 doi:10.20944/preprints202207.0343.v1

Trajectory of Massive Particles around a Static Black Hole in f(R) Gravity

Surajit Mandal

Department of Physics, Jadavpur University, Kolkata, West Bengal 700032, India

E-mail: surajitmandalju@gmail.com

Abstract: In this paper, we investigated the trajectory of the massive particle in the vicinity of a general spherical symmetric black hole. Also, in the framework of general sphericalily symmetric black hole, Pseudo-Newtonian potential (PNP) and effective potentials has been investigated. As an example, static spherically symmetric black hole in f(R) gravity is considered and presented the brief discussion on the structure of spacetime and horizons. We calculated energy and angular momentum in the framework of general relativity as well as in Pseudo-Newtonian theory. A graphical comparison of angular momentum in this both framework has been studied.

Keywords: Massive particle; Static black hole; f(R) gravity; Pseudo-Newtonian Potential.

 doi:10.20944/preprints202207.0343.v1

Contents

1 Introduction

Study of motion of a massive particle around a black hole has a relevance to enquire the behavior of the gravitational field in the vicinity of a black hole [1–3]. Such an enquiry has started around 1915 after the discovery of Einetein's general theory of relativity and publication of Schwarzschild's vacuum solution Einstein field equations (1916) [4–8]. Also the motion of a massive particle near the Schwarzchild black hole can be found in [1–3]. Pseudo-Newtonian Potential (PNP) can reproduce all the properties of accretion disk around a black hole [9–12]. Few years back, S. Chakraborty and S. Chakraborty gave a general formalism of the trajectory of test particles in the vicinity of a general spherically symmetric non-rotating black hole [13] and as an example they considered non-rotating charged Reissner-Nordström black hole. It is to be noted that, the general formulation of the trajectory is not limited to standard general relativity, this formulation can also be applicable and extendable for any black hole solution in the modified theories of gravity. They did not adress the analysis by using modified theories of gravity. Here, in this paper, as an example we have considered a static and spherically symmetric black hole in f(R) gravity.

Modified theories of gravity helps people to study the problems such as the accelerated expansion [14, 15] and the dark matter origin [16] of our universe. f(R) theory of gravity is one of modified theory of gravity [17]. In f(R) modified gravity, at large distances, the geometry of the space-time will be different from that of general theory of relativity. Analysis of accretion disks around a black hole can be one of the possible way that can differentiates the possible deviations from general relativity. It is well established that black hole can grow in mass due to the accretion process and the extensive study on mass accretion near rotating black holes in general relativity can be found in [18]. Thereafter a study on steady state accretion disk around black hole was made in [19–21]. The study on thermal corrections and phase transition for static black hole in f(R) gravity was made in [22]. Prateek Sharma an others [23] have studied the geodesics of a Static Charged black hole in f(R) gravity. A brief study on static and spherically symmetric black holes in f(R) gravity theories can be found [24]. Recently, Saheb Soroushfar and Sudhaker Upadhyay have studied on accretion disks near a static black hole in f(R) theory of gravity [25].

The paper is organized in eight parts. In Section 2, we outline the motion of a massive particle near a general spherically symmetric black hole. Effective potential, trajectory of massive particle at equatorial plane, condition for turning point and circular orbit are also presented for the black hole in general framework. In section 3, we compared Pseudo-Newtonian potential (PNP) with the effective potential. A brief study on the structure of spacetime in f(R) gravity is presented in section 4. The horizon of this black hole and graphical analysis of the horizon function with respect to radial coordinate r are studied in section 5. We calculated the energy and general relativistic angular momentum in section 6 and also determined the energy and angular momentum in the framework of Pseudo-Newtonian theory in section 7. In this section, a graphical comparison study of angular in the framework of GR and PNP theory has been presented. Finally, we make our conclusion in section 8.

2 Motion of massive particles in the vicinity of a general black hole

For static spherically symmetric spacetime the line element is,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{2.1}$$

here, $d\Omega_2^2 = d\theta^2 + Sin^2\theta d\phi^2$ represents the metric on unit two sphere and to describe a black hole solution f(r) should satisty the conditions such as : (i) It should have a zero for some positive values of r (let $r^\star$) in such a way that time dilation will be infinite at $r^\star$. (ii) The Kretschmann scalar ($\mathcal{K} = R^{\mu\nu\sigma\lambda}R_{\mu\nu\sigma\lambda}$) must be positive at $r = r^\star$ but it should be

 doi:10.20944/preprints202207.0343.v1

diverge at r = 0. This indicates that the spacetime of eq. (2.1) has a curvature singularity at $r = r^{\star}$.
Now, the Lagrangian for metric (2.1) is,

$$2\mathcal{L} = -f(r)\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2(\dot{\theta}^2 + Sin^2\theta\dot{\phi}^2) \tag{2.2}$$

here dot is the differentiation with respect to affine parameter λ along the geodesics. This Lagrangian has two cyclic coordinates t and ϕ, gives the two conserved quantity, namely enargy E and momentum h and defined as

$$E = -\frac{p^0}{m} = constant, \;\; h = \frac{p^\phi}{m} = constant \tag{2.3}$$

At equatorial plane ($\theta = \frac{\pi}{2}$), momentum components in explicit form :

$$p^0 = E\frac{m}{f(r)} \tag{2.4}$$

$$p^r = m\frac{dr}{d\lambda} \tag{2.5}$$

$$p^\theta = 0 \tag{2.6}$$

$$p^\phi = \frac{mh}{r^2} \tag{2.7}$$

Using the above momentum components and the energy-momentum conservation relation $p^\mu p_\mu = -m^2$ we get,

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V_{eff}^2(r) \tag{2.8}$$

where

$$V_{eff}^2(r) = f(r)\left(1 + \frac{h^2}{r^2}\right) \tag{2.9}$$

is the effective potential for massive particle moving around the black hole.
Now differentiating eq. (2.8) we get

$$\frac{d^2r}{d\lambda^2} = -\frac{1}{2}\frac{dV_{eff}^2}{dr} \tag{2.10}$$

Now from eqs. (2.4)-(2.7) we can write the momentum in the ϕ direction as

$$\frac{d\phi}{d\lambda} = \frac{h}{r^2} \tag{2.11}$$

The differential equation for the trajectory of the massive particle at the equatorial plane can be calculated from eqs. (2.9), (2.10) and (2.11) as

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{h^2}\left[E^2 - f(r)\left(1 + \frac{h^2}{r^2}\right)\right] \tag{2.12}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{h^2}P(r) \tag{2.13}$$

where

$$P(r) = E^2 - f(r)\left(1 + \frac{h^2}{r^2}\right) \tag{2.14}$$

The significant conclusion for the trajectory of massive particle near black hole is :
a) Effective potential must be less than the energy of the particle.
b) Particles will comes from infinity for those values of r in which $P(r) > 0$ and go to the origin. This is generally known as terminating escape orbit.
c) If particles comes from finite distance for an one positive zero of P(r) then either it fall directly to the origin (called terminating bound orbit) or it can follow the escape orbit with impact parameter$\frac{h}{E}$.
d) There arises two cases for two positive zeros of P(r) : i) If P(r) takes positive values between two zeros then this type of trajectory is known as periodic bound orbit ; and ii) When P(r) takes negative values between two zeros then trajectory will be one among the escape orbit and terminating bound orbit.
e) Turnating point of the trajectory can be calculated when P(r) = 0. In this case it satisfies the following condition :

$$E^2 = f(r)\left(1 + \frac{h^2}{r^2}\right) \tag{2.15}$$

f) Circular orbits can be obtained when $\frac{dV^2_{eff}}{dr} = 0$. Therefore we get

$$\frac{f'(r)}{f(r)} = \frac{2h^2}{r(r^2 + h^2)} \tag{2.16}$$

3 Pseudo-Newtonian potential (PNP) and effective potentials

Pesudo-Newtonian gravitational potential can be stated as [12]

$$\psi = \int \frac{l_c^2}{r^3} dr \tag{3.1}$$

here r denotes the radial coordinate and l_c represents the ratio of the conserved angular momentum L_c to the energy per particle mass E_c and is known specific angular momentum in general relativity.
Gravitational potential ψ_N in view of Newtonian theory,

$$\psi_N = \int \frac{l_{cN}^2}{r^3} dr \tag{3.2}$$

where l_{cN} is the Newtonian angular momentum per particle mass. To match the Pesudo-Newtonian angular momentum per particle mass with the general relativistic angular momentum, here we take PN potential (3.1). From eqs. (2.15) and (2.16), energy per particle mass E_c and relativistic conserved angular momentum L_c for circular orbit, respectively, are obtained by

$$E_c = \frac{\sqrt{2}f(r)}{\sqrt{2f(r) - rf'(r)}} \tag{3.3}$$

and

$$L_c = \sqrt{\frac{r^3 f'(r)}{2f(r) - rf'(r)}} \tag{3.4}$$

Therefore,

$$l_c = \frac{1}{f(r)}\sqrt{\frac{r^3 f'(r)}{2}} \tag{3.5}$$

Using (3.1) pseudo-Newtonian ψ can be calculated as

$$\psi = c - \frac{1}{2f(r)} \tag{3.6}$$

here c has no physical meaning and can be estimated from the result of Schwarzschild geometry.
In Schwarzchild black hole geometry, the well known Paczynski-Witta gravitational potential is in the form [26]

$$\psi_{PW} = -\frac{M}{r - 2M} \tag{3.7}$$

Substituting this in (3.6) we obtain Pseudo-Newtonian potential as

$$\psi = \frac{1}{2}\left[1 - \frac{1}{2f(r)}\right] \tag{3.8}$$

Gravitational potential will be zero for static radius (r_s), so $f(r_s) = 1$ in (3.8). Hence the circular orbit of massive particles can have the radius r_c in following range [27]

$$r_a < r_c < r_s$$

here r_a denotes the photon circular orbit.
It is clear to us from (3.8) that PNP diverges at event horizon then reaches maximum point at $r = r_{max}$ where $f'(r_{max}) = 0$ and it will decreases at larger distance from r_{max}. This clearly indicates that the gravitational field due to this PNP will be repulsive for distance larger than r_{max}. When the metric shows asymptotically flat behavior then $\psi \to 0$ [27, 28]. Keplarian motion in the radial direction gives

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = e - v_{eff} \tag{3.9}$$

where e and v_{eff} stands for pseudo Newtonian energy and pseudo Newtonian effective potential per particle mass respectively. Pseudo Newtonian effective potential has the following form [29]

$$v_{eff} = \psi + \frac{l^2}{2r^2} \tag{3.10}$$

where ψ is the PNP calculated in (3.8) and l stands for pseudo Newtonian angular momentum per particle mass. Circular orbits (keplarian) can be obtained when $\frac{dv_{eff}}{dr} = 0$ and as a result we get,

$$l_c^2 = \frac{r^3 f'(2)}{2[f(r)]^2} \tag{3.11}$$

It can also be estimated with the help of eqs. (3.3) and (3.4) as

$$l_c = \frac{L_c}{E_c} \tag{3.12}$$

Therefore energy expression will be

$$e_c = \frac{1}{2}\left[1 + \frac{rf'(r) - 2f(r)}{2[f(r)]^2}\right] \tag{3.13}$$

Another way of calculating e_c is as follows :

$$e_c = \frac{1}{2}\left[1 - \frac{1}{E_c^2}\right] \tag{3.14}$$

It is important to note that general relativistic angular momentum will be identical with the angular momentum obtained from pseudo-Newtonian effective potential theory.

Extrema condition of effective potential v_{eff} gives us the information about the stability of circular orbit, discussed above, so for stable and unstable circular orbit we have, respectively

$$\frac{\partial l_c^2}{\partial r} > 0$$

and

$$\frac{\partial l_c^2}{\partial r} < 0$$

Since both GTR and PNP theory provides same angular momentum, so their stability criteria would be same. For innermost and outermost marginally stable circular orbits we have the following condition :

$$2r[f'(r)]^2 = f(r)[3f'(r) + rf''(r)] \tag{3.15}$$

To get stability we should follow the below condition :

$$2r[f'(r)]^2 < f(r)[3f'(r) + rf''(r)] \tag{3.16}$$

Few important points :
a) l_c^2 and e_c will diverge at event horizon and E_c and L_c exist when $[2f(r) - rf'(r)] > 0$.

b) l_c^2 and e_c will vanish at the static radius (f(r)= 1) while E_c and L_c depend on different values of f(r).

4 Structure of spacetime in f(R) gravity

In this section, we introduce metric tensor and horizon of static black hole in the context of $f(R)$ gravity. $f(R)$ gravity in 4-dimensions is described by the action :

$$\mathcal{I} = \frac{1}{2k} \int d^4x\sqrt{-g}f(R) + \mathcal{I}_m \tag{4.1}$$

where k is Einsteins constant, R is the Ricci scalar and the matter part is defined by $\mathcal{I}_m$. Considering variational principle in account, the action (4.1) gives the following field equations :

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f'(R) = kT_{\mu\nu} \tag{4.2}$$

with $\Box = \nabla_\alpha\nabla^\alpha$ and $f' = \frac{df(R)}{dR}$
The line element for 4-dimensional static spherically symmetric black hole in f(R) gravity can be written as [30]

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + Sin^2\theta d\phi^2) \tag{4.3}$$

where metric function becomes

$$g(r) = 1 - \frac{2M}{r} + \beta r - \frac{1}{3}\Lambda r^2 \tag{4.4}$$

Here, M is the mass, β and Λ are the real constant and the cosmological constant respectively [30, 31]. Non-zero cosmological constant has importance in various perspective of static spherically symmetric black holes can be found in [32, 33]. The idea of static radius is evidential here as it can address a natural boundary of gravitationally bound frameworks in an extending universe ruled by a cosmological constant, as discussed in various circumstances [34–39].

5 Horizons of spacetime in f(R) gravity

Evet horizon can be calculated when metric function (4.4) vanishes, i.e, g(r) = 0. This gives the following equation :

$$\Lambda r^3 - 3\beta r^2 - 3r + 6M = 0 \tag{5.1}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 22 July 2022 doi:10.20944/preprints202207.0343.v1

Real positive roots of this equation gives the position of the event horizon. Interestingly computation shows that among the three different roots of eq. (5.1), only one are real positive root which gives us the radius of the horizon of static black hole in f(R) gravity. The real positive roots is :

$$r = \frac{\beta}{\Lambda} - \frac{2^{\frac{1}{3}}(-9\beta^2 - 9\Lambda)}{3\Lambda\left(P + \sqrt{4(-9\beta^2 - 9\Lambda)^3 + P^2}\right)^{\frac{1}{3}}} + \frac{\left(P + \sqrt{4(-9\beta^2 - 9\Lambda)^3 + P^2}\right)^{\frac{1}{3}}}{32^{\frac{1}{3}}\Lambda} \tag{5.2}$$

where $P = 54\beta^3 + 81\beta\Lambda - 162M\Lambda^2$.

Figure 1 depicts the profile of metric function g(r) versus radial coordinate r for three different values of the parameter β and Λ. Figure 1(a) is for changing cosmological constant Λ but fixed real constant $\beta = 1$ while the figure 1(b) is for varying real constant β by taking the constant value of $\Lambda = 1$.

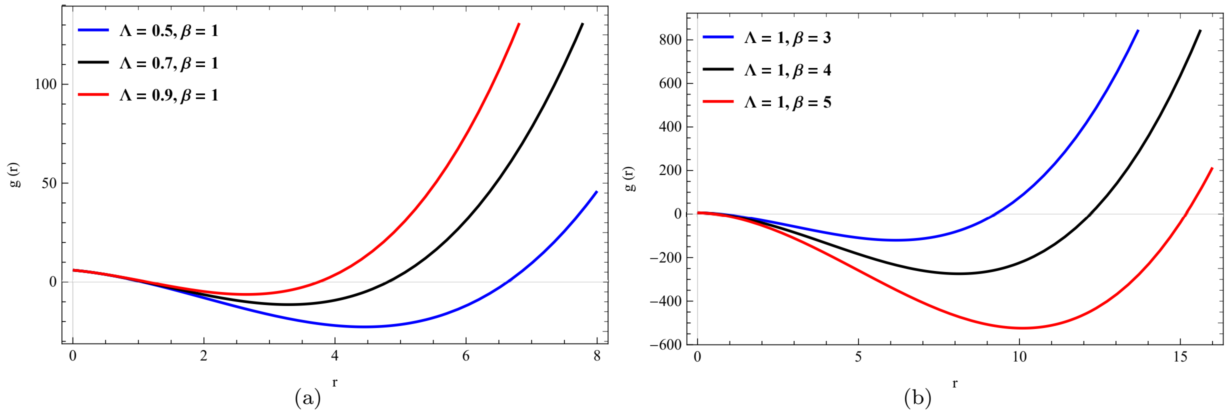

Figure 1. In 1(a), the behavior of g(r) versus r by changing Λ for a fixed $\beta = 1$. In 1(b), the behavior of g(r) with respect to r by changing β for a fixed $\Lambda = 1$ and .

6 Relativistic theory : energy and angular momentum

For circular orbit the energy and relativistic conserved angular momentum are respectively

$$E_c = \frac{\sqrt{r}(1 - \frac{2M}{r} + \beta r - \frac{1}{3}\Lambda r^2)}{\sqrt{r - 3M + \frac{1}{2}\beta r^2}} \tag{6.1}$$

and

$$L_c = \frac{\sqrt{r}\sqrt{6M + 3\beta r^2 - 2\Lambda r^3}}{\sqrt{6}\sqrt{r - 3M + \frac{1}{2}\beta r^2}} \tag{6.2}$$

It is evident that both E_c and L_c depends on various parameters like real constant β, cosmological constant Λ, radial coordinate r and mass (M) of the black hole. In figure 2(a) the variation of relativistic angular momentum L_c with respect to changing r and β for a fixed Λ and in figure 2(b), the variation of relativistic angular momentum L_c with respect to changing r and Λ for a fixed β, has been depicted. However, in figure 3(a), the variation of energy E_c with respect to changing r and β for a fixed Λ and in figure 3(b), the variation of energy E_c with respect to changing r and Λ for a fixed β, has been plotted.

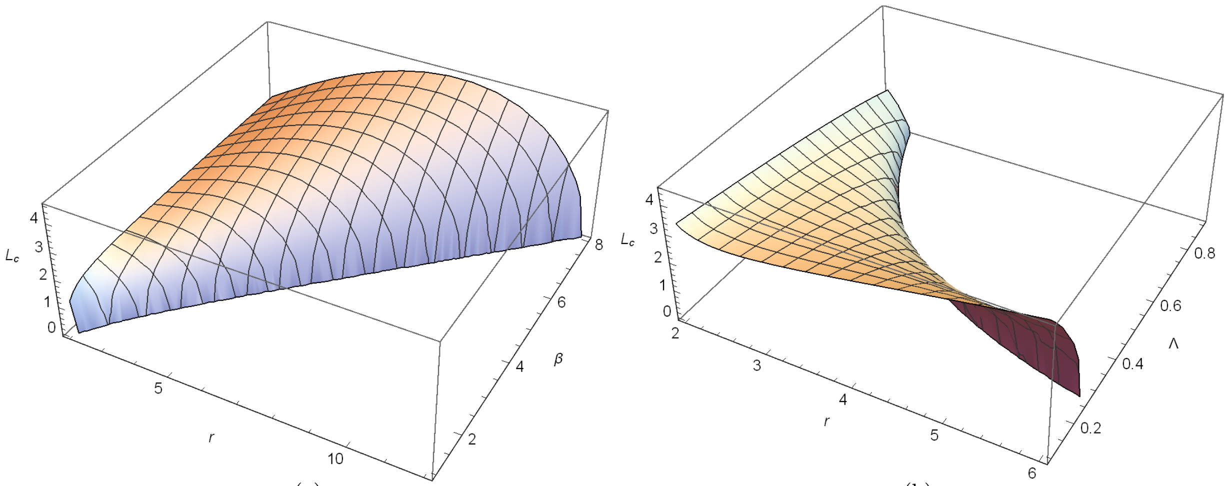

Figure 2. In 2(a), the variation of relativistic angular momentum L_c with respect to changing r and β for a fixed $\Lambda = 1$. In 2(b), the variation of relativistic angular momentum L_c with respect to changing r and Λ for a fixed $\beta = 1$. Here, M = 1.

Now, relativistic specific angular momentum $l_c (= \frac{L_c}{E_c})$ can be written as

$$l_c = \frac{\sqrt{r}\sqrt{6M + 3\beta r^2 - 2\Lambda r^3}}{\sqrt{6}(1 - \frac{2M}{r} + \beta r - \frac{1}{3}\Lambda r^2)} \tag{6.3}$$

Effective potential V_{eff} takes the following form :

$$V_{eff} = \sqrt{\left(1 - \frac{2M}{r} + \beta r - \frac{1}{3}\Lambda r^2\right)\left(1 + \frac{h^2}{r^2}\right)} \tag{6.4}$$

The nature of V_{eff} versus r by changing h, Λ and β has plotted in figure 4

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 22 July 2022 doi:10.20944/preprints202207.0343.v1

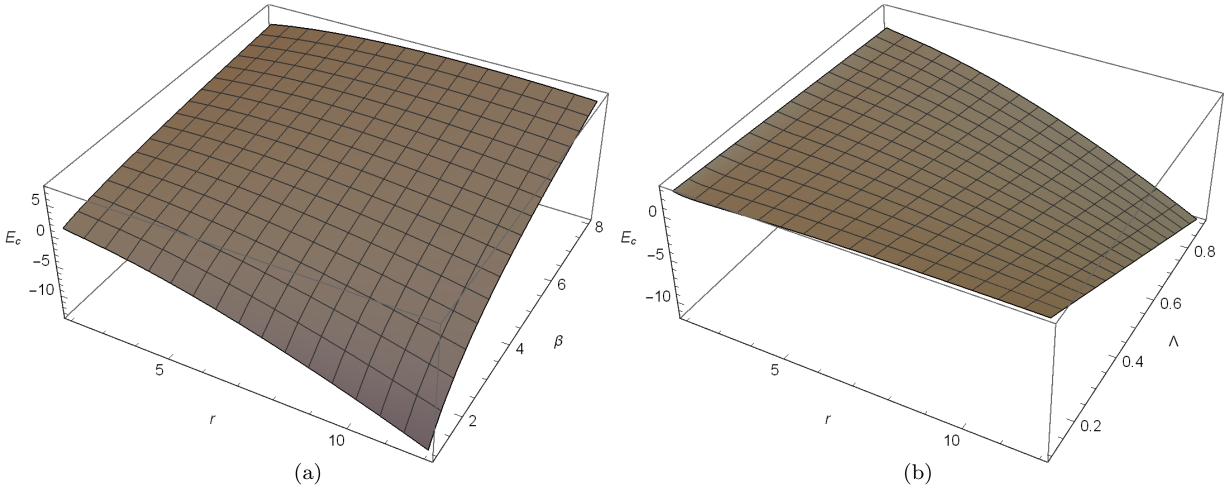

Figure 3. In 3(a), the variation of energy E_c with respect to changing r and β for a fixed $\Lambda = 1$. In 3(b), the variation of energy E_c with respect to changing r and Λ for a fixed $\beta = 1$. Here, M = 1.

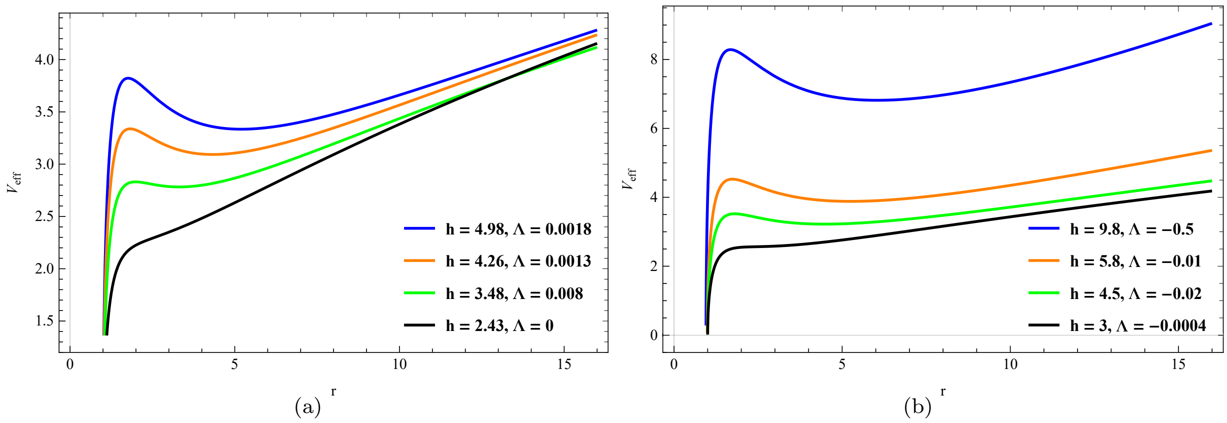

Figure 4. In 4(a) and 4(b), The behavior of V_{eff} with respect to r by varying h, Λ but constant $\beta = 1$. Here, M = 1.

7 Pseudo-Newtonian theory : potential, energy and angular momentum

PNP and effective potentials are

$$\psi = \frac{-6M + 3\beta r^2 - \Lambda r^3}{2\left(3r - 6 + 3\beta r^2 - \Lambda r^3\right)} \tag{7.1}$$

and

$$V_{eff} = \psi + \frac{l^2}{r^2} = \frac{-6M + 3\beta r^2 - \Lambda r^3}{2\left(3r - 6 + 3\beta r^2 - \Lambda r^3\right)} + \frac{l^2}{r^2} \tag{7.2}$$

The graph of PNP ψ for variation of both r and β with $\Lambda = 1$ is presented in figure 5(a). In figure 5(b), variation of PNP ψ for variation of both r and Λ with $\beta = 1$ has plotted. Moreover figure 6 depicts the dependence of PNP for different values of β (left figure) and Λ (right figure). Figure 5(a) shows that PNP will be decreases when real constant β will takes higher values while from figure 5(b) it is clear that PNP will be increases for increasing values of cosmological constant Λ.

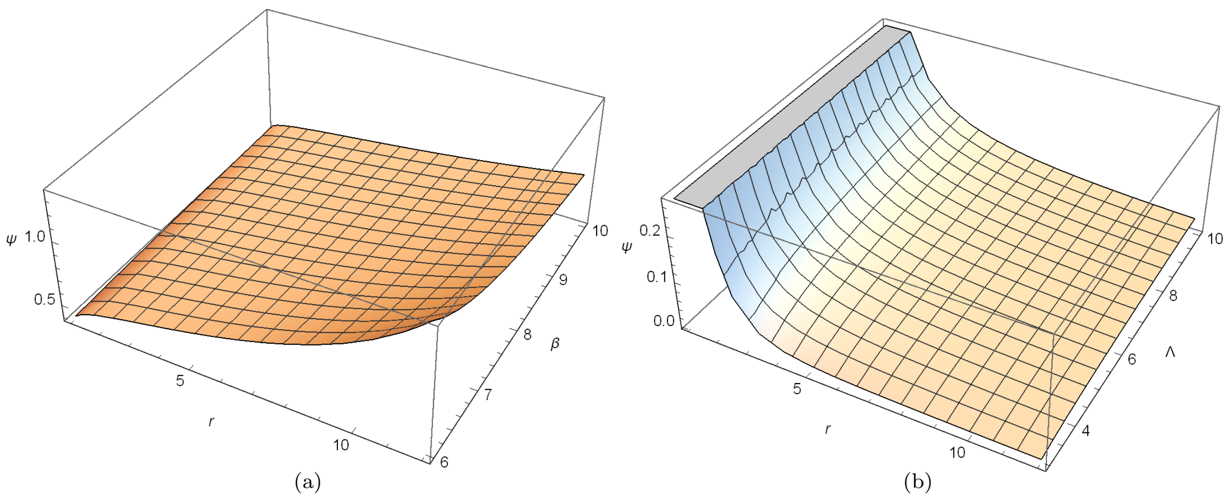

Figure 5. In 5(a), the variation of PNP ψ with respect to changing r and β for a fixed Λ = 1. In 5(b), the variation of PNP ψ with respect to changing r and Λ for a fixed $\beta = 1$. Here, M = 1.

Now, we obtain pseudo-Newtonian angular momentum as following :

$$l_c = \frac{\sqrt{r}\sqrt{6M + 3\beta r^2 - 2\Lambda r^3}}{\sqrt{6}(1 - \frac{2M}{r} + \beta r - \frac{1}{3}\Lambda r^2)} \tag{7.3}$$

Figure 7(a) shows the variation of l_c for the variation of both r and β with fixed Λ and Figure 7(b) shows the variation of l_c for the variation of both r and Λ with fixed β. Also a comparative plot of L_c(from eq.(6.2)) and l_c(from eq.(7.3)) has been depicted in figure 8(a), 8(b), 8(c) and 8(d) for two different values of β but fixed Λ (top pannel) and two different values of Λ but fixed β (bottom panel). This graphs tellus us that the angular momentum curve for Pseudo-Newtonian theory takes same form as the angular momentum curve in general relativistic theory.

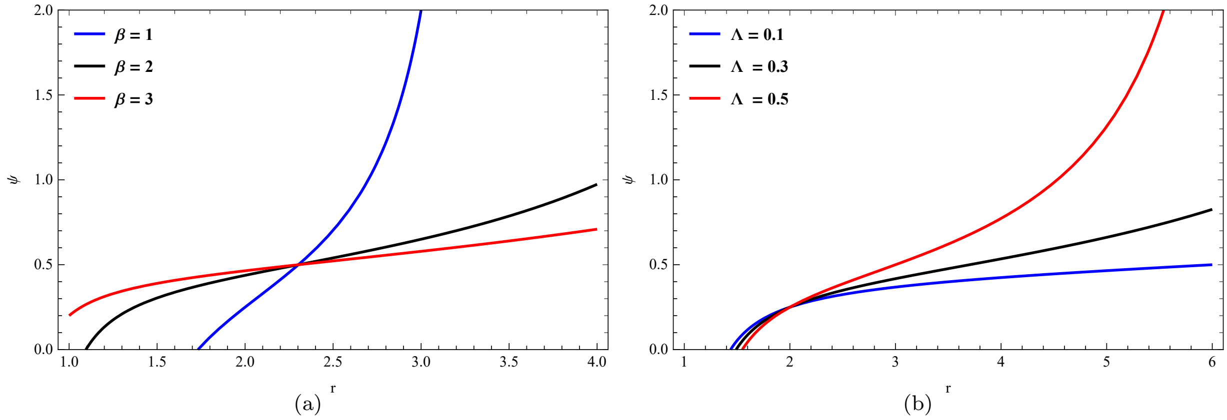

Figure 6. In 5(a), the variation of PNP ψ with respect to r for changing β for a fixed $\Lambda = 1$. In 5(b), the variation of PNP ψ with respect to r for changing Λ for a fixed $\beta = 1$. Here, M = 1.

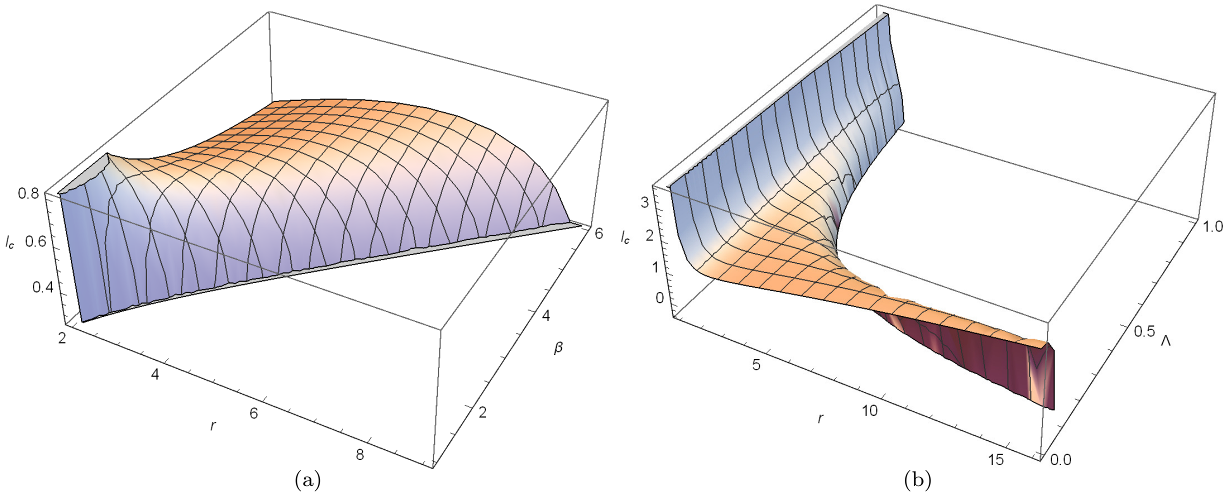

Figure 7. In 7(a), the variation of l_c with respect to r and β for fixed $\Lambda = 1$. In 7(b), the variation of l_c with respect to r and Λ for fixed $\beta = 1$. Here, M = 1.

and

$$e_c = \frac{9}{2}\frac{\left(-Mr + \frac{1}{2}\beta r^3 - \frac{2}{3}\Lambda r^4 + 4M^2 - 4M\beta r^2 - \frac{4}{3}M\Lambda r^3 + \beta^2 r^4 - \frac{2}{3}\Lambda\beta r^5 + \frac{1}{9}\Lambda^2 r^6\right)}{\left(3r - 6M + 3\beta r^2 - \Lambda r^3\right)^2} \tag{7.4}$$

 doi:10.20944/preprints202207.0343.v1

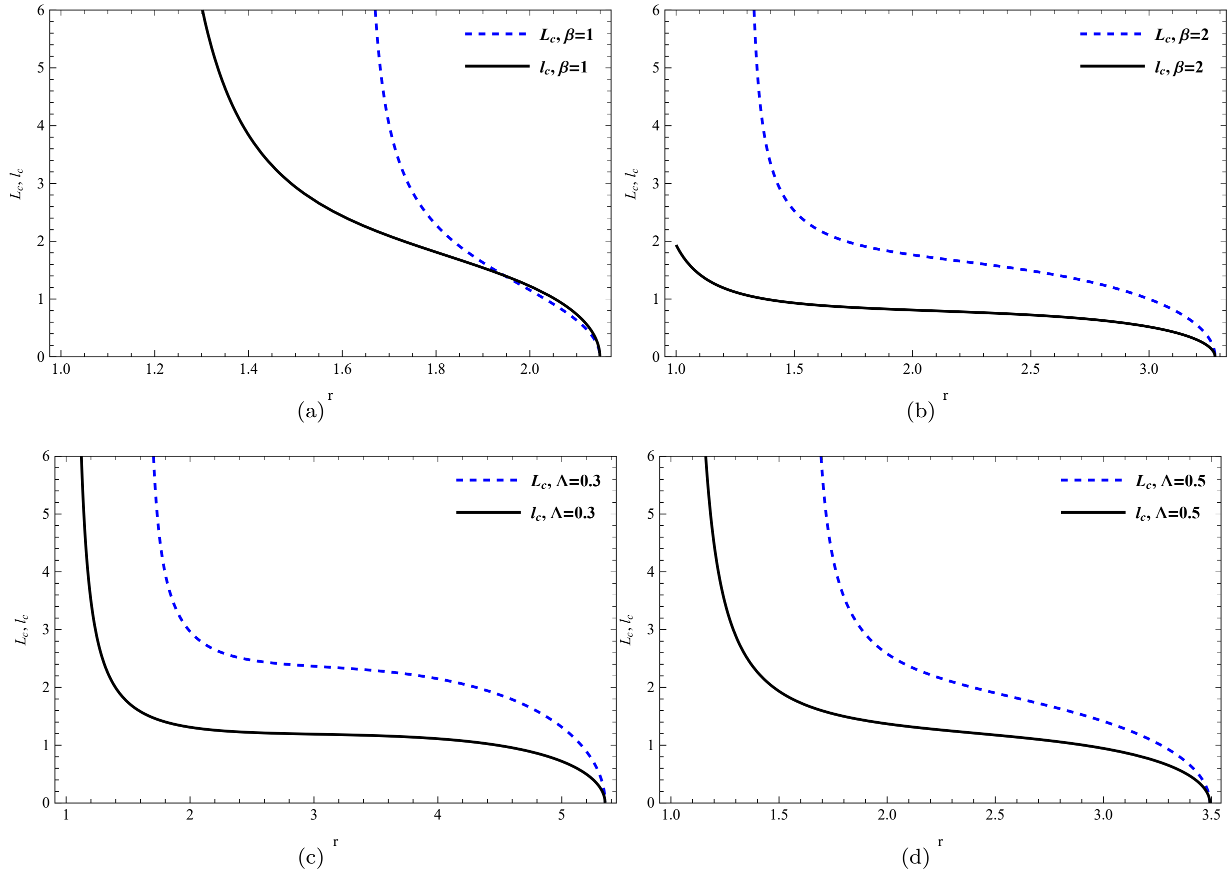

Figure 8. In 8(a) and 8(b), the variation of L_c and l_c with respect to r for changing $\beta = 1, 2$ but fixed $\Lambda = 1$. In 8(c) and 8(d), the variation of L_c and l_c with respect to r for changing $\Lambda = 0.3, 0.5$ but fixed $\beta = 1$. Here, M = 1.

The marginally stable circular orbits can be calculated from the positive root of the following equation :

$$9\Lambda\beta r^5 + (24\Lambda - 9\beta^2)r^4 - (27\beta + 90M\Lambda)r^3 + 108M\beta r^2 - 18Mr + 108M^2 = 0 \quad (7.5)$$

We obtain stable circular orbit for the condition :

$$9\Lambda\beta r^5 + (24\Lambda - 9\beta^2)r^4 - (27\beta + 90M\Lambda)r^3 + 108M\beta r^2 - 18Mr + 108M^2 > 0 \quad (7.6)$$

8 Conclusion

Now we summarize our work here. In this paper, we presented a general formulation of the trajectory of massive particle around a static and spherically symmetric black hole in four-dimensional space time. We also discussed the classification of the trajectories of massive particles by considering the possible zeros (positive) of the following function :

$$P(r) = E^2 - f(r)\Big(1 + \frac{h^2}{r^2}\Big)$$

the function f(r) is different for different black holes. Once we know about f(r) then we can estimate the possible trajectories of massive particles around that particular black hole. Effective potential, trajectory of massive particle at equatorial plane, condition for turning point and circular orbit has been presented for the black hole in general framework. In addition to this, we compared Pseudo-Newtonian Potential (PNP) with the effective potential. We outline spacetime structure in f(R) gravity and have noticed that horizon function of this black hole gives one positive root. To get a deeper insight, we plot horizon function with respect to radial coordinate r. We calculate energy and angular momentum in relativistic treatments. Here, we found that the energy and angular momentum depends on many parameters like real constant, cosmological constant, radial coordinate and mass of the black hole. We have done a graphical analysis to check the dependency of energy and angular momentum on these parameters. We also determined the effective potential for this black hole and make a graphical analysis.

Within the context of Pseudo-Newtonian theory, we calculate Pseudo-Newtonian Potential (PNP), angular momentum and energy. Here also we found that PNP, energy and angular momentum depends on various parameters like real constant, cosmological constant, radial coordinate and mass of the black hole. We have provided a graphical analysis of energy and angular momentum in this framework also. The graph of PNP for different values of β and Λ shows that PNP will be decreases when real constant β will takes higher values and also it will be increases for increasing values of cosmological constant Λ. More importantly, we have done a comparative plot of relativistic angular momentum L_c and Pesudo-Newtonian angular momentum l_c for different values of real constant and cosmological constant. These graphs illustrates that, in both the GR treatment and PNP treatment, the angular momentum curve takes same form and these plots justify the PNP theory. We also noticed that marginally stable circular orbit depends on the parameters like real constant, cosmological constant, radial coordinate and mass of the black hole. More interestingly, the general formulation of standard general gravity theory has applied very well to a black hole solution in f(R) gravity.

It is important to note that the analysis of the trajectory can also be applicable for black hole in higher dimension. As an example the metric for a N-dimensionalblack hole can be wrriten in the following form :

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{N-2} \tag{8.1}$$

where $d\Omega^2_{N-2}$ define the metric on unit N-2 sphere, given by

$$d\Omega^2_1 = d\phi$$

and

$$d\Omega^2_{j+1} = d\theta^2_j + Sin^2\theta_j d\Omega^2_j,\ j \geq 1$$

here the motion of any test particle (massive or photon) should be restricted to the equatorial plane ($\theta_j = \frac{\pi}{2}, j \geq 1$) because of the spherical symmetry of the space-time. Therefore, for future work, an extension of the above approach for non-spherical black hole case (especially axi-symmetric) would be interesting.

Data availibility

Data sharing is not applicable to this article, as no data sets were generated or analyzed during the current study.

Acknowledgement

The author is thankful to Surajit Das, department of physics, Cooch Behar Panchanan Barma University, Coochbehar, West Bengal, for various suggestions which developed the presentation of the paper.

References

[1] Ray.D.Inverno Introducing Einsteins General Theory of Relativity (Clarenden Press,Oxford)(2003).

[2] Narlikar.J.V Lectures on General Relativity and Cosmology (The Macmillan Company of India) (1978).

[3] Schutz.Bernerd A First Course in General Relativity (Cambridge University Press)(1995).

[4] Einstein. A Relativity: the special and general theory (Methuen, London) (1920).

[5] Einstein.A Preuss, Akad.Wiss.Berlin, Sitzber, 142 (1917).

[6] Einstein.A The Meaning of Relativity(Methuen, London) (1951).

[7] Einstein.A Preuss, Akad.Wiss.Berlin, Sitzber, 778,799,831 and 844 (1915).

[8] Schwarzchild.K Sitzber, Deut.Akad.Wiss.Berlin.KI.Math-Phys. Tech. 189 (1923).

[9] Siddhartha Sankar Sarkar and Ritabrata Biswas, Eur. Phys. J. C (2019) 79: 380.

[10] Shubhrangshu Ghosh and Banibrata Mukhopadhyay, the Astrophysical Journal, 667:367Y374, 2007 September 20.

[11] arXiv:gr-qc/0504025v1.

[12] Mukhopadhyay.B Astrophysical Journal 581 (2002) 427.

[13] Sumanta Chakraborty, Subenoy Chakraborty, Can.j. phys. 89: 689-695(2011).

[14] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D70, 043528 (2004); A. Dobado and A. L. Maroto Phys. Rev. D 52, 1895 (1995); A. de la Cruz-Dombriz and A. Dobado, Phys. Rev. D 74: 087501 (2006).

[15] S. Nojiri and S. D. Odintsov, Phys. Rev. D 68, 123512 (2003); S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. 4 115, (2007).

[16] J. A. R. Cembranos, Phys. Rev. Lett. 102, 141301 (2009).

[17] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010).

[18] I. D. Novikov and K. S. Thorne, in Black Holes, ed. C. DeWitt and B. DeWitt, New York: Gordon and Breach (1973).

[19] D.N. Page and K.S. Thorne, Astrophys. J. 191, 499 (1974).

[20] I.D. Novikov and K.S. Thorne, Astron. Astrophys. 24, 33 (1973).

[21] K.S. Thorne, Astrophys. J. 191, 507 (1974).

[22] M.Rostami, J.Sadeghi and S.Miraboutalebi, Physics of the Dark Universe Volume 29, September 2020, 100590.

[23] Sharma, P.; Nandan, H.; Nashed, G.G.L.; Giri, S.; Abebe, A. Geodesics of a Static Charged Black Hole Spacetime in f(R) Gravity. Symmetry 2022, 14, 309. https://doi.org/10.3390/sym14020309.

[24] Santiago Esteban Perez Bergliaffa and Yves Eduardo Chifarelli de Oliveira Nunes, Phys. Rev. D 84, 084006.

[25] Saheb Soroushfar and Sudhaker Upadhyay, Eur. Phys. J. Plus 135, 338 (2020).

[26] Paczynski.B and Witta.p Astron.Astrophys. 88 (1980) 23.

[27] Stuchlk.Z and Kovar.J Int.J.Mod.Phys.D 17 (2008) 2089.

[28] Stuchlk.Z, Slany and Kovar.J Class.Quantum.Grav 26 (2009) 215013.

[29] Minser.C.W, Thorne.K.S and Wheeler.J.A, Gravitation (Freeman, San Francisco, 1973).

[30] R. Saffari and S. Rahvar, Phys. Rev. D 77, 104028 (2008).

[31] S. Soroushfar, R. Saffari, J. Kunz and C. Lammerzahl, Phys. Rev. D 92, 044010 (2015).

[32] Z. Stuchlik, and S. Hledik, Phys.Rev. D 60 (1999) 044006.

[33] Z. Stuchlik, Bulletin of the Astronomical Institute of Czechoslovakia 34, 129 (1983).

[34] Z. Roupas, Universe 5(1), 12 (2019).

[35] I. Arraut, Universe 3(2), 45 (2017).

[36] S. Bhattacharya, K. F. Dialektopoulos, A. E. Romano, C. Skordis and T. N. Tomaras, JCAP 7, 018 (2017).

[37] V. Faraoni, Physics of the Dark Universe 11, 11 (2016).

[38] Z. Stuchlk, S. Hledk and J. Novotn, Phys. Rev. D 94, 103513(2016).

[39] Z. Stuchlk and J. Schee, JCAP 2011, 18 (2011).