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Article

Projection onto the Set of Rank-constrained Structured Matrices for Reduced-order Controller Design

Masaaki Nagahara [1,†,‡], Yu Iwai [2,‡] and Noboru Sebe [3,*]

1 The University of Kitakyushu; nagahara@ieee.org
2 The University of Kitakyushu; a9531007@eng.kitakyu-u.ac.jp
3 Kyushu Institute of Technology; sebe@ics.kyutech.ac.jp
* Correspondence: nagahara@ieee.org

Abstract: In this paper, we propose an efficient numerical computation method of reduced-order controller design for linear time-invariant systems. The design problem is described by linear matrix inequalities (LMIs) with a rank constraint on a structured matrix, due to which the problem is NP-hard. Instead of the heuristic method that approximates the matrix rank by the nuclear norm, we propose a numerical projection onto the rank-constrained set based on the alternating direction method of multipliers (ADMM). Then the controller is obtained by alternating projection between the rank-constrained set and the LMI set. We show the effectiveness of the proposed method compared with existing heuristic methods, by using 95 benchmark models from the COMPL$_e$ib library.

Keywords: reduced-order control; rank constraint; linear matrix inequality; alternating projection; convex optimization.)

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1. Introduction

It is well-known that a stabilizing output-feedback controller and an H^∞ controller of a linear time-invariant system can be obtained by solving linear matrix inequalities (LMIs) assuming that the order of the controller is more than or equal to that of the controlled plant model [1,2]. Since the set of optimization variables described by LMIs is convex, the problem can be efficiently solved by convex optimization solvers such as Sedumi [3], SDPT3 [4], and MOSEK [5]. Also, LMIs are easily coded with YALMIP [6] and CVX [7] on MATLAB, and CVXPY [8] on Python.

Practically, it is preferred for implementation to use a low-order controller, especially a static controller, of a high-order plant, which we call a *reduced-order controller*. To obtain a reduced-order controller that has a lower order than the plant is however known to be NP-hard [9] due to a rank constraint [10]. Therefore, we need to employ a heuristic method to efficiently obtain an *approximated* reduced-order controller. Actually, a couple of heuristic methods have been proposed; the XY-centring algorithm [11], the cone complementarity linearization algorithm [12], and alternating projection methods [13,14], to name a few.

More recently, the nuclear norm minimization with LMIs has been proposed to cope with this hard problem [15–19]. This is based on the fact that the nuclear norm of a matrix well approximates the matrix rank [20]. Since the nuclear norm is a convex function and the set described by LMIs is also a convex set, the problem boils down to a convex optimization problem that can be solved very efficiently.

Although the nuclear norm heuristic is widely used for rank-constrained problems, we show by numerical examples in this paper that this cannot be efficient for some plants. Instead, we propose a new method to solve the reduced-order controller design problem by extending the alternating projection method proposed in [13]. The idea is to compute a more precise projection onto the set of rank-constrained structured matrices by the alternating direction method of multipliers (ADMM) [21] . By numerical examples in Section 4, we show that the proposed method significantly improves the precision of the solution compared to the nuclear norm minimization [15] and the original alternating projection method [13].

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The organization of this paper is as follows: In Section 2, we show two reduced-order control problems that are described as rank-constrained LMI problems. In Section 3, we propose the alternating projection algorithm to solve the rank-constrained LMI problem. Numerical examples are shown in Section 4 to illustrate the effectiveness of the proposed method with 95 benchmark models from the COMPL$_e$ib library [22]. A summary is given in Section 5.

We note that the MATLAB programs to check the numerical examples and the results of stability tests for 95 benchmark models shown in Section 4 are available at the web page of [23].

Notation

Let A be a matrix. The transpose of A is denoted by $A^\top$, the trace by $\mathrm{tr}(A)$, and the rank by $\mathrm{rank}(A)$. The i-th singular value of A is denoted by $\sigma_i(A)$. In this paper, we use two kinds of matrix norms: one is the Frobenius norm $\|A\|$ defined by

$$\|A\| \triangleq \sqrt{\mathrm{tr}(A^\top A)} = \sqrt{\sum_{i=1}^{n} \sigma_i^2(A)}, \tag{1}$$

and the other is the nuclear norm $\|A\|_*$ defined by

$$\|A\|_* \triangleq \mathrm{tr}\left(\sqrt{A^\top A}\right) = \sum_{i=1}^{n} \sigma_i(A), \tag{2}$$

where $\sqrt{A^\top A}$ is a positive semidefinite matrix that satisfies $\left(\sqrt{A^\top A}\right)^2 = A^\top A$. Matrix inequalities $A \succ 0$, $A \succeq 0$, $A \prec 0$, and $A \preceq 0$ respectively mean A is positive definite, positive semidefinite, negative definite, and negative semidefinite. For $A \in \mathbb{R}^{n\times m}$ with $r = \mathrm{rank}(A) < n$, $A^\perp$ is a matrix that satisfies

$$A^\perp \in \mathbb{R}^{(n-r)\times n}, \quad A^\perp A = 0, \quad A^\perp A^{\perp\top} \succ 0. \tag{3}$$

By $\mathcal{S}_n$, we denote the set of $n \times n$ real symmetric matrices. For a closed subset Ω of $\mathcal{S}_n$, the projection operator of $X \in \mathcal{S}_n$ onto Ω is denoted by Π_Ω, that is,

$$\Pi_\Omega(X) \in \arg\min_{Z\in\Omega} \|Z - X\|. \tag{4}$$

2. Reduced-order controller design problems

In this section, we show two examples of reduced-order controller design.

2.1. Reduced-order stabilizing controllers

Let us consider the following linear time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad t \geq 0, \tag{5}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. For this system, we consider an output-feedback controller $u = Ky$, whose order is assumed to be $n_c < n$. Then, the reduced-order output-feedback controller design is described as the following feasibility problem [10].

Problem 1 (Stabilizing controller)**.** *Find $X_1, X_2 \in \mathcal{S}_n$ such that the rank constraint*

$$\mathrm{rank}\begin{bmatrix} X_1 & I \\ I & X_2 \end{bmatrix} \leq n + n_c, \tag{6}$$

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and LMIs

$$-\begin{bmatrix} X_1 & I \\ I & X_2 \end{bmatrix} \preceq 0, \tag{7}$$

$$B^{\perp}(AX_1 + X_1A^{\top})B^{\perp\top} \preceq -\epsilon I, \tag{8}$$

$$C^{\top\perp}(X_2A + A^{\top}X_2)C^{\top\perp\top} \preceq -\epsilon I \tag{9}$$

hold for some $\epsilon > 0$[1].

2.2. Reduced-order H^∞ controllers

Let us consider the following generalized plant:

$$G: \begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \tag{10}$$

where we assume (A, B_2) is stabilizable and (C_2, A) is detectable. Let T_{zw} denote the feedback connection (or the linear fractional transformation) [24] of G and a controller K such that $u = Ky$. Then, the problem is to seek a controller K of order $n_c \leq n$ such that the H^∞ norm of T_{zw} satisfies $\|T_{zw}\|_\infty < \gamma$ with a given $\gamma > 0$. This problem is described as LMIs with a rank constraint [10,25].

Problem 2 (H^∞ controller). *Find* $X_1, X_2 \in \mathcal{S}_n$ *such that the rank constraint* (6)*, the LMI* (7)*, and the following LMIs:*

$$\begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix}^{\perp} \begin{bmatrix} AX_1 + X_1A^{\top} & X_1C_1^{\top} & B_1 \\ C_1X_1 & -\gamma I & D_{11} \\ B_1^{\top} & D_{11}^{\top} & -\gamma I \end{bmatrix} \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix}^{\perp\top} \preceq -\epsilon I, \tag{11}$$

$$\begin{bmatrix} C_2^{\top} \\ D_{21}^{\top} \\ 0 \end{bmatrix}^{\perp} \begin{bmatrix} X_2A + A^{\top}X_2 & X_2B_1 & C_1^{\top} \\ B_1^{\top}X_2 & -\gamma I & D_{11}^{\top} \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} C_2^{\top} \\ D_{21}^{\top} \\ 0 \end{bmatrix}^{\perp\top} \preceq -\epsilon I \tag{12}$$

hold for some $\epsilon > 0$.

3. Algorithms

In this section, we derive algorithms to solve Problems 1 and 2. First, we define function $F(X_1, X_2)$ such that $F(X_1, X_2) \preceq 0$ is equivalent to the LMIs to be solved. For Problem 1 for example, $F(X_1, X_2) \preceq 0$ means that the LMIs (7)–(9) hold.

3.1. Nuclear norm minimization

We first introduce an algorithm using *nuclear norm minimization* [18] to approximately solve the rank-constrained LMIs in Problems 1 and 2. The idea is to approximate the matrix rank by the nuclear norm (2), the sum of the singular values of the matrix. Using the nuclear norm heuristic, Problem 1 is relaxed to the following problem:

$$\underset{(X_1,X_2)\in\mathcal{S}_n^2}{\text{minimize}} \left\| \begin{bmatrix} X_1 & I \\ I & X_2 \end{bmatrix} \right\|_* \text{ subject to } F(X_1, X_2) \preceq 0. \tag{13}$$

This is a convex optimization problem and easily solved. Similarly, Problem 2 can be reduced to convex optimization.

[1] In general, the inequality "$\preceq -\epsilon I$" can be "$\prec 0$," however for the projection-based algorithm described in Section 3, we introduce small $\epsilon > 0$ to make the subsets closed.

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3.2. Alternating projection

Instead of the nuclear norm heuristic, we propose alternating projection [26] to solve the rank-constrained LMI problems. For this, we define the two closed subsets of $\mathcal{S}_n^2$:

$$\Omega_r \triangleq \left\{ (X_1, X_2) \in \mathcal{S}_n^2 : \operatorname{rank} \begin{bmatrix} X_1 & I \\ I & X_2 \end{bmatrix} \leq r \right\}, \tag{14}$$

$$\Lambda \triangleq \left\{ (X_1, X_2) \in \mathcal{S}_n^2 : F(X_1, X_2) \preceq 0 \right\}. \tag{15}$$

Then the problem is to find a pair of matrices (X_1, X_2) in $\Omega_r \cap \Lambda$ with $r = n + n_c$. For this, we adapt alternating projection between Ω_r and Λ. The iterative algorithm is given by

$$\begin{aligned} Z[k] &= \Pi_{\Omega_r}(X[k]), \\ X[k+1] &= \Pi_\Lambda(Z[k]), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{16}$$

where $X[0] = (X_1[0], X_2[0]) \in \mathcal{S}_n^2$ is a given initial guess of (X_1, X_2). The computation of the projection operators Π_{Ω_r} and Π_Λ are shown in the following subsections.

Remark 1. *We can also adopt Dykstra algorithm [27] that gives an element in $\Omega_r \cap \Lambda$ that is a* projection *(i.e., one of the nearest points) on $\Omega_r \cap \Lambda$ from the initial guess $X[0] \in \mathcal{S}_n^2$. The algorithm is described as follows:*

$$\begin{aligned} Z[k] &= \Pi_{\Omega_r}(X[k] + P[k]), \\ P[k+1] &= X[k] + P[k] - Z[k], \\ X[k+1] &= \Pi_\Lambda(Z[k] + Q[k]), \\ Q[k+1] &= Z[k] + Q[k] - X[k+1], \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{17}$$

where we set $P[0] = Q[0] = 0$.

3.3. Projection onto the set Ω_r of rank-constrained structured matrices

Here we consider the projection of (X_1, X_2) onto the set Ω_r of rank-constrained structured matrices in (14). This projection can be written by definition as

$$\Pi_{\Omega_r}(X_1, X_2) \in \operatorname*{arg\,min}_{(Z_1, Z_2) \in \Omega_r} \|Z_1 - X_1\|^2 + \|Z_2 - X_2\|^2. \tag{18}$$

We note that since the set Ω_r is closed but non-convex there may exist multiple solutions for the minimization in (18).

3.3.1. Approximated projection

The paper [13] also considers alternating projection with an approximated projection onto the set Ω_r. Namely, they consider the following iteration:

$$\begin{aligned} Z[k] &= \Pi_{\mathcal{R}}(\Pi_{\mathcal{D}}(X[k])), \\ X[k+1] &= \Pi_\Lambda(Z[k]), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \mathcal{D} &\triangleq \left\{ Z \in \mathcal{S}_{2n} : Z = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, X_1, X_2 \in \mathcal{S}_n \right\}, \\ \mathcal{R} &\triangleq \{ Z \in \mathcal{S}_{2n} : \operatorname{rank}(Z + J) \leq r \}, \\ J &\triangleq \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \end{aligned} \tag{20}$$

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We note that Ω_r in (14) is the intersection of $\mathcal{D}$ and $\mathcal{R}$. Namely, the algorithm in [13] *approximates* the projection Π_{Ω_r} by a composite projection $\Pi_{\mathcal{R}}\Pi_{\mathcal{D}}$.

3.3.2. Precise projection

Instead of the approximated projection by [13], we propose a precise projection based on convex optimization. For the minimization problem (18), we introduce the indicator function $\mathcal{I}_r$ defined by

$$\mathcal{I}_r(Z) \triangleq \begin{cases} 0, & \text{if } \operatorname{rank}(Z) \le r, \\ +\infty, & \text{otherwise.} \end{cases} \tag{21}$$

Then the minimization problem in (18) is equivalently described as

$$\begin{aligned} \underset{(Z_1,Z_2)\in\mathcal{S}_n^2,\tilde{Z}\in\mathcal{S}_{2n}}{\text{minimize}} \quad & \|Z_1 - X_1\|^2 + \|Z_2 - X_2\|^2 + \mathcal{I}_r(\tilde{Z}) \\ \text{subject to} \quad & \tilde{Z} = \begin{bmatrix} Z_1 & I \\ I & Z_2 \end{bmatrix} \end{aligned} \tag{22}$$

To solve this optimization problem, we first consider the projection $\Pi_{\mathcal{C}_r}(Z)$ onto the set of rank-r matrices

$$\mathcal{C}_r \triangleq \{Z \in \mathbb{R}^{2n\times 2n} : \operatorname{rank}(Z) \le r\}. \tag{23}$$

The projection $\Pi_{\mathcal{C}_r}$ is easily computed via the singular value decomposition $Z = U\Sigma V^\top$. Define Σ_r by setting all but r largest (in magnitude) diagonal entries of Σ to 0. Then, the projection $\Pi_{\mathcal{C}_r}(Z)$ is given by

$$\Pi_{\mathcal{C}_r}(Z) = U\Sigma_r V^\top. \tag{24}$$

Now, the optimization problem in (22) can be efficiently solved by adapting the alternating direction method of multipliers (ADMM) algorithm [21]. The iterative algorithm is given by

$$Z_1[i+1] = \left(1 + \frac{\rho}{2}\right)^{-1}\left(X_1 + \frac{\rho}{2}M_{11}[i]\right), \tag{25}$$

$$Z_2[i+1] = \left(1 + \frac{\rho}{2}\right)^{-1}\left(X_2 + \frac{\rho}{2}M_{22}[i]\right), \tag{26}$$

$$\tilde{Z}[i+1] = \Pi_{\mathcal{C}_r}\left(\begin{bmatrix} Z_1[i+1] & I \\ I & X_2[i+1] \end{bmatrix} - W[i]\right), \tag{27}$$

$$W[i+1] = W[i] + \tilde{Z}[i+1] - \begin{bmatrix} Z_1[i+1] & I \\ I & Z_2[i+1] \end{bmatrix}, \quad i = 0, 1, 2, \ldots, \tag{28}$$

where $\rho > 0$ is the step size, and $M_{11}[i], M_{22}[i] \in \mathbb{R}^{n\times n}$ are defined as

$$\begin{bmatrix} M_{11}[i] & M_{12}[i] \\ M_{21}[i] & M_{22}[i] \end{bmatrix} \triangleq \tilde{Z}[i] + W[i]. \tag{29}$$

We show in the appendix section how to obtain this iteration algorithm for solving (22).

3.4. *Projection onto the set Λ described by LMIs*

The projection of (X_1, X_2) onto the set Λ in (15) can be described as convex optimization with LMIs [28,29].

$$\begin{aligned} \underset{Z_1,Z_2,W\in\mathcal{S}_n}{\text{minimize}} \quad & \operatorname{tr}(W) \\ \text{subject to} \quad & F(Z_1, Z_2) \preceq 0, \\ & \begin{bmatrix} W & (Z - X)^\top \\ Z - X & I \end{bmatrix} \succeq \epsilon I, \end{aligned} \tag{30}$$

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Table 1. Stabilizing static controller results: successful (✓) and fail (—)

model	TF1	NN1	NN11	NN12	HE6	HE7	DIS1	ROC7
proposed method	✓	✓	✓	✓	✓	✓	✓	—
method by [13]	—	—	✓	✓	✓	✓	✓	—
nuclear norm	—	—	—	—	—	—	—	✓

where $Z = [Z_1\ Z_2]^\top$ and $X = [X_1\ X_2]^\top$.

4. Numerical Examples

In this section, we show some control examples to illustrate the effectiveness of the proposed algorithm. We use benchmark models listed in the COMPL$_e$ib library [22]. MATLAB programs for the numerical computation in this section can be downloaded from [23]. For numerical optimization in the examples, we use SDPT3 [4] on MATLAB.

4.1. Stabilizing static controllers

Let us choose the `AC4` model from COMPL$_e$ib, which was considered in [30] for autopilot control of an air-to-air missile. The state-space matrices of this model are given by

$$A = \begin{bmatrix} -0.876 & 1 & -0.1209 & 0 \\ 8.9117 & 0 & -130.75 & 0 \\ 0 & 0 & -150 & 0 \\ -1 & 0 & 0 & -0.05 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 150 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{31}$$

Here, we have $n = 4$, $m = 1$, and $p = 2$. We find a static controller that stabilizes this system, and hence we set $n_c = 0$. For this, we solve Problem 1 by three methods: the nuclear norm minimization discussed in Subsection 3.1, and the alternating projection in Subsection 3.2 with the approximated projection onto Ω_r (Subsection 3.3.1) and with the proposed precise projection (Seubsection 3.3.2).

By the nuclear norm minimization, we obtain matrices X_1 and X_2 that satisfy the LMIs (7)–(9), and achieve $\|X_1X_2 - I\| = 5.05 \times 10^{-2}$. We note that small $\|X_1X_2 - I\|$ implies that $X_1X_2 \approx I$, or

$$\operatorname{rank}\begin{bmatrix} X_1 & I \\ I & X_2 \end{bmatrix} = n + \operatorname{rank}(X_1X_2 - I) \approx n. \tag{32}$$

We also compute X_1 and X_2 by the alternating projection with the approximated projection discussed in Subsection 3.3.1. We obtain $\|X_1X_2 - I\| = 5.70 \times 10^{-3}$ after 100 iterations. Then, by the proposed precise projection shown in Subsection 3.3.2, we obtain $\|X_1X_2 - I\| = 7.47 \times 10^{-4}$ after 100 iterations.

Figure 1 shows the residuals $\|X_1[k]X_2[k] - I\|$ for $k = 1, 2, 3, \ldots$ just after Π_Λ by the two alternating projection methods. Note that each$(X_1[k], X_2[k])$ satisfies the LMIs in Λ. The figure shows the proposed method computes the two matrices X_1 and X_2 that satisfy the rank constraint more precisely than the method in [13]. On the other hand, Figure 2 shows the residuals $\|X_1[k]X_2[k] - I\|$ after the composite projection $\Pi_\mathcal{R}\Pi_\mathcal{D}$ in (19), and after Π_{Ω_r} by the proposed precise projection. It is clear that there are a large gap between the two methods. This gap is due to the approximation error of $\Pi_\mathcal{R}\Pi_\mathcal{D}$ in the approximated projection (19).

To see the difference between the proposed method and the other methods more clearly, we also consider all benchmark models whose order are less than 1000 from COMPL$_e$ib. There are 95 benchmark models to be checked. We summarize the results of 8 benchmark models in Table 1. These benchmark models have different stability results by the three methods. For the other 87 models, the stability results are the same. For example, for the `TF1` model, only the proposed method can find the static controller while the other methods cannot.

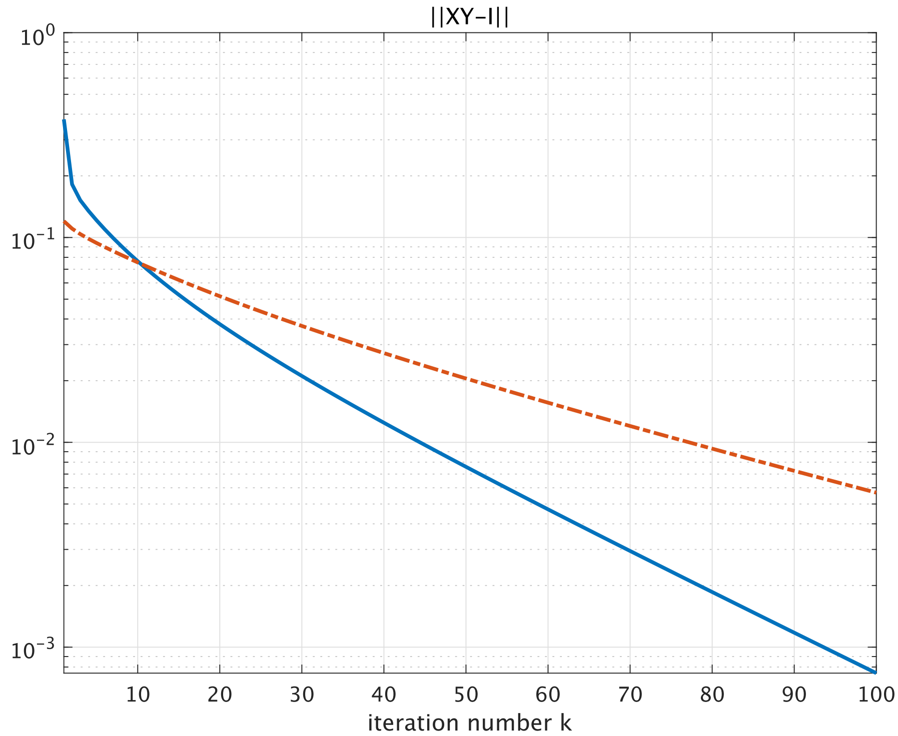

Figure 1. $\|X_1[k]X_2[k] - I\|$ just after Π_Λ by Grigoriadis and Skelton [13] (dash) and the proposed method (solid)

Table 2. Upper bounds γ of the H^∞ norm by static controller

model	AC4	NN1	NN12	HE6
γ	1.000	74.72	28.09	520.0

4.2. H^∞ static controllers

Next, we consider the reduced-order H^∞ control problem formulated in Problem 2. Here we choose AC4, NN1, NN12, and HE6 from COMPL$_e$ib. For these plant models, we seek H^∞ static controller by using the bisection method on γ. Namely, we first give a sufficiently large upper bound $\overline{\gamma}$, for example $\overline{\gamma} = 100$, and a sufficiently small lower bound $\underline{\gamma}$ (e.g., $\underline{\gamma} = 0$). Then we set $\gamma = (\underline{\gamma} + \overline{\gamma})/2 = 50$ and solve Problem 2. If there is a feasible solution, then we update the upper bound to $\overline{\gamma} = \gamma = 50$, otherwise we set the lower bound to $\underline{\gamma} = \gamma = 50$. We note that the problem is assumed to be infeasible if a solution of Problem 2 is not found after 15 iterations of (16). We repeat this process until sufficient accuracy is achieved.

Table 2 shows the obtained upper bounds of γ for the chosen models. The obtained static controllers are given as follows:

$$
\begin{aligned}
K_{\mathrm{AC4}} &= \begin{bmatrix} -0.3228 & -0.07534 \end{bmatrix}, \\
K_{\mathrm{NN1}} &= \begin{bmatrix} 3.5 & 60.22 \end{bmatrix}, \\
K_{\mathrm{NN12}} &= \begin{bmatrix} 22.81 & -35.61 \\ -5.179 & 8.137 \end{bmatrix}, \\
K_{\mathrm{HE6}} &= \begin{bmatrix} 83.25 & -0.5581 & -0.5931 & 0.1238 & 0.1546 & -0.02533 \\ -24.27 & 7.876 & 0.7263 & 0.0309 & 0.3272 & -0.6042 \\ -15.85 & 0.1609 & -6.578 & -2.062 & 1.677 & 0.1478 \\ 65.77 & -1.413 & 9.07 & -16.61 & -0.9596 & -0.04191 \end{bmatrix}.
\end{aligned}
\tag{33}
$$

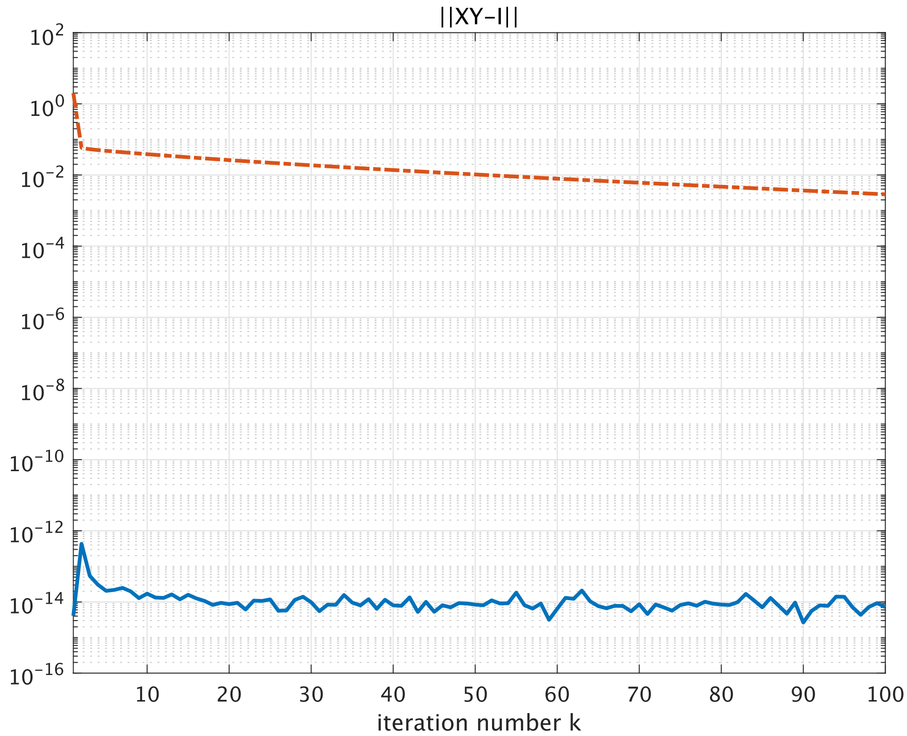

Figure 2. $\|X_1[k]X_2[k] - I\|$ just after $\Pi_{\mathcal{R}}\Pi_{\mathcal{D}}$ by [13] (dash) and Π_{Ω_r} by the proposed method (solid)

It is easy to check the obtained static controllers really achieve the H^∞ norm listed in Table II. These numerical examples demonstrate the effectiveness of the proposed method.

5. Conclusion

In this paper, we have proposed a novel design method of reduced-order controllers based-on projection onto the set of rank-constrained structured matrices. We have compared the proposed method with the nuclear norm minimization and the approximated alternating projection method [13] by numerical examples, and the proposed method has shown better performance. Future work includes reduced-order controller design with sparsity constraints on the controller realization, which is a challenging problem that should take two non-convex rank and sparsity constraints.

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Appendix F ADMM algorithm

The minimization problem in (18) is described as a standard form for ADMM [21], and the iteration algorithm is directly obtained by

$$\begin{aligned}
\begin{bmatrix} Z_1[i+1] \\ Z_2[i+1] \end{bmatrix} &= \arg\min_{Z_1, Z_2} f_1(Z_1, Z_2; \tilde{Z}[i], W[i]), \\
\tilde{Z}[i+1] &= \arg\min_{\tilde{Z}} f_2(\tilde{Z}; Z_1[i+1], Z_2[i+1], W[i]), \\
W[i+1] &= W[i] + \tilde{Z}[i+1] - \begin{bmatrix} X[i+1] & I \\ I & Y[i+1] \end{bmatrix}, \\
i &= 0, 1, 2, \ldots,
\end{aligned} \tag{A34}$$

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where

$$f_1(Z_1, Z_2; \tilde{Z}, W) \triangleq \|Z_1 - X_1\|^2 + \|Z_2 - X_2\|^2 + \frac{\rho}{2}\left\|\tilde{Z} - \begin{bmatrix} Z_1 & I \\ I & Z_2 \end{bmatrix} + W\right\|^2, \tag{A35}$$

$$f_2(\tilde{Z}; Z_1, Z_2, W) \triangleq \mathcal{I}_r(\tilde{Z}) + \frac{\rho}{2}\left\|\tilde{Z} - \begin{bmatrix} Z_1 & I \\ I & Z_2 \end{bmatrix} + W\right\|^2. \tag{A36}$$

First, for function f_1, we have

$$f_1(Z_1, Z_2; \tilde{Z}[i], W[i]) = c\left\|Z_1 - c^{-1}\left(X_1 + \frac{\rho}{2}M_{11}[i]\right)\right\|^2 + c\left\|Z_2 - c^{-1}\left(X_2 + \frac{\rho}{2}M_{22}[i]\right)\right\|^2 + \text{const.} \tag{A37}$$

where $c \triangleq 1 + \rho/2$, and $M_{11}[i]$, $M_{22}[i]$ are defined in (29). From (A37), we have the first two steps (25) and (26).

Next, by the definition of projection, a minimizer of $f_2(\tilde{Z}; Z_1[i+1], Z_2[i+1], W[i])$ in (A36) is obtained by the right-hand side of (27).