $(0, 2\ell)$, which also has a mean ℓ, analytical solutions can also be found. The general case $\lambda > 0$, $Q_0 > 0$ looks complicated and not particularly insightful. However, for arbitrary initial conditions and no inflow:

$$Q(t) = 1 - (1 - Q_0)\, e^{c_4 t}, \tag{19a}$$
$$z(t) = (1 - Q_0)\left(e^{c_4 t} - 1\right)/c_4\ , \tag{19b}$$

where $c_4 = Q_0/2\ell$. Since $c_4 > 0$ when $Q_0 > 0$, the accumulation decreases more rapidly with time.

Fig. 1 shows a comparison of the exponential and uniform models with the same input parameters. It can be seen in the first row of the figure that under uniform trip lengths the accumulation tends to zero much more slowly than in the exponential case for earlier times, while the opposite is true for later times. The second row shows an initially empty network being loaded at a constant intensity, where one can see that both distributions tend to the same equilibrium point but at slightly different speeds. Notice that models Eq. (18) and Eq. (19) are valid only in $0 \le Q(t) \le 1$, and as soon as we hit one of these boundaries one should stop the evaluation; see points "a" in the figure. Similarly, because the uniform distribution is only defined in $(0, 2\ell)$, as soon as $z(t) = 2\ell$ then Eq. (19) no longer applies.

We can conclude that the effect of the trip-length distribution affects the time to reach steady state, but not the equilibrium point itself. This is confirmed in the next section.

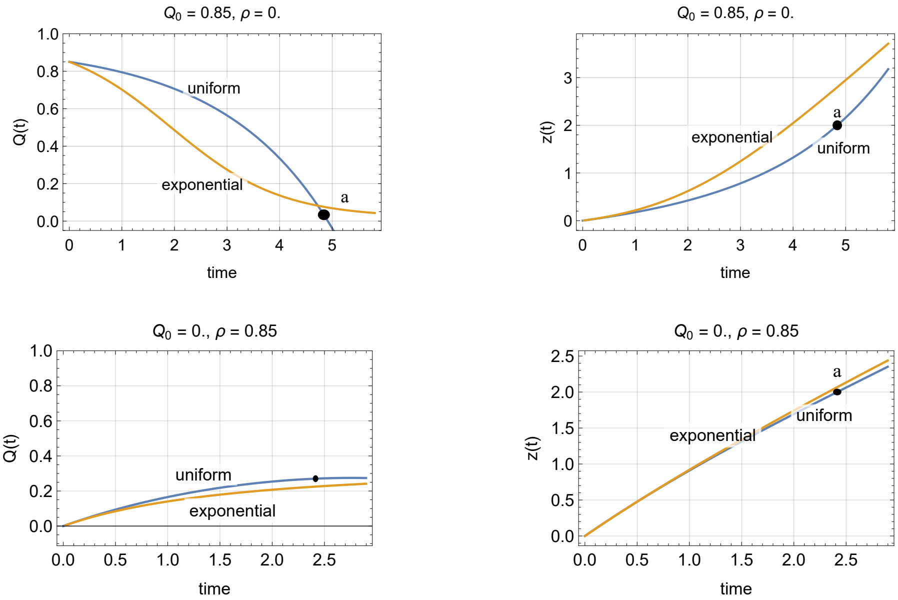

Figure 1: Comparison of the exponential and uniform models with the same input parameters and $\ell = 1$.

2.2. *Steady-state*

In this section all input variables are time-independent, i.e. $\lambda(t) = \lambda, f_s(\cdot) = f(\cdot)$, and all other steady-state variables denoted with a star.

Proposition 2.2. *The accumulation-based model under steady-state conditions is valid regardless of the distribution of trip lengths.*

Proof. In steady-state the network production $Q^* v^*$ has to match the incoming production $\lambda\ell$,

$$Q^* v^* = \lambda\ell, \tag{20}$$

where $v^* = V(Q^*)$. From Eq. (10) we can see that the steady-state condition $Q'(t) = 0$ implies

$$\mu^* = \lambda, \tag{21}$$

as expected. Combining Eq. (21) and Eq. (20) gives:

$$\mu^* = Q^* v^* / \ell, \tag{22}$$

which establishes the result.

□

This result implies that in steady state one may use the simple accumulation-based model to evaluate traffic conditions, regardless of the distribution of trip lengths. For instance, as shown previously in Laval et al. (2017), solving for Q^* in Eq. (20) gives two equilibrium solutions; the first solution is stable and in the free-flow regime, and in the Greenshields approximation it reads as:

$$Q^*(\rho) = \frac{1}{2}\left(1 - \sqrt{1-\rho}\right), \tag{23}$$

while the second solution, $\frac{1}{2}\left(1 + \sqrt{1-\rho}\right)$, is in the congested regime and acts as a repellor; see Laval et al. (2017) for details.

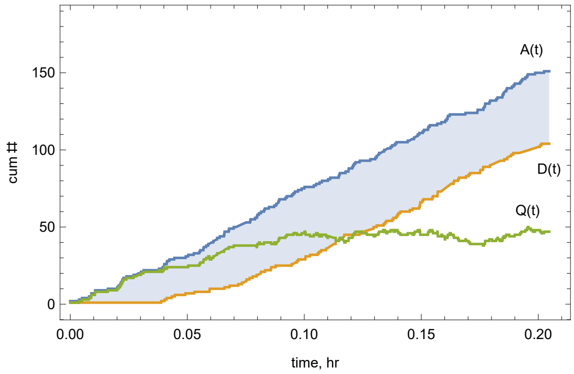

Figure 2: The network accumulation $Q(t)$ is the difference between cumulative vehicle arrivals to the network, $A(t)$, and departures from the network, $D(t)$.

Proposition 2.3 (Steady-state density)**.** *In steady-state, the density is given by:*

$$k^*(x) = [1 - F(x)]\,\lambda/v^*. \tag{24}$$

Proof. Since in steady state characteristics are straight lines of slope $-v^*$, the density $k(t,x) = \lambda \int_0^t f(x + z_s^t)\,ds$ can be computed explicitly with the change of viable $z = (t-s)v^*$ to obtain:

$$k(t,x) = [F(x + tv^*) - F(x)]\,\lambda/v^*. \tag{25}$$

Taking the limit $t \to \infty$ we obtain the steady-state density $k^*(x) = k(t \to \infty, x)$ as claimed. □

It follows that the probability density function (PDF) of remaining trip distances, $f_X(x) = k^*(x)/Q^*$, is given by:

$$f_X(x) = [1 - F(x)]/\mathrm{E}\{L\}, \tag{26}$$

and it can be shown that their moments are given by Eick et al. (1993) as:

$$\mathrm{E}\{X\} = \frac{\mathrm{E}\{L\}}{2}\left(1 + C_L^2\right), \tag{27}$$

$$\mathrm{E}\{X^m\} = \frac{\mathrm{E}\left\{L^{m+1}\right\}}{(m+1)\mathrm{E}\{L\}}. \tag{28}$$

Notice that for the exponential distribution, Eq. (26) gives $f_X = f$, as expected given the memoryless property of this distribution[3]. This means that the expected remaining trip length is $\mathrm{E}\{X\} = \mathrm{E}\{L\}$, as can be verified in Eq. (28) with $C_L^2 = 1$. For other distributions this is not the case, however, where in general we find that $\mathrm{E}\{X\} < \mathrm{E}\{L\}$. [4]

3. Queueing approximation for the stochastic problem

In this section, we present a queuing approximation for the stochastic version of our problem. For simplicity, we start with empty initial conditions and assume that the only sources of randomness are the arrival process and trip lengths, while the travel speed remains deterministic given the accumulation, as in the previous sections. The network

[3]For the exponential distribution with mean ℓ, $F(l) = 1 - \exp(-l/\ell)$, which gives $f_X(x) = f(x) = \exp(-x/\ell)/\ell$.
[4]For distributions more positively skewed than the exponential distribution, $\mathrm{E}\{X\} > \mathrm{E}\{L\}$, but this is unlikely to arise in practice.

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accumulation (or queue) $Q(t) \equiv A(t) - D(t)$ is the difference between cumulative vehicle arrivals to the network, $A(t)$, and departures from the network, $D(t)$; see Fig. 2. The mean and variance of the accumulation is then given by:

$$\mathrm{E}\{Q(t)\} = \mathrm{E}\{A(t)\} - \mathrm{E}\{D(t)\}, \tag{29a}$$

$$\mathrm{Var}\{Q(t)\} = \mathrm{Var}\{A(t)\} + \mathrm{Var}\{D(t)\} - 2\,\mathrm{Cov}\{A(t), D(t)\}. \tag{29b}$$

In traditional queuing systems the departure rate is independent of the accumulation, which acts as a buffer between arrivals and departures: as long as the queue is not zero, departures tend to be independent of arrivals. As argued in Newell (1982), the more the accumulation in the system, the more independent the departure and arrival process will be and is typically assumed that the covariance term $\mathrm{Cov}\{A(t), D(t)\} = 0$. This is true even for limited storage, which can be treated as a boundary condition. But in our case, by definition of the MFD there is a dependency between accumulation and service rate and therefore the covariance term $\mathrm{Cov}\{A(t), D(t)\}$ in Eq. (29b) cannot be neglected. In any case, as long as the accumulation is not close to zero, its distribution should be approximately normal with mean and variance given above. The challenge here is to characterize the terms $\mathrm{Var}\{D(t)\}$ and $\mathrm{Cov}\{A(t), D(t)\}$ due to the nontrivial dependencies in the reservoir model.

As customary in the queueing literature, we introduce the variance-to-mean ratio of arrivals, I_A, departures, I_D, and accumulations, I_Q:

$$I_A(t) \equiv \frac{\mathrm{Var}\{A(t)\}}{\mathrm{E}\{A(t)\}}, \quad I_D(t) \equiv \frac{\mathrm{Var}\{D(t)\}}{\mathrm{E}\{D(t)\}}, \quad I_Q(t) \equiv \frac{\mathrm{Var}\{Q(t)\}}{\mathrm{E}\{Q(t)\}} \tag{30}$$

Notice that $I_A = 1$ for Poisson arrivals, and that if service times are independent, then I_D gives the squared coefficient of variation of service times (Newell, 1982), and equals one for exponential service times. It will be convenient to define:

$$I_{AD}(t) \equiv \frac{\mathrm{Cov}\{A(t), D(t)\}}{\mathrm{E}\{D(t)\}} \tag{31}$$

which is one for the M/G/∞ queue (Newell, 1982), and is in the range $0 \le I_{AD} \le I_D$. With these definitions the variance-to-mean ratio of the accumulation can be written as:

$$I_Q = \frac{\mathrm{E}\{A(t)\}\, I_A - \mathrm{E}\{D(t)\}\,(2I_{AD} - I_D)}{\mathrm{E}\{A(t)\} - \mathrm{E}\{D(t)\}}, \tag{32}$$

which is the main focus of this section. Although this ratio is time dependent we will be interested in its state-state behavior and what follows.

The traditional queuing model most similar to our problem is the M/G/∞ queue, with Poisson arrivals (so $I_A = 1$), a general service time distribution, and an infinite number of servers. In our case, each commuter is its own server and the service time play the role of the travel time from origin to destination, which is governed by the reservoir model dynamics. As it turns out, these dynamics will make a big difference: it is known that for the M/G/∞ queue the distribution of departures is Poisson, so $I_D = I_{AD} = 1$ and therefore accumulations are also Poisson, as can be seen in (32) that gives $I_Q = 1$. But our discrete-event simulation (described in appendix) results presented in Fig. 3 and Fig. 4 show that $I_Q > 1$ depending on the intensity ρ. The simulation experiments consider, separately, three trip-length distributions: exponential, uniform and the "square" distribution arising when origins and destinations are uniformly distributed on a square; see Aghamohammadi and Laval (2020). The parameters of these distributions were set such that they all have the same mean, but different coefficient of variation: the C_L^2 is 1, 1/27 and 1/4 for the exponential, uniform and square distribution, respectively.

Fig. 3(Top) shows a typical simulation output in terms of cumulative count curves for departures (left) and accumulations (right) for several realizations of the experiment. Arrivals are omitted as they look very similar to the departure plot. From these outputs one can calculate the sample means of $\mathrm{E}\{A(t)\}$, $\mathrm{Var}\{A(t)\}$, etc. to estimate the ratios I_A, I_D, I_Q and I_{AD}, all functions of time. The bottom part of the figure shows simulation time series showing the evolution of $\mathrm{E}\{Q(t)\}/Q^*$ and $I_Q \approx \mathrm{Var}\{Q(t)\}/Q^*$ with the exponential and uniform distributions for different values of $\mathrm{E}\{\rho\}$. It becomes clear that the distribution with the highest coefficient of variation, the exponential distribution, implies longer relaxation times to reach the steady state, but this steady state is unaffected by the distribution of trip lengths.

Fig. 4 shows the steady-state time-average values of parameters I_A, I_D, I_{AD} and I_Q as a function of the average intensity $\mathrm{E}\{\rho\}$. It can be seen that all the columns in the figure look very similar, which indicates that the impact of the

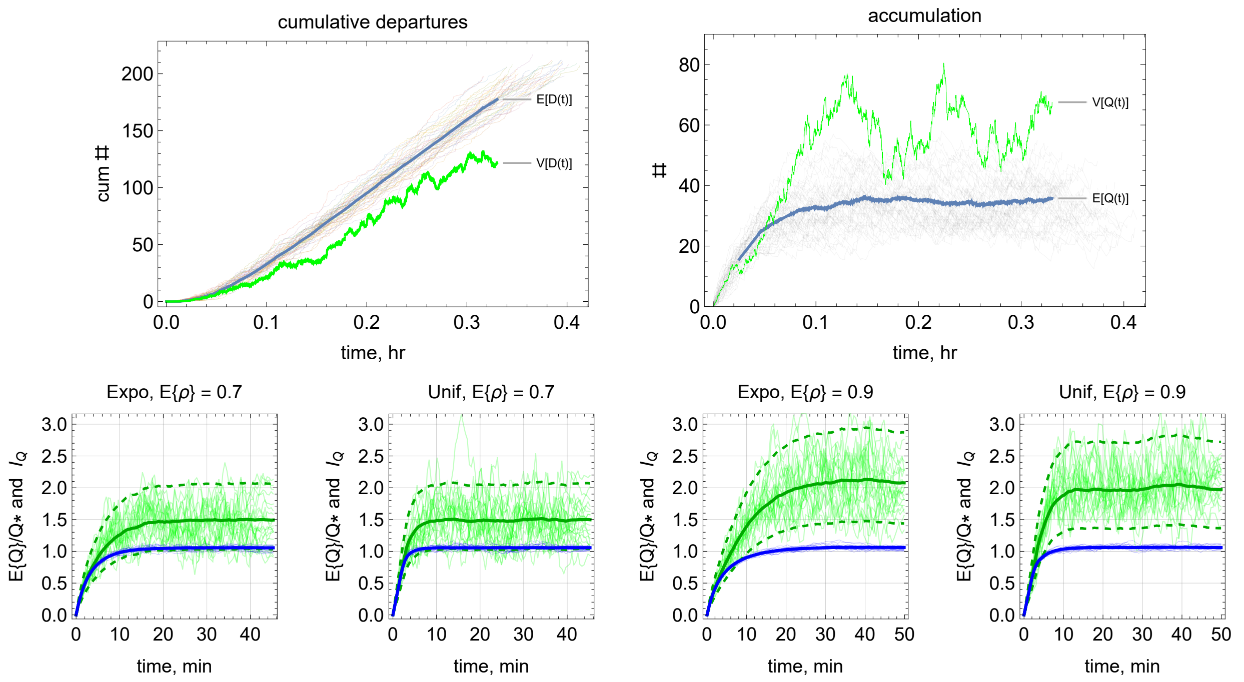

Figure 3: Top: Typical simulation output in terms of cumulative count curves for departures (left) and accumulations (right), for 50 realizations of the experiment. Notice in the left panel that $\mathrm{E}\{D(t)\} > \mathrm{Var}\{D(t)\}$, which is typically the case, and from the right panel that $\mathrm{E}\{Q(t)\} < \mathrm{Var}\{Q(t)\}$. Bottom: Simulation time series with the exponential and uniform distributions for different values of $\mathrm{E}\{\rho\}$, showing the evolution of $\mathrm{E}\{Q(t)\}/Q^*$ in blue and $I_Q \approx \mathrm{Var}\{Q(t)\}/Q^*$ in green. The solid thin lines represent these quantities for a single experiment as in the top part of the figure, while solid thick lines correspond to the average of 1000 such experiments. The dashed green lines give the 0.05 and 0.95 percentiles from the chi-square distribution with parameter 50-1=49.

trip length distribution is almost negligible when it comes to the statistical properties of I_A, I_D, I_{AD} and I_Q. Notice that around intensity 0.9 there is a consistent drop in all the parameters, followed by a sudden increase at $\mathrm{E}\{\rho\} > 1$. The drop is a consequence of arrival flows being throttled in congestion (see appendix), which causes a reduction in the variance of the relevant variables since the system is forced to maintain the same accumulation. The sudden increase can be explained by the non-stationary of the system when $\rho > 1$. Notice that for $\mathrm{E}\{\rho\} < 0.9$ the first three columns of the figure are as expected, with $1 \approx I_A > I_D \approx I_{AD} \approx 0.9$ as we have Poisson arrivals, whose variability is dampened slightly due to the interactions in the bathtub, leading to smaller I_D and I_{AD}.

But not as expected is the behavior of I_Q observed in the last column of Fig. 4, which is not captured by (32). As shown next, this can be explained by the reservoir dynamics brought about by the MFD causing the variance to diverge to infinity near the critical point $\rho = 1$. (Notice that this is not the case for the M/G/∞ queue because it has no exit capacity constraint such as $\hat{\mu}$ here; see (15).) To see this, we series expand (23) around the critical point $\rho = 1$ up to first order and compute its variance assuming Poisson arrivals, i.e. $\mathrm{Var}\{\rho\} = \mathrm{E}\{\rho\}$. This gives $\mathrm{Var}\{Q^*(\rho)\} \approx (Q^{*'}(\mathrm{E}\{\rho\}))^2\mathrm{E}\{\rho\} = \mathrm{E}\{\rho\}/(16(1-\mathrm{E}\{\rho\}))$, which goes to infinity as $\mathrm{E}\{\rho\} \to 1$. This behavior is typical of systems that undergo a phase transition (Halperin and Hohenberg, 1969), e.g. from liquid to solid or from free-flow to congestion in our case. The usual way to deal with this divergence in the phase transition literature is to assume a power-law divergence, in this case of the type:

$$\mathrm{Var}\{Q^*(\rho)\} = a\mathrm{E}\{\rho\}(1-\mathrm{E}\{\rho\})^{-b}, \tag{33}$$

where $a > 0, b > 0$ are constants and b is known as a critical exponent. Assuming $\mathrm{E}\{Q^*(\rho)\} \approx Q^*(\mathrm{E}\{\rho\})$ we can compute:

$$I_Q \approx \frac{1}{2}\left(1 + (1-\mathrm{E}\{\rho\})^{-1/2}\right) \tag{34}$$

where we have used $a = 1/4, b = 1/2$ that provide a good fit to the data. This is shown by the red line in Fig. 4, which

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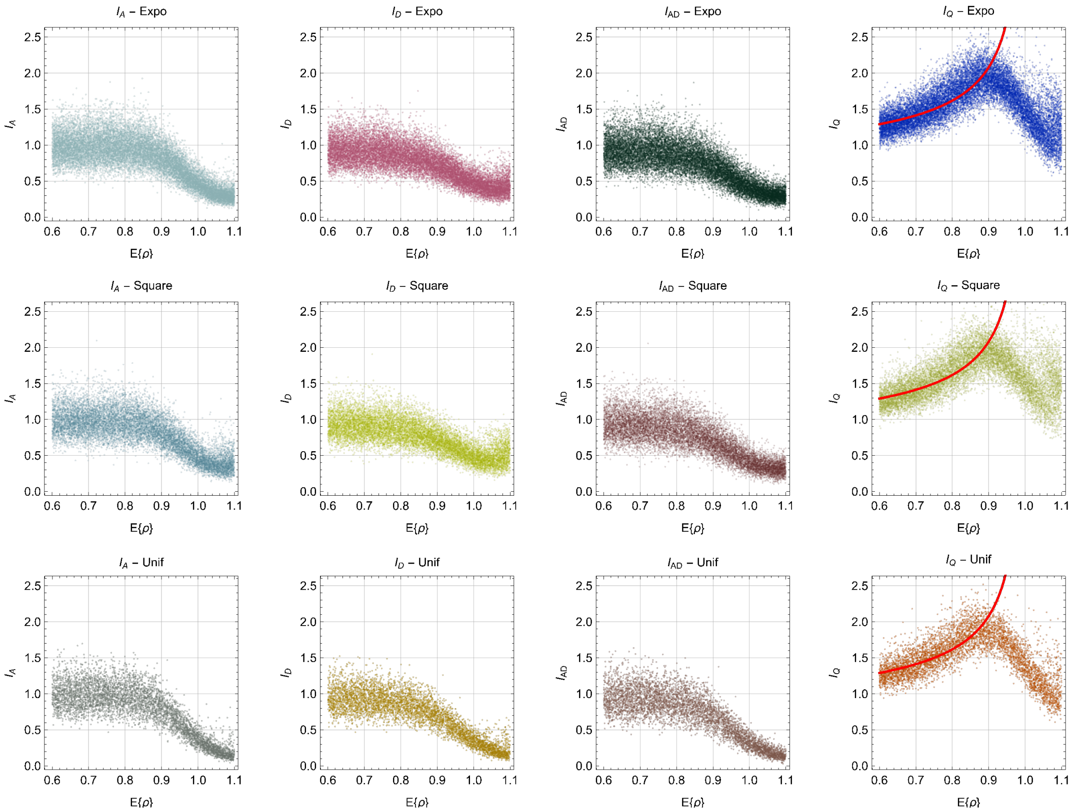

Figure 4: Steady-state time-average values of parameters I_A, I_D, I_Q and I_{AD} as a function of the intensity ρ. Each row corresponds to a different distribution of trip length: exponential, uniform and square distribution. The red line corresponds to Eq. (34)

corresponds to Eq. (34). The good fit to the simulated data is apparent, and confirms that the steady-state behavior of the system is not affected by the different trip-length distributions. Therefore, we can conclude that the statistical properties of congestion are largely independent of the trip-length distribution.

4. Discussion

This paper has presented additional results of the generalized bathtub model for urban networks. Section 2 was devoted to deterministic arrivals, as is invariably the case in the literature, which included a simpler derivation of the model, exact solutions for uniformly distributed trip lengths, and the characterization of the steady-state. While the variability of trip lengths has an impact on the time to reach steady-state, it was found that under steady-state conditions the accumulation-based model remains valid regardless of the distribution of trip lengths. This is good news because in steady state or in the common case of slowly-varying demand, one may use the accumulation-based model, which is a much more parsimonious model that only requires a good estimation of the average trip length.

Section 3 examines the case of Poisson arrivals drawing the analogy with the well-know M/G/∞ queue, which predicts Poisson accumulations. In contrast, we found that accumulations exhibit a much larger variance than predicted by the M/G/∞ queue, which can be explained by the nonlinear dynamics imposed by the MFD, where the critical density acts as a critical point of a dynamical system subject to phase transitions. As in the case with deterministic arrivals, we find that the trip length distribution has an impact only on the transient accumulation trajectory, but it does not affect its variance-to-mean ratio in steady-state. This indicates that the proposed approximation method for the variance-to-mean ratio I_Q in Eq. (34) will be robust, and perhaps universal as it is typically the case with power-law relationships arising in phase transitions. This universality would imply that the proposed approximation should be accurate independently of the shape of the MFD (so long as it is concave) and of other details such as the degree of randomness in the arrival process. That traffic flow behaves as a fluid undergoing phase transitions has been known for a few decades now (Nagatani, 2002, Helbing, 2001, Chowdhury et al., 2000, Nagatani, 2020) but its consequences have not permeated the MFD . More research is needed to explore the analogy with phase transitions to better understand the complex power-law dynamics taking place and the possible universal aspects of this problem.

The high values for I_Q found in this paper are unusually high for traditional queuing systems, but perhaps as expected given our experience as commuters on congested networks. This index can be seen as a proxy for the reliability of the network, a topic that is worth investigating with the tools provided here. We have seen that the demand capping has a strong effect in lowering the I_Q, which strongly suggests that perimeter control can be used effectively to improve reliability. Additional research is warranted to investigate different implementations of the demand capping assumption in its impact in the reliability of the urban network.

As pointed out in the introduction, assuming exponentially distributed trip lengths has been considered unrealistic, starting with Arnott (2013) who argues that this is akin to assuming that commuters do not know their trip distance until initiating the trip, which clearly does not make any sense. However, this is an unfair criticism since the argument follows from the memoryless property of the exponential distribution, which is only a sufficient condition, not a necessary one. This means that there are stochastic processes, such as bus bunching, that exhibit the exponential distribution and therefore appear to be memoryless but in fact they are generated by physically sound mechanisms. In addition, the empirical trip length distributions from American and European cities reported in Martínez and Jin (2021), Thomas and Tutert (2013) strongly suggest that the exponential distribution exhibits the correct overall trend of the data. Adding the main result of this paper that the statistical properties of congestion are largely independent of the distribution choice, which only affects the transient behavior of the system, makes a compelling argument in favor of the exponential assumption. More research is needed, however, to settle the debate.

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Appendix A. Simulation experiments

This section explains the simulation experiments presented in section 3 of the paper, used to estimate the parameters I_A, I_D, I_{AD} and I_Q, and to verify the accuracy of the approximation presented here. Our simulation model implements the discrete-event simulation proposed in Mariotte et al. (2017) with the exceptions that arrivals are driven by a Poisson processes instead of being deterministic. In the simulation, each vehicle has its own trip length and reservoir speeds are updated each time a vehicle either enters or leaves the system. As customary in the literature, in congestion the arrival flow is capped; i.e, restricted to remain at or below the prevailing flow on the network; see Daganzo (2007), Leclercq and Paipuri (2019), Mariotte and Leclercq (2019). We use a Greenshields speed-accumulation relationship, $V(Q) = 80(1 - Q/120)$, where the free-flow speed is 80 km/hr and the jam accumulation is 120 vehs.

A single experiment consists of an initially empty system with Poisson arrivals with a constant rate λ. Each arrival has a trip length drawn from the distribution f, which can be the exponential, uniform distributions and the "square" distribution of trip length when origins and destinations are uniformly distributed on a square; see Aghamohammadi and Laval (2020) for details. Notice that all these distributions have the same mean of $\ell = 3$, but different coefficient of variation: the C_L^2 is 1, 1/27 and 1/4 for the exponential, uniform and square distribution, respectively. The simulation stops once a steady state has been reached for 30 minutes.