 doi:10.20944/preprints202207.0175.v1

ORTHOGONAL FRAMES IN KREIN SPACES

FERRER OSMIN[1], SIERRA ARLEY[2] and OSVALDO POLO[3]

ABSTRACT. In the present paper we introduce the concept of orthogonal frames in Krein spaces, prove the independence of the choice of the fundamental symmetry, and from this we obtain a number of interesting properties that they satisfy. We show that there is no distinction between orthogonal frames in a Krein space and orthogonal frames in its associated Hilbert. Furthermore, we characterize frames dual to a given frame, which is a useful tool for constructing examples.

1. Introduction

The following definitions and results can be consulted at [12, 13].

Definition 1. Let $\mathfrak{F}$ be a vector space over the field $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$. A **inner product** in $\mathfrak{F}$ is a function

$$[\cdot,\cdot] : \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}$$

which satisfies the following properties:

i) $[\alpha x, y] = \alpha[x, y]$ for all $x, y \in \mathfrak{F}$ and every $\alpha \in \mathbb{K}$,
ii) $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in \mathfrak{F}$,
iii) $[x, y] = \overline{[y, x]}$ for all $x, y \in \mathfrak{F}$.

To space $(\mathfrak{F}, [\cdot,\cdot])$ is called **inner product spaces**. Similarly, the space $(\mathfrak{F}, -[\cdot,\cdot])$ is also an inner product space and is known as the anti-space of $(\mathfrak{F}, [\cdot,\cdot])$. For the condition $iii)$ above, it is clear that $[x, x] \in \mathbb{R}$ for all $x \in \mathbb{R}$, so, by the law of trichotomy of real numbers, it is possible to give the following definition.

Definition 2. Let $\mathfrak{V}$ be a vector subspace of $\mathfrak{F}$. If $\mathfrak{V}$ has only positive vectors ($[x, x] > 0$) (negatives, ($[x, x] < 0$)) and the null vector, is said to be **defined positive (negative)**. Also, if it has both positive and negative elements, it is said to be a space with indefinite inner product; otherwise it is said to be a space with semi-definite inner product.

Definition 3. It is said that two vectors $x, y \in \mathfrak{F}$ are orthogonal ($x \perp y$) if $[x, y] = 0$ and that two sets $\mathfrak{V}, \mathfrak{W} \subseteq \mathfrak{F}$ are orthogonal ($\mathfrak{V} \perp \mathfrak{W}$) if $x \perp y$ for all $x \in \mathfrak{V}$, $y \in \mathfrak{W}$; in particular if $\mathfrak{V}$ is reduced to a single vector x, is simply written $x \perp \mathfrak{W}$. In addition, if $\mathfrak{V}$ is a subset of $\mathfrak{F}$, the orthogonal complement of $\mathfrak{V}$ is given by $\mathfrak{V}^{\perp} := \{x \in \mathfrak{F} : x \perp \mathfrak{V}\}$ in such a way that $\mathfrak{V} \subseteq \mathfrak{V}^{\perp\perp}$ and $\mathfrak{V}^{\perp} = \mathfrak{V}^{\perp\perp\perp}$.

Definition 4. Let $\mathfrak{V}, \mathfrak{V}'$ be subspaces of $\mathfrak{F}$ such that $\mathfrak{V} \cap \mathfrak{V}' = \{0\}$. The direct sum of $\mathfrak{V}$ and $\mathfrak{V}'$ is denoted $\mathfrak{V}[\dot{+}]\mathfrak{V}'$. In addition, if $\mathfrak{V} \perp \mathfrak{V}'$ then it is called **orthogonal direct sum** and we write $\mathfrak{V} \oplus \mathfrak{V}'$.

Key words and phrases. Krein space, fundamental symmetry, Hilbert associated, frames.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 July 2022 doi:10.20944/preprints202207.0175.v1

Definition 5. (Fundamental descomposition) Let $(\mathfrak{F},[\cdot,\cdot])$ be a space with inner product. We say that $\mathfrak{F}$ admits a fundamental decomposition if subspaces exist $\mathfrak{F}^0 \subset \mathfrak{F}$, $\mathfrak{F}^+ \subset \mathfrak{F}$ and $\mathfrak{F}^- \subset \mathfrak{F}$ such that $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^+ \oplus \mathfrak{F}^-$, where $(\mathfrak{F}^0,[\cdot,\cdot])$ is a neutral space, $(\mathfrak{F}^+,[\cdot,\cdot])$ is positive definite, and $(\mathfrak{F}^-,[\cdot,\cdot])$ is defined negative. In this case, we call $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^+ \oplus \mathfrak{F}^-$ a **fundamental descomposition**.

The subspace $\mathfrak{V}^0 := \mathfrak{V} \cap \mathfrak{V}^{\perp}$ is called the isotropic part of $\mathfrak{V}$ and its non-zero elements are known as *isotropic vectors*. If $\mathfrak{V}^0 = \{0\}$ it is said that $\mathfrak{V}$ is a *non-degenerate subspace*, otherwise it is called a degenerate subspace.

Definition 6. A **Krein space** is a space with non-degenerate inner product $(\mathfrak{K},[\cdot,\cdot])$ which admits a fundamental decomposition $\mathfrak{K} = \mathfrak{K}^+ \oplus \mathfrak{K}^-$ con $(\mathfrak{K}^+,[\cdot,\cdot])$ and $(\mathfrak{K}^-,-[\cdot,\cdot])$ Hilbert spaces.

Definition 7. Let $(\mathfrak{K},[\cdot,\cdot])$ be a Krein space with fundamental decomposition $\mathfrak{K} = \mathfrak{K}^+ \oplus \mathfrak{K}^-$, then we know that there are unique operators

$$\mathfrak{P}^+ : (\mathfrak{K},[\cdot,\cdot]) \longrightarrow (\mathfrak{K}^+,[\cdot,\cdot]), \quad \mathfrak{P}^- : (\mathfrak{K},[\cdot,\cdot]) \longrightarrow (\mathfrak{K}^-,-[\cdot,\cdot])$$

in the following way $\mathfrak{P}^+(k) = k^+$ and $\mathfrak{P}^-(k) = k^-$ for all $k \in \mathfrak{K}$ where $k^+ \in \mathfrak{K}^+$, $k^- \in \mathfrak{K}^-$ and $k = k^+ + k^-$. To operators $\mathfrak{P}^+$ and $\mathfrak{P}^-$ are known as **fundamental projectors.** The operator $\mathfrak{J} : (\mathfrak{K},[\cdot,\cdot]) \longrightarrow (\mathfrak{K},[\cdot,\cdot])$ defined by $\mathfrak{J} := \mathfrak{P}^+ - \mathfrak{P}^-$, that is, for all $k \in \mathfrak{K}$,

$$\mathfrak{J}k = \mathfrak{P}^+k - \mathfrak{P}^-k = k^+ - k^-,$$

is called the **fundamental symmetry** of Krein space $\mathfrak{K}$ associated to the fundamental decomposition. From now on we will write $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ to denote Krein space $(\mathfrak{K},[\cdot,\cdot])$ with fundamental symmetry $\mathfrak{J}$ associated to the fundamental decomposition $\mathfrak{K} = \mathfrak{K}^+ \oplus \mathfrak{K}^-$.

Proposition 8. *Let $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ be a Krein space, then $\mathfrak{J} : (\mathfrak{K},[\cdot,\cdot]) \longrightarrow (\mathfrak{K},[\cdot,\cdot])$ is a symmetric operator, $\mathfrak{J}$-isometric, self-adjoint and invertible with $\mathfrak{J} = \mathfrak{J}^{-1}$.*

Proposition 9. *Let $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ be a Krein space. We define the function $[\cdot,\cdot]_{\mathfrak{J}} : \mathfrak{K}\times\mathfrak{K} \longrightarrow \mathbb{C}$ by means of the rule,*

$$[k_1,k_2]_{\mathfrak{J}} = [\mathfrak{J}k_1,k_2], \quad \textit{for all } k_1,k_2 \in \mathfrak{K},$$

then $[\cdot,\cdot]_{\mathfrak{J}}$ is a positive definite inner product, known as **$\mathfrak{J}$-inner product.**

Definition 10. Let $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ be a Krein space. The fundamental symmetry $\mathfrak{J}$ induces a norm in $\mathfrak{K}$ defined by

$$\|k\|_{\mathfrak{J}} := \sqrt{[k,k]_{\mathfrak{J}}}, \text{ for all } k \in \mathfrak{K}.$$

This norm is known as the **$\mathfrak{J}-$norm** of $\mathfrak{K}$. In a more explicit form

$$\|k\|_{\mathfrak{J}} := ([k^+,k^+] - [k^-,k^-])^{\frac{1}{2}},$$

for all $k \in \mathfrak{K}$.

Theorem 11. *Let $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ be a Krein space. Then $(\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}})$ is a Hilbert space, known as the Hilbert space associated to the Krein space $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 July 2022 doi:10.20944/preprints202207.0175.v1

Example 12. The usual inner product which gives Hilbert space structure to $\ell_2(\mathbb{N})$ is defined by

$$\langle \cdot,\cdot\rangle_{\ell_2} : \ell_2(\mathbb{N}) \times \ell_2(\mathbb{N}) \to \mathbb{C},\ \langle \{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}}\rangle_{\ell_2} := \sum_{n\in\mathbb{N}} \alpha_n\overline{\beta_n},$$

for all $\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})$. However, to $\ell_2(\mathbb{N})$ we can also see it as Krein space with an inner product whose $\mathfrak{J}$-inner product coincides with the usual one. In this sense, we define the following mapping,

$$[\cdot,\cdot]_{\ell_2} : \ell_2(\mathbb{N}) \times \ell_2(\mathbb{N}) \to \mathbb{C},\ [\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}}]_{\ell_2} := \sum_{n\in\mathbb{N}} (-1)^n\alpha_n\overline{\beta_n},$$

for all $\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})$, Thus, if $\{e_n\}_{n\in\mathbb{N}}$ is the canonical orthonormal basis of $\ell_2(\mathbb{N})$ then $\ell_2(\mathbb{N})$ accepts the following fundamental decomposition:

$$\ell_2(\mathbb{N}) = \ell_2^+(\mathbb{N}) \oplus \ell_2^-(\mathbb{N}),$$

where $\ell_2^+(\mathbb{N}) = \overline{\text{span}}\{e_{2n} : n \in \mathbb{N}\}$ and $\ell_2^-(\mathbb{N}) = \overline{\text{span}}\{e_{2n+1} : n \in \mathbb{N}\}$ with associated fundamental symmetry:

$$\mathfrak{J}_{\ell_2} : (\ell_2(\mathbb{N}), [\cdot,\cdot]_{\ell_2}) \to (\ell_2(\mathbb{N}), [\cdot,\cdot]_{\ell_2})$$

given by $\mathfrak{J}_{\ell_2}(\{\alpha_n\}_{n\in\mathbb{N}}) = \{(-1)^n\alpha_n\}_{n\in\mathbb{N}}$ for all $\{\alpha_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})$. Therefore, $[\cdot,\cdot]_{\mathfrak{J}_{\ell_2}} = \langle\cdot,\cdot\rangle_{\ell_2}$.

From now on, whenever we see $\ell_2(\mathbb{N})$ as Krein space we shall underst and it to be endowed with a fundamental symmetry $\mathfrak{J}_{\ell_2}$ such that $[\cdot,\cdot]_{\mathfrak{J}_{\ell_2}} = \langle\cdot,\cdot\rangle_{\ell_2}$. An example of such symmetry is the one developed above, and a more trivial example is the symmetry given by the identity operator in $\ell_2(\mathbb{N})$. Thus, we will write $\mathfrak{R}_2(\mathbb{N})$ instead of $\ell_2(\mathbb{N})$ when viewed as Krein space with such properties and the fundamental symmetry by $\mathfrak{J}_{\mathfrak{R}_2}$, to avoid confusion.

Definition 13. Let $(\mathfrak{K}_1, [\cdot,\cdot]_1, \mathfrak{J}_1)$ and $(\mathfrak{K}_2, [\cdot,\cdot]_2, \mathfrak{J}_2)$ be Krein spaces. The linear operator $T : (\mathfrak{K}_1, [\cdot,\cdot]_1) \longrightarrow (\mathfrak{K}_2, [\cdot,\cdot]_2)$ it is said **bounded** if there is a real number $c > 0$ such that for all $k \in \mathfrak{K}_1$,

$$\|Tk\|_{\mathfrak{J}_2} \le c\|k\|_{\mathfrak{J}_1}.$$

In the following $\mathcal{B}(\mathfrak{K})$ denote the space of linear and bounded operators in $(\mathfrak{K}, [\cdot,\cdot])$.

Remark 14. Given a bounded linear operator $T : (\mathfrak{K}_1, [\cdot,\cdot]_1) \longrightarrow (\mathfrak{K}_2, [\cdot,\cdot]_2)$, we can define the operator $T^\circ : (\mathfrak{K}_1, [\cdot,\cdot]_{\mathfrak{J}_1}) \longrightarrow (\mathfrak{K}_2, [\cdot,\cdot]_{\mathfrak{J}_2})$, as $T^\circ(k_1) := T(k_1)$ for all k_1 in $\mathfrak{K}_1$. So note that T° is bounded linear and in essence different from the operator T.

Definition 15. Let $(\mathfrak{K}_1, [\cdot,\cdot]_1, \mathfrak{J}_1)$ and $(\mathfrak{K}_2, [\cdot,\cdot]_2, \mathfrak{J}_2)$ be Krein spaces. The **adjoint** of the bounded linear operator $T : (\mathfrak{K}_1, [\cdot,\cdot]_1) \longrightarrow (\mathfrak{K}_2, [\cdot,\cdot]_2)$, is the only bounded linear operator $T^* : (Dom(T^*) \subset \mathfrak{K}_2, [\cdot,\cdot]_2) \longrightarrow (\mathfrak{K}_1, , [\cdot,\cdot]_1)$ such that for all $k_1 \in \mathfrak{K}_1$ and $k_2 \in Dom(T^*)$,

$$[Tk_1, k_2]_2 = [k_1, T^*k_2]_1.$$

Remark 16. Given a Krein space $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$, it is of great importance for what follows, to denote with $\mathfrak{J}^{\mathfrak{J}}$ to the linear and bounded mapping defined from Krein space $(\mathfrak{K}, [\cdot,\cdot])$ to the associated Hilbert space $(\mathfrak{K}, [\cdot,\cdot]_{\mathfrak{J}})$ as $\mathfrak{J}^{\mathfrak{J}}(k) := \mathfrak{J}(k)$ for all $k \in \mathfrak{K}$. We also define linear and bounded mappings: $\mathfrak{id}_{\mathfrak{J}} : (\mathfrak{K}, [\cdot,\cdot]_{\mathfrak{J}}) \to (\mathfrak{K}, [\cdot,\cdot])$, $\mathfrak{id}_{\mathfrak{J}}(k) := \mathfrak{id}_{\mathfrak{K}}(k) = k$, for all $k \in \mathcal{K}$, $\mathfrak{id}^{\mathfrak{J}} : (\mathfrak{K}, [\cdot,\cdot]) \to (\mathfrak{K}, [\cdot,\cdot]_{\mathfrak{J}})$, $\mathfrak{id}^{\mathfrak{J}}(k) := \mathfrak{id}_{\mathfrak{K}}(k) = k$, for all $k \in \mathfrak{K}$ and $\mathfrak{J}_{\mathfrak{J}} : (\mathfrak{K}, [\cdot,\cdot]_{\mathfrak{J}}) \to (\mathfrak{K}, [\cdot,\cdot])$, $\mathfrak{J}_{\mathfrak{J}}(k) := \mathfrak{J}(k)$, for all $k \in \mathfrak{K}$. and adjoints are given by $(\mathfrak{id}_{\mathfrak{J}})^* = \mathfrak{J}^{\mathfrak{J}}$ and $(\mathfrak{id}^{\mathfrak{J}})^* = \mathfrak{J}_{\mathfrak{J}}$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 July 2022 doi:10.20944/preprints202207.0175.v1

Proposition 17. [12, 13] *Let* $(\mathfrak{K}_1, [\cdot,\cdot]_1, \mathfrak{J}_1)$, $(\mathfrak{K}_2, [\cdot,\cdot]_2, \mathfrak{J}_2)$ *be Krein spaces and consider* $T : (\mathfrak{K}_1, [\cdot,\cdot]_1) \longrightarrow (\mathfrak{K}_2, [\cdot,\cdot]_2)$ *a bounded linear operator, then* $T^* = \mathfrak{J}_{1\mathfrak{J}_1}(T^\circ)^*\mathfrak{J}_2{}^{\mathfrak{J}_2}$.

The following theorem is of great importance insofar as it allows us to consider the existence of frames in Krein spaces, so it will be taken into account for what follows:

Theorem 18. *Let* $(\mathfrak{K} = \mathfrak{K}^+ \oplus \mathfrak{K}^-, [\cdot,\cdot], \mathfrak{J})$ *be a Krein space. If* $\{x_n^+\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}^+$ *and* $\{x_m^-\}_{m\in\mathbb{N}} \subseteq \mathfrak{K}^-$ *are frames for Hilbert spaces* $(\mathfrak{K}^+, [\cdot,\cdot]_\mathfrak{J})$ *and* $(\mathfrak{K}^-, [\cdot,\cdot]_\mathfrak{J})$ *respectively, then the sequence*

$$\left\{x_{n,m} := x_n^+ \oplus x_m^-\right\}_{n,m\in\mathbb{N}} \subseteq \mathfrak{K}^+ \oplus \mathfrak{K}^-,$$

is a frame in Hilbert space $(\mathfrak{K}^+ \oplus \mathfrak{K}^- = \mathfrak{K}, [\cdot,\cdot]_\mathfrak{J})$.

Proof. As $\{x_n^+\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}^+$ and $\{x_m^-\}_{m\in\mathbb{N}} \subseteq \mathfrak{K}^-$ are frames, then the pre-frame operators $T^\circ_{\mathfrak{K}^+} : \left(\ell_2(\mathbb{N}), \langle\cdot,\cdot\rangle_{\ell_2}\right) \longrightarrow (\mathfrak{K}^+, [\cdot,\cdot]_\mathfrak{J})$ *and* $T^\circ_{\mathfrak{K}^-} : \left(\ell_2(\mathbb{N}), \langle\cdot,\cdot\rangle_{\ell_2}\right) \longrightarrow (\mathfrak{K}^-, [\cdot,\cdot]_\mathfrak{J})$, exist, are bounded and surjective . Also, note that the pre-frame operator associated to the sequence $\mathbb{X} = \left\{x_{n,m} := x_n^+ \oplus x_m^-\right\}$ is given by

$$T^\circ_\mathbb{X} : \left(\ell_2(\mathbb{N}), \langle\cdot,\cdot\rangle_{\ell_2}\right) \longrightarrow \left(\mathfrak{K}^+ \oplus \mathfrak{K}^-, [\cdot,\cdot]_\mathfrak{J}\right),\ T^\circ_\mathbb{X} = T^\circ_{\mathfrak{K}^+} \oplus T^\circ_{\mathfrak{K}^-},$$

and therefore $T^\circ_\mathbb{X}$ is well-defined, bounded and surjective, since $T^\circ_{\mathfrak{K}^+}$ and $T^\circ_{\mathfrak{K}^-}$ have these properties. □

Definition 19. [5] Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ be a Krein space. A sequence $\mathbb{X} = \{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathfrak{K}$ is a frame in $(\mathfrak{K}, [\cdot,\cdot])$ if there are constants $0 < A \le B < +\infty$ such that

$$A\,\|k\|^2_\mathfrak{J} \le \sum_{n\in\mathbb{N}} |[k, x_n]|^2 \le B\,\|k\|^2_\mathfrak{J} \quad \forall k \in \mathfrak{K}. \tag{1.1}$$

The constants A and B are called bounds of the frame.

Definition 20. [7] Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ be a Krein space and the sequence $\mathbb{X} = \{x_n\}_{n\in\mathbb{N}}$ a frame in $(\mathfrak{K}, [\cdot,\cdot])$. It is said that $\mathbb{Y} = \{y_n\}_{n\in\mathbb{N}}$ is a dual frame to $\mathbb{X}$ if and only if,

$$\sum_{n\in\mathbb{N}} [k, y_n]x_n = k, \quad \text{for all} \quad k \in \mathfrak{K}.$$

Proposition 21. *Let* $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ *be a Krein space. If* $\mathbb{X}$ *is a frame in* $(\mathfrak{K}, [\cdot,\cdot])$. *Then the operator* $\mathcal{T}_\mathbb{X} : (\mathfrak{R}_2(\mathbb{N}), [\cdot,\cdot]_\mathfrak{R}) \to (\mathfrak{K}, [\cdot,\cdot])$, $\mathcal{T}_\mathbb{X}(\{c_n\}_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} c_n x_n$ *for all* $\{c_n\}_{n\in\mathbb{N}} \in \mathfrak{R}_2(\mathbb{N})$, *is well defined and bounded. This operator is called* ***pre-frame operator*** *associated to* $\mathbb{X}$.

Proof. It is sufficient to note that the following diagram commutes:

$$\begin{array}{ccc} (\mathfrak{R}_2(\mathbb{N}), [\cdot,\cdot]) & \overset{\mathcal{T}_\mathbb{X}}{\to} & (\mathfrak{K}, [\cdot,\cdot]) \\ \mathfrak{id}^{\mathfrak{J}_{\mathfrak{R}_2}} \downarrow & & \uparrow \mathfrak{id}_\mathfrak{J} \\ (\ell_2(\mathbb{N}), \langle\cdot,\cdot\rangle_{\ell_2}) & \overset{\mathcal{T}^\circ_\mathbb{X}}{\to} & (\mathfrak{K}, [\cdot,\cdot]_\mathfrak{J}) \end{array}$$

where $\mathcal{T}^\circ_\mathbb{X} : (\ell_2(\mathbb{N}), \langle\cdot,\cdot\rangle_{\ell_2}) \to (\mathfrak{K}, [\cdot,\cdot]_\mathfrak{J})$ is the pre-frame operator associated to $\mathbb{X}$ in $(\mathfrak{K}, [\cdot,\cdot]_\mathfrak{J})$. □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 July 2022 doi:10.20944/preprints202207.0175.v1

Proposition 22. *Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ be a Krein space and $\mathbb{X}$ a frame in $(\mathfrak{K}, [\cdot,\cdot])$. Then the adjoint operator of the pre-frame operator is given by*

$$\mathcal{T}_{\mathbb{X}}^{*} : (\mathfrak{K}, [\cdot,\cdot]) \to (\mathfrak{R}_2(\mathbb{N}), [\cdot,\cdot]_{\mathfrak{R}_2}), \quad \mathcal{T}_{\mathbb{X}}^{*}(k) = \mathfrak{J}_{\mathfrak{R}_2}(\{[k, x_n]\}_{n\in\mathbb{N}}), \ \text{for all } k \in \mathfrak{K},$$

*and is known as the **operator analysis** associated to $\mathbb{X}$.*

Definition 23. Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ be a Krein space and $\mathbb{X}$ a frame in$(\mathfrak{K}, [\cdot,\cdot])$. The operator

$$\mathcal{S}_{\mathbb{X}} : (\mathfrak{K}, [\cdot,\cdot]) \to (\mathfrak{K}, [\cdot,\cdot]), \ \text{defined by } \mathcal{S}_{\mathbb{X}} = \mathcal{T}_{\mathbb{X}} \mathfrak{J}_{\mathfrak{R}_2} \mathcal{T}_{\mathbb{X}}^{*},$$

is called **frame operator** associated with $\mathbb{X}$.

Remark 24. Let k in $\mathfrak{K}$ any, then the frame operator is given by:

$$\begin{aligned} \mathcal{S}_{\mathbb{X}}(k) =& \mathcal{T}_{\mathbb{X}} J_{\mathfrak{R}_2} \mathcal{T}_{\mathbb{X}}^{*}(k) = \mathcal{T}_{\mathbb{X}} \mathfrak{J}_{\mathfrak{R}_2}(\mathfrak{J}_{\mathfrak{R}_2}(\{[k, x_n]\}_{n\in\mathbb{N}})) \\ =& \mathcal{T}_{\mathbb{X}} \mathfrak{J}_{\mathfrak{R}_2}^{2}(\{[k, y_n]\}_{n\in\mathbb{N}}) = \mathcal{T}_{\mathbb{X}}(\{[k, x_n]\}_{n\in\mathbb{N}}) \\ =& \sum_{n\in\mathbb{N}} [k, x_n] x_n. \end{aligned}$$

Next we define an operator that will allow us to study orthogonal frames in Krein spaces.

Definition 25. Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ be a Krein space and $\mathbb{X}$, $\mathbb{Y}$ frames in $(\mathfrak{K}, [\cdot,\cdot])$. We define the operator

$$\mathfrak{W}_{\mathbb{X},\mathbb{Y}} : (\mathfrak{K}, [\cdot,\cdot]) \to (\mathfrak{K}, [\cdot,\cdot]), \quad \text{by} \quad \mathfrak{W}_{\mathbb{X},\mathbb{Y}} := \mathcal{T}_{\mathbb{X}} \mathfrak{J}_{\mathfrak{R}_2} \mathcal{T}_{\mathbb{Y}}^{*}.$$

The operator $\mathfrak{W}_{\mathbb{X},\mathbb{Y}}$ can be seen as follows:

$$\begin{aligned} \mathfrak{W}_{\mathbb{X},\mathbb{Y}}(k) =& \mathcal{T}_{\mathbb{X}} \mathfrak{J}_{\mathfrak{R}_2}(\mathcal{T}_{\mathbb{Y}}^{*}(k)) = \mathcal{T}_{\mathbb{X}} \mathfrak{J}_{\mathfrak{R}_2}(\mathfrak{J}_{\mathfrak{R}_2}(\{[k, y_n]\}_{n\in\mathbb{N}})) \\ =& \mathcal{T}_{\mathbb{X}} J_{\mathfrak{R}_2}^{2}(\{[k, y_n]\}_{n\in\mathbb{N}}) \\ =& \mathcal{T}_{\mathbb{X}}(\{[k, y_n]\}_{n\in\mathbb{N}}) \\ =& \sum_{n\in\mathbb{N}} [k, y_n] x_n, \quad \forall k \in \mathfrak{K}. \end{aligned}$$

Note that given two frames $\mathbb{X}$ and $\mathbb{Y}$ in a Krein space $\mathfrak{K}$, then it is clear that $\mathbb{Y}$ is dual to $\mathbb{X}$ if and only if the above operator satisfies that $\mathfrak{W}_{\mathbb{X},\mathbb{Y}} = \mathfrak{id}_{\mathfrak{K}}$. In addition, note that if $\mathbb{X} = \mathbb{Y}$ then $\mathfrak{W}_{\mathbb{X},\mathbb{Y}} = \mathcal{S}_{\mathbb{X}}$, that is, the operator $\mathfrak{W}_{\mathbb{X},\mathbb{Y}}$ turns out to be the frame operator associated with $\mathbb{X}$.

In the following section we present the main results of this work, extending the notion of orthogonal frames of Hilbert spaces to Krein spaces.

2. Orthogonal frames in indefinite metric spaces

In a Hilbert space $\mathfrak{H}$ it is said that $\mathbb{X}$ is a frame orthogonal to $\mathbb{Y}$ if and only if $ran\,((T_{\mathbb{X}}^{\circ})^{*}) \perp ran\,((T_{\mathbb{Y}}^{\circ})^{*})$, or equivalently, if and only if $\mathfrak{W}_{\mathbb{X},\mathbb{Y}}^{0} \equiv \mathbf{0}$. Now, the following theorem allows us to think about orthogonal frames in Krein spaces, attending to the above definition for the associated Hilbert space.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 July 2022 doi:10.20944/preprints202207.0175.v1

Theorem 26. *(**Existence of orthogonal frames in Krein spaces**) Let $(\mathfrak{K}=\mathfrak{K}^+\oplus\mathfrak{K}^-,[\cdot,\cdot],\mathfrak{J})$ be a Krein space. If $\{x_n^+\}_{n\in\mathbb{N}}$ is a frame orthogonal to $\{y_n^+\}_{n\in\mathbb{N}}$ in Hilbert space $(\mathfrak{K}^+,[\cdot,\cdot]_{\mathfrak{J}})$ and $\{x_m^-\}_{m\in\mathbb{N}}$ is a frame orthogonal to $\{y_m^-\}_{m\in\mathbb{N}}$ in Hilbert space $(\mathfrak{K}^-,[\cdot,\cdot]_{\mathfrak{J}})$, then the frame $\left\{x_{n,m}:=x_n^+\oplus x_m^-\right\}_{n,m\in\mathbb{N}}$ is orthogonal to the frame $\left\{y_{n,m}:=y_n^+\oplus y_m^-\right\}_{n,m\in\mathbb{N}}$ in Hilbert space $\left(\mathfrak{K}^+\oplus\mathfrak{K}^-=\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}}\right)$.*

Proof. Let $\mathbb{X}^+:=\{x_n^+\}_{n\in\mathbb{N}}$, $\mathbb{Y}^+:=\{y_n^+\}_{n\in\mathbb{N}}$, $\mathbb{X}^-:=\{x_m^-\}_{m\in\mathbb{N}}$, $\mathbb{Y}^-:=\{y_m^-\}_{m\in\mathbb{N}}$, $\mathbb{X}:=\{x_{n,m}\}_{n,m\in\mathbb{N}}$ and $\mathbb{Y}:=\{y_{n,m}\}_{n,m\in\mathbb{N}}$. Then $\mathbb{X}$, $\mathbb{Y}$ are frames in $\left(\mathfrak{K}^+\oplus\mathfrak{K}^-=\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}}\right)$ by the Theorem (18). In addition, by hypothesis it is satisfied that

$$\mathfrak{W}^{\circ}_{\mathbb{X}^+,\mathbb{Y}^+}:=\mathcal{T}^{\circ}_{\mathbb{X}^+}\left(\mathcal{T}^{\circ}_{\mathbb{Y}^+}\right)^*\equiv\mathbf{0},\ \ \mathfrak{W}^{\circ}_{\mathbb{X}^-,\mathbb{Y}^-}:=\mathcal{T}^{\circ}_{\mathbb{X}^-}\left(\mathcal{T}^{\circ}_{\mathbb{Y}^-}\right)^*\equiv\mathbf{0}.$$

So, what we have to prove is $\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}:=\mathcal{T}^{\circ}_{\mathbb{X}}\left(\mathcal{T}^{\circ}_{\mathbb{Y}}\right)^*\equiv\mathbf{0}$. Indeed, note that for any $k^+\oplus k^-\in\mathfrak{K}^+\oplus\mathfrak{K}^-$,

$$\begin{aligned}\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}\left(k^+\oplus k^-\right) &= \mathcal{T}^{\circ}_{\mathbb{X}}\left(\mathcal{T}^{\circ}_{\mathbb{Y}}\right)^*\left(k^+\oplus k^-\right)\\ &= \left(\mathcal{T}^{\circ}_{\mathbb{X}^+}\oplus\mathcal{T}^{\circ}_{\mathbb{X}^-}\right)\left(\left(\mathcal{T}^{\circ}_{\mathbb{Y}^+}\right)^*\oplus\left(-\left(\mathcal{T}^{\circ}_{\mathbb{Y}^-}\right)^*\right)\right)\left(k^+\oplus k^-\right)\\ &= \left(\mathcal{T}^{\circ}_{\mathbb{X}^+}\oplus\mathcal{T}^{\circ}_{\mathbb{X}^-}\right)\left(\left(\mathcal{T}^{\circ}_{\mathbb{Y}^+}\right)^*\left(k^+\right)\oplus\left(-\left(\mathcal{T}^{\circ}_{\mathbb{Y}^-}\right)^*\left(k^-\right)\right)\right)\\ &= \mathcal{T}^{\circ}_{\mathbb{X}^+}\left(\left(\mathcal{T}^{\circ}_{\mathbb{Y}^+}\right)^*\left(k^+\right)\right)-\mathcal{T}^{\circ}_{\mathbb{X}^-}\left(\left(\mathcal{T}^{\circ}_{\mathbb{Y}^-}\right)^*\left(k^-\right)\right)\\ &= \mathfrak{W}^{\circ}_{\mathbb{X}^+,\mathbb{Y}^+}(k^+)-\mathfrak{W}^{\circ}_{\mathbb{X}^-,\mathbb{Y}^-}(k^-)\\ &= 0.\end{aligned}$$

Therefore $\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}\equiv\mathbf{0}$. Then $\mathbb{X}$ is orthogonal to $\mathbb{Y}$ in $(\mathfrak{K}^+\oplus\mathfrak{K}^-=\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}})$. □

Definition 27. (**Orthogonal frames in Krein**) Let $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ be a Krein space and $\mathbb{X}$, $\mathbb{Y}$ frames in $(\mathfrak{K},[\cdot,\cdot])$. We say that $\mathbb{X}$ is orthogonal to $\mathbb{Y}$ in $(\mathfrak{K},[\cdot,\cdot])$ if it is satisfied that $ran(\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}})\perp ran(\mathcal{T}^*_{\mathbb{Y}})$, where orthogonality is with respect to $[\cdot,\cdot]_{\mathfrak{R}_2}$.

Proposition 28. *Let $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ be a Krein space and $\mathfrak{J}_{\mathfrak{R}_2}$, $\mathfrak{J}'_{\mathfrak{R}'_2}$ fundamental symmetries of $\ell_2(\mathbb{N})$ such that $[\cdot,\cdot]_{\mathfrak{R}_2}=\langle\cdot,\cdot\rangle_{\ell_2}=[\cdot,\cdot]_{\mathfrak{R}'_2}$. Then, $\mathbb{X}$ is orthogonal to $\mathbb{Y}$ in Krein space $(\mathfrak{K},[\cdot,\cdot])$ with respect to $[\cdot,\cdot]_{\mathfrak{R}_2}$ if and only if $\mathbb{X}$ is orthogonal to $\mathbb{Y}$ in Krein space $(\mathfrak{K},[\cdot,\cdot])$ with respect to $[\cdot,\cdot]_{\mathfrak{R}'_2}$.*

Proof. ($\Rightarrow$) Suppose that $\mathbb{X}$ is orthogonal to $\mathbb{Y}$ in $(\mathfrak{K},[\cdot,\cdot])$ and with respect to$[\cdot,\cdot]_{\mathfrak{R}_2}$, that is,

$$ran(\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}})\perp ran(\mathcal{T}^*_{\mathbb{Y}}).$$

Let's see what $ran(\mathfrak{J}'_{\mathfrak{R}'_2}\mathcal{T}^*_{\mathbb{X}})\perp ran(\mathcal{T}^*_{\mathbb{Y}})$, in effect, let $\{\alpha_n\}_{n\in\mathbb{N}}\in ran(\mathfrak{J}'_{\mathfrak{R}'_2}\mathcal{T}^*_{\mathbb{X}})$ and $\{\beta_n\}_{n\in\mathbb{N}}\in ran(\mathcal{T}^*_{\mathbb{Y}})$ be any, then $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are of the form

$$\{\alpha_n\}_{n\in\mathbb{N}}=\mathfrak{J}'_{\mathfrak{R}'_2}\mathcal{T}^*_{\mathbb{X}}(k_1)\text{ y }\{\beta_n\}_{n\in\mathbb{N}}=\mathcal{T}^*_{\mathbb{Y}}(k_2),\ k_1,k_2\in\mathfrak{K}.$$

Then,

$$\begin{aligned}[\{\alpha_n\}_{n\in\mathbb{N}},\{\beta_n\}_{n\in\mathbb{N}}]_{\mathfrak{R}_2} &= [\mathfrak{J}'_{\mathfrak{R}'_2}\mathcal{T}^*_{\mathbb{X}}(k_1),\mathcal{T}^*_{\mathbb{Y}}(k_2)]_{\mathfrak{R}_2}=[\mathcal{T}^*_{\mathbb{X}}(k_1),\mathcal{T}^*_{\mathbb{Y}}(k_2)]_{\mathfrak{J}'_{\mathfrak{R}'_2}}\\ &= [\mathcal{T}^*_{\mathbb{X}}(k_1),\mathcal{T}^*_{\mathbb{Y}}(k_2)]_{\mathfrak{J}_{\mathfrak{R}_2}}=[\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}(k_1),\mathcal{T}^*_{\mathbb{Y}}(k_2)]_{\mathfrak{R}_2}=0.\end{aligned}$$

 doi:10.20944/preprints202207.0175.v1

Therefore $ran(\mathfrak{J}'_{\mathfrak{R}'_2}\mathcal{T}^*_{\mathbb{X}}) \perp ran(\mathcal{T}^*_{\mathbb{Y}})$ and with this $\mathbb{X}$ is orthogonal to $\mathbb{Y}$ in $(\mathfrak{K}, [\cdot,\cdot])$ with respect to $[\cdot,\cdot]_{\mathfrak{R}'_2}$.

($\Leftarrow$) Similar to the previous test. □

Proposition 29. *Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ be a Krein space and $\mathbb{X}, \mathbb{Y}$ frames in $(\mathfrak{K}, [\cdot,\cdot])$, then the following statements are equivalent*

i) *$\mathbb{X}$ is orthogonal to $\mathbb{Y}$ in $(\mathfrak{K}, [\cdot,\cdot])$,*
ii) *$\mathbb{Y}$ is orthogonal to $\mathbb{X}$ in $(\mathfrak{K}, [\cdot,\cdot])$,*
iii) *$W_{\mathbb{X},\mathbb{Y}} \equiv \mathbf{0}$,*
iv) *$W_{\mathbb{Y},\mathbb{X}} \equiv \mathbf{0}$.*

Proof. *i)* $\Rightarrow$ *ii)* Suppose that $\mathbb{X}$ is orthogonal to $\mathbb{Y}$, that is, $ran(\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}) \perp ran(\mathcal{T}^*_{\mathbb{Y}})$ with respect to $[\cdot,\cdot]_{\mathfrak{R}_2}$. Let's see what $\mathbb{Y}$ is orthogonal to $\mathbb{X}$. Let $\{\alpha_n\}_{n\in\mathbb{N}} \in ran(\mathcal{T}^*_{\mathbb{X}})$ and $\{\beta_n\}_{n\in\mathbb{N}} \in ran(\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}})$ be any. Then, there are vectors $k_1, k_2 \in \mathfrak{K}$ such that $\{\alpha_n\}_{n\in\mathbb{N}} = \mathcal{T}^*_{\mathbb{X}}(k_2)$ and $\{\beta_n\}_{n\in\mathbb{N}} = \mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}(k_1)$,

$$\begin{aligned}
[\{\beta_n\}_{n\in\mathbb{N}}, \{\alpha_n\}_{n\in\mathbb{N}}]_{\mathfrak{R}_2} &= [\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}(k_1), \mathcal{T}^*_{\mathbb{X}}(k_2)]_{\mathfrak{R}_2} \\
&= [\mathcal{T}^*_{\mathbb{Y}}(k_1), \mathfrak{J}^*_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}(k_2)]_{\mathfrak{R}_2} \\
&= [\underbrace{\mathcal{T}^*_{\mathbb{Y}}(k_1)}_{\in ran(\mathcal{T}^*_{\mathbb{Y}})}, \underbrace{\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}(k_2)}_{\in ran(\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}})}]_{\mathfrak{R}_2} \\
&= 0
\end{aligned}$$

Therefore $ran(\mathfrak{J}_{\mathfrak{R}_2}T^*_{\mathbb{Y}}) \perp ran(\mathcal{T}^*_{\mathbb{X}})$ with respect to $[\cdot,\cdot]_{\mathfrak{R}_2}$. We conclude that $\mathbb{Y}$ is orthogonal to $\mathbb{X}$.

ii) $\Rightarrow$ *iii)* Suppose that $ran(\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}) \perp ran(\mathcal{T}^*_{\mathbb{X}})$ of inner product $[\cdot,\cdot]_{\mathfrak{R}_2}$. Let's see what $\mathfrak{W}_{\mathbb{X},\mathbb{Y}} \equiv \mathbf{0}$. Indeed, let $k_1, k_2 \in \mathfrak{K}$ be any,

$$\begin{aligned}
[\mathfrak{W}_{\mathbb{X},\mathbb{Y}}(k_1), k_2] &= [\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}(k_1), k_2] \\
&= [\underbrace{\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}(k_1)}_{\in ran(\mathfrak{J}_{\mathfrak{R}_2}T^*_{\mathbb{Y}})}, \underbrace{\mathcal{T}^*_{\mathbb{X}}(k_2)}_{\in ran(\mathcal{T}^*_{\mathbb{X}})}]_{\mathfrak{R}_2} \\
&= 0
\end{aligned}$$

iii) $\Rightarrow$ *iv)* $\mathbf{0} \equiv \mathbf{0}^* = \mathfrak{W}^*_{\mathbb{X},\mathbb{Y}} = (\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}})^* = (\mathcal{T}^*_{\mathbb{Y}})^*\mathfrak{J}^*_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}} = \mathcal{T}_{\mathbb{Y}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}} = \mathfrak{W}_{\mathbb{Y},\mathbb{X}}$.

iv) $\Rightarrow$ *i)* Suppose that $\mathfrak{W}_{\mathbb{Y},\mathbb{X}} \equiv \mathbf{0}$. Let's see what $\mathbb{X}$ is orthogonal to $\mathbb{Y}$, that is, $ran(\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}) \perp ran(\mathcal{T}^*_{\mathbb{Y}})$ with respect to $[\cdot,\cdot]_{\mathfrak{R}_2}$. Let $\{\alpha_n\}_{n\in\mathbb{N}} = \mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}(k_1)$, $\{\beta_n\}_{n\in\mathbb{N}} = \mathcal{T}^{[*]}_{\mathbb{Y}}(k_2)$ for some $k_1, k_2 \in \mathfrak{K}$. Let's note that,

$$\begin{aligned}
[\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}}]_{\mathfrak{R}_2} &= [J_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}(k_1), \mathcal{T}^*_{\mathbb{Y}}(k_2)]_{\mathfrak{R}_2} \\
&= [\mathcal{T}_{\mathbb{Y}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}(k_1), k_2] \\
&= [\mathfrak{W}_{\mathbb{Y},\mathbb{X}}(k_1), k_2] \\
&= [0, k_2] \\
&= 0.
\end{aligned}$$

Therefore $\mathbb{X}$ is orthogonal to $\mathbb{Y}$. □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 July 2022 doi:10.20944/preprints202207.0175.v1

Remark 30. If $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ is a Krein space and $\mathbb{X}$, $\mathbb{Y}$ are orthogonal frames in $(\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}})$ we know that the operator $\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}:(\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}})\longrightarrow(\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}})$ defined by $\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}:=\mathcal{T}^{\circ}_{\mathbb{X}}(\mathcal{T}^{\circ}_{\mathbb{Y}})^*$ turns out to be the operator $\mathbf{0}$. Thus it is useful to ask about the relationship between $\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}$ and $\mathfrak{W}_{\mathbb{X},\mathbb{Y}}$ where $\mathfrak{W}_{\mathbb{X},\mathbb{Y}}:(\mathfrak{K},[\cdot,\cdot])\longrightarrow(\mathfrak{K},[\cdot,\cdot])$ is given by $\mathfrak{W}_{\mathbb{X},\mathbb{Y}}:=\mathcal{T}_{\mathbb{X}}J_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}$, same for the relationships between operators $\mathfrak{W}_{\mathbb{X},\mathfrak{J}\mathbb{Y}}$, $\mathfrak{W}_{\mathfrak{J}\mathbb{X},\mathbb{Y}}$ and $\mathfrak{W}_{\mathfrak{J}\mathbb{X},\mathfrak{J}\mathbb{Y}}$. The following result establishes some relationships between these operators.

Proposition 31. *Let* $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ *be a Krein space and* $\mathbb{X}$*,* $\mathbb{Y}$ *frames in* $(\mathfrak{K},[\cdot,\cdot])$*. Then they are equivalent:*

i) $\mathbb{X}$*,* $\mathbb{Y}$ *are orthogonal frames in* $(\mathfrak{K},[\cdot,\cdot])$*,*
ii) $\mathbb{X}$*,* $\mathfrak{J}\mathbb{Y}$ *are orthogonal frames in* $(\mathfrak{K},[\cdot,\cdot])$*,*
iii) $\mathfrak{J}\mathbb{X}$*,* $\mathfrak{J}\mathbb{Y}$ *are orthogonal frames in* $(\mathfrak{K},[\cdot,\cdot])$*,*
iv) $\mathfrak{J}\mathbb{X}$*,* $\mathbb{Y}$ *are orthogonal frames in* $(\mathfrak{K},[\cdot,\cdot])$*.*

Proof. Let $\mathbb{X}$, $\mathbb{Y}$ be frames in $(\mathfrak{K},[\cdot,\cdot])$ then it is clear that $\mathfrak{J}\mathbb{X}$, $\mathfrak{J}\mathbb{Y}$, are also frames in $(\mathfrak{K},[\cdot,\cdot])$.

i) $\Rightarrow$ *ii)* Suppose that $\mathbb{X}$, $\mathbb{Y}$ are orthogonal in $(\mathfrak{K},[\cdot,\cdot])$, i.e., $\mathfrak{W}_{\mathbb{X},\mathbb{Y}}\equiv\mathbf{0}$. However,

$$\mathfrak{W}_{\mathbb{X},\mathfrak{J}\mathbb{Y}} = \mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathfrak{J}\mathbb{Y}}=\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\left(\mathfrak{J}\mathcal{T}_{\mathbb{Y}}\right)^*=\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}\mathfrak{J}^*=\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}\mathfrak{J}=\mathfrak{W}_{\mathbb{X},\mathbb{Y}}\mathfrak{J}=\mathbf{0}\mathfrak{J}\equiv\mathbf{0}.$$

ii) $\Rightarrow$ *iii)* Suppose that $\mathbb{X}$, $\mathfrak{J}\mathbb{Y}$ are orthogonal in $(\mathfrak{K},[\cdot,\cdot])$. We want to prove that $\mathfrak{W}_{\mathfrak{J}\mathbb{X},\mathfrak{J}\mathbb{Y}}\equiv\mathbf{0}$. In effect,

$$\mathfrak{W}_{\mathfrak{J}\mathbb{X},\mathfrak{J}\mathbb{Y}}=\mathcal{T}_{\mathfrak{J}\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathfrak{J}\mathbb{Y}}=\mathfrak{J}\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathfrak{J}\mathbb{Y}}=\mathfrak{J}\mathfrak{W}_{\mathbb{X},\mathfrak{J}\mathbb{Y}}=\mathfrak{J}\mathbf{0}\equiv\mathbf{0}.$$

iii) $\Rightarrow$ *iv)* Suppose that $\mathfrak{W}_{\mathfrak{J}\mathbb{X},\mathfrak{J}\mathbb{Y}}\equiv\mathbf{0}$. Let us see that $\mathfrak{W}_{\mathfrak{J}\mathbb{X},\mathbb{Y}}\equiv\mathbf{0}$. In effect,

$$\mathfrak{W}_{\mathfrak{J}\mathbb{X},\mathbb{Y}}=\mathcal{T}_{\mathfrak{J}\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}=\mathfrak{J}\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}=\mathfrak{J}\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}\mathfrak{id}_{\mathfrak{K}}=\mathfrak{J}\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}\mathfrak{J}\mathfrak{J}=\mathfrak{W}_{\mathfrak{J}\mathbb{X},\mathfrak{J}\mathbb{Y}}\mathfrak{J}=\mathbf{0}\mathfrak{J}\equiv\mathbf{0}.$$

iv) $\Rightarrow$ *i)* Suppose that $\mathfrak{W}_{\mathfrak{J}\mathbb{X},\mathbb{Y}}\equiv\mathbf{0}$. Then note that

$$\mathfrak{W}_{\mathbb{X},\mathbb{Y}}=\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}T^*_{\mathbb{Y}}=\mathfrak{id}_{\mathfrak{K}}\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}=\mathfrak{J}\mathfrak{J}\mathcal{T}_{\mathbb{X}}J_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}=\mathfrak{J}\mathcal{T}_{\mathfrak{J}\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{Y}}=\mathfrak{J}\mathfrak{W}_{\mathfrak{J}\mathbb{X},\mathbb{Y}}=\mathfrak{J}\mathbf{0}\equiv\mathbf{0}.$$

□

Theorem 32. *Let* $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ *be a Krein space and* $\mathbb{X}$*,* $\mathbb{Y}$ *frames in* $(\mathfrak{K},[\cdot,\cdot])$*. Then,* $\mathbb{X}$*,* $\mathbb{Y}$ *are orthogonal in* $(\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}})$ *if and only if* $\mathbb{X}$*,* $\mathbb{Y}$ *are orthogonal in* $(\mathfrak{K},[\cdot,\cdot])$*.*

Proof. $(\Rightarrow)$ Suppose that $\mathbb{X}$, $\mathbb{Y}$ are orthogonal in $(\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}})$, i.e., $\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}\equiv\mathbf{0}$. Now, note that the following diagram commutes:

$$\begin{array}{ccc}(\mathfrak{K},[\cdot,\cdot]) & \overset{\mathfrak{W}_{\mathbb{X},\mathbb{Y}}}{\to} & (\mathfrak{K},[\cdot,\cdot]) \\ \mathfrak{J}^{\mathfrak{J}}\downarrow & & \uparrow\mathfrak{id}_{\mathfrak{J}} \\ (\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}}) & \overset{\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}}{\to} & (\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}})\end{array}$$

in effect, let $k\in\mathfrak{K}$ be anyone, then

$$\begin{aligned}\mathfrak{id}_{\mathfrak{J}}\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}\mathfrak{J}^{\mathfrak{J}}(k) &= \mathfrak{id}_{\mathfrak{J}}\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}\mathfrak{J}(k)=\mathfrak{id}_{\mathfrak{J}}\left(\textstyle\sum_{n\in\mathbb{N}}[\mathfrak{J}(k),y_n]_{\mathfrak{J}}x_n\right)\\ &= \mathfrak{id}_{\mathfrak{K}}\left(\textstyle\sum_{n\in\mathbb{N}}[\mathfrak{J}^2(k),y_n]x_n\right)=\textstyle\sum_{n\in\mathbb{N}}[k,y_n]x_n=\mathfrak{W}_{\mathbb{X},\mathbb{Y}}.\end{aligned}$$

Therefore $\mathfrak{W}_{\mathbb{X},\mathbb{Y}}=id_J\mathbf{0}\mathfrak{J}^{\mathfrak{J}}\equiv\mathbf{0}$ and $\mathbb{X}$, $\mathbb{Y}$ are orthogonal frames in $(\mathfrak{K},[\cdot,\cdot])$.

($\Leftarrow$) Suppose that $\mathfrak{W}_{\mathbb{X},\mathbb{Y}} \equiv \mathbf{0}$. Let's see what $\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}} \equiv \mathbf{0}$. For this note that the following diagram commutes:

$$\begin{array}{ccc} (\mathfrak{K}, [\cdot,\cdot]_{\mathfrak{J}}) & \stackrel{\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}}}{\rightarrow} & (\mathfrak{K}, [\cdot,\cdot]_{\mathfrak{J}}) \\ \mathfrak{I}_{\mathfrak{J}} \downarrow & & \uparrow \mathfrak{id}^{\mathfrak{J}} \\ (\mathfrak{K}, [\cdot,\cdot]) & \stackrel{\mathfrak{W}_{\mathbb{X},\mathbb{Y}}}{\rightarrow} & (\mathfrak{K}, [\cdot,\cdot]) \end{array}$$

Indeed, for any $k \in \mathfrak{K}$,

$$\mathfrak{id}^{\mathfrak{J}}\mathfrak{W}_{\mathbb{X},\mathbb{Y}}\mathfrak{I}_{\mathfrak{J}}(k) = \mathfrak{id}^{\mathfrak{J}}\mathfrak{W}_{\mathbb{X},\mathbb{Y}}\mathfrak{J}(k) = \mathfrak{id}^{\mathfrak{J}}\left(\sum_{n\in\mathbb{N}}[\mathfrak{J}(k), y_n]x_n\right) = \mathfrak{id}_{\mathfrak{K}}\left(\sum_{n\in\mathbb{N}}[k, y_n]_{\mathfrak{J}}x_n\right) = \sum_{n\in\mathbb{N}}[k, y_n]_{\mathfrak{J}}x_n = \mathfrak{W}_{\mathbb{X},\mathbb{Y}}$$

So, $\mathfrak{W}^{\circ}_{\mathbb{X},\mathbb{Y}} = \mathfrak{id}^{\mathfrak{J}}\mathbf{0}\mathfrak{I}_{\mathfrak{J}} \equiv \mathbf{0}$. This concludes the proof of the theorem. □

Proposition 33. *Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ be a Krein space, $\mathbb{X}$ a frame in $(\mathfrak{K}, [\cdot,\cdot])$ with bounds $B \geq A > 0$ and $\psi \in \mathcal{B}(\mathfrak{K})$. Then, $\psi(\mathbb{X})$ is a frame in $(\mathfrak{K}, [\cdot,\cdot])$ if and only if ψ is invertible.*

Proof. ($\Rightarrow$) Suppose that $\psi(\mathbb{X})$ is a frame in $(\mathfrak{K}, [\cdot,\cdot])$, then the frame operator $\mathcal{S}_{\psi(\mathbb{X})}$ exists and is invertible. Also,

$$\mathcal{S}_{\psi(\mathbb{X})} = \mathcal{T}_{\psi\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\psi\mathbb{X}}$$

and for any $\{\alpha_n\}_{n\in\mathbb{N}} \in \mathfrak{R}_2(\mathbb{N})$ is satisfied

$$\begin{aligned} \mathcal{T}_{\psi\mathbb{X}}\left(\{\alpha_n\}_{n\in\mathbb{N}}\right) &= \sum_{n\in\mathbb{N}}\alpha_n\psi(x_n) = \lim_{M\to\infty}\sum_{n=1}^{M}\alpha_n\psi(x_n) = \lim_{M\to\infty}\psi\left(\sum_{n=1}^{M}\alpha_n x_n\right) \\ &= \psi\left(\lim_{M\to\infty}\sum_{n=1}^{M}\alpha_n x_n\right) = \psi\left(\sum_{n\in\mathbb{N}}\alpha_n x_n\right) = \psi\mathcal{T}_{\mathbb{X}}. \end{aligned}$$

Then,

$$\mathcal{S}_{\psi(\mathbb{X})} = \psi\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\left(\psi\mathcal{T}_{\mathbb{X}}\right)^* = \psi\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\left(\mathcal{T}^*_{\mathbb{X}}\psi^*\right) = \psi\left(\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}\right)\psi^* = \psi\mathcal{S}_{\mathbb{X}}\psi^*$$

and since $\mathcal{S}_{\mathbb{X}}$ is invertible then ψ is invertible.

($\Leftarrow$) Suppose that ψ is invertible. Let us see that $\psi(\mathbb{X})$ is frame in $(\mathfrak{K}, [\cdot,\cdot])$. Since ψ is invertible then

$$\psi\psi^{-1} = \mathfrak{id}_{\mathfrak{K}} = \psi^{-1}\psi \quad \text{o} \quad \left(\psi^{-1}\right)^*\psi^* = \mathfrak{id}_{\mathfrak{K}} = \psi^*\left(\psi^{-1}\right)^*.$$

So, note that for any $k \in \mathfrak{K}$, $\psi^*(k) \in \mathfrak{K}$,

$$\begin{aligned} A\|\psi^{-1}\|^{-2}\|k\|^2_{\mathfrak{J}} &= A\|\psi^{-1}\|^{-2}\|(\psi^{-1})^*\psi^*(k)\|^2_{\mathfrak{J}} \leq A\|\psi^{-1}\|^{-2}\left(\|(\psi^{-1})^*\|\|\psi^*(k)\|^2_{\mathfrak{J}}\right)^2 \\ &= A\|\psi^{-1}\|^{-2}\|(\psi^{-1}\|^2\|\psi^*(k)\|^2_{\mathfrak{J}} = A\|\psi^*(k)\|^2_{\mathfrak{J}} \leq \sum_{n\in\mathbb{N}}|[\psi^*(k), x_n]|^2 \\ &\leq \sum_{n\in\mathbb{N}}|[k, \psi(x_n)]|^2 \leq B\|\psi^*(k)\|^2_{\mathfrak{J}} \leq B\|\psi^*\|^2\|k\|^2_{\mathfrak{J}} = B\|\psi\|^2\|k\|^2_{\mathfrak{J}}. \end{aligned}$$

Therefore, $A\|\psi^{-1}\|^{-2}\|k\|_{\mathfrak{J}}^2 \leq \sum_{n\in\mathbb{N}} |[k,\psi(x_n)]|^2 \leq B\|\psi\|^2\|k\|_{\mathfrak{J}}^2$ and then $\psi(\mathbb{X})$ is a frame in $(\mathfrak{K},[\cdot,\cdot])$ with bounds $B\|\psi\|^2 \geq A\|\psi^{-1}\|^{-2} > 0$. □

Theorem 34. *Let $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ be a Krein space, $\mathbb{X}$, $\mathbb{Y}$ orthogonal frames in $(\mathfrak{K},[\cdot,\cdot])$ and ψ_1, $\psi_2 \in \mathcal{B}(\mathfrak{K})$ invertible. Then $\psi_1(\mathbb{X})$, $\psi_2(\mathbb{Y})$ are orthogonal frames in $(\mathfrak{K},[\cdot,\cdot])$.*

Proof. Suppose that $\mathbb{X}$, $\mathbb{Y}$ are orthogonal frames in $(\mathfrak{K},[\cdot,\cdot])$ and ψ_1, $\psi_2 \in \mathcal{B}(\mathfrak{K})$ are invertible, then by the above result $\psi_1(\mathbb{X})$, $\psi_2(\mathbb{Y})$ are frames in $(\mathfrak{K},[\cdot,\cdot])$. It remains to prove that $\mathfrak{W}_{\psi_1(\mathbb{X}),\psi_2(\mathbb{Y})} \equiv \mathbf{0}$. In effect,

$$\mathfrak{W}_{\psi_1(\mathbb{X}),\psi_2(\mathbb{Y})} = \mathcal{T}_{\psi_1(\mathbb{X})}\mathfrak{J}_{\mathfrak{K}_2}\mathcal{T}^*_{\psi_1(\mathbb{Y})} = \psi_1\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{K}_2}\mathcal{T}^*_{\mathbb{Y}}\psi_2^* = \psi_1\mathfrak{W}_{\mathbb{X},\mathbb{Y}}\psi_2^* = \psi_1\mathbf{0}\psi_2^* \equiv \mathbf{0}.$$

□

Theorem 35. *Let $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ be a Krein space and $\mathbb{X}$, $\mathbb{Y}$ are frames in $(\mathfrak{K},[\cdot,\cdot])$. Then the following are equivalent*

i) $\mathfrak{W}_{\mathbb{X},\mathbb{Y}} = \mathfrak{id}_{\mathfrak{K}}$,
ii) $\mathfrak{W}_{\mathbb{Y},\mathbb{X}} = \mathfrak{id}_{\mathfrak{K}}$,
iii) $\mathfrak{W}_{\mathfrak{J}\mathbb{Y},\mathfrak{J}\mathbb{X}} = \mathfrak{id}_{\mathfrak{K}}$,
iv) $[f,g] = \sum_{n\in\mathbb{N}}[f,x_n][y_n,g], \ \ \forall f,g \in \mathfrak{K}$.

Proof. i) ⇒ ii)

$$\mathfrak{W}_{\mathbb{Y},\mathbb{X}} = \mathfrak{W}^*_{\mathbb{X},\mathbb{Y}} = (\mathfrak{id}_{\mathfrak{K}})^* = \mathfrak{id}_{\mathfrak{K}}.$$

ii) ⇒ iii)

$$\mathfrak{W}_{\mathfrak{J}\mathbb{Y},\mathfrak{J}\mathbb{X}} = \mathcal{T}_{\mathfrak{J}\mathbb{Y}}\mathfrak{J}_{\mathfrak{K}_2}\mathcal{T}^*_{J\mathbb{X}} = \mathfrak{J}\mathcal{T}_{\mathbb{Y}}\mathfrak{J}_{\mathfrak{K}_2}\mathcal{T}^*_{\mathbb{X}}\mathfrak{J}^* = \mathfrak{J}\mathcal{T}_{\mathbb{Y}}\mathfrak{J}_{\mathfrak{K}_2}\mathcal{T}^*_{\mathbb{X}}\mathfrak{J} = \mathfrak{J}\mathfrak{W}_{\mathbb{Y},\mathbb{X}}\mathfrak{J} = \mathfrak{J}\mathfrak{id}_{\mathfrak{K}}\mathfrak{J} = \mathfrak{J}\mathfrak{J} = \mathfrak{J}^2 = \mathfrak{id}_{\mathfrak{K}}.$$

iii) ⇒ iv) Let f, $g \in \mathfrak{K}$ be any,

$$\begin{aligned}[f,g] &= [\mathfrak{J}(f),\mathfrak{J}(g)] = [\mathfrak{W}_{\mathfrak{J}\mathbb{Y},\mathfrak{J}\mathbb{X}}(\mathfrak{J}(f)),\mathfrak{J}(g)] = \left[\sum_{n\in\mathbb{N}}[\mathfrak{J}(f),\mathfrak{J}(x_n)]\mathfrak{J}(y_n),\mathfrak{J}(g)\right] \\ &= \sum_{n\in\mathbb{N}}[\mathfrak{J}(f),\mathfrak{J}(x_n)][\mathfrak{J}(y_n),\mathfrak{J}(g)] = \sum_{n\in\mathbb{N}}[f,x_n][y_n,g].\end{aligned}$$

iv) ⇒ i) For any g, $f \in \mathfrak{K}$,

$$\begin{aligned}\left[g, f-\sum_{n\in\mathbb{N}}[f,y_n]x_n\right] &= [g,f] - \left[g,\sum_{n\in\mathbb{N}}[f,y_n]x_n\right] = [g,f] - \sum_{n\in\mathbb{N}}\overline{[f,y_n]}[g,x_n] \\ &= [g,f] - \sum_{n\in\mathbb{N}}[g,x_n][y_n,f] = [g,f]-[g,f] = 0.\end{aligned}$$

Then $\sum_{n\in\mathbb{N}}[f,y_n]x_n = f$, $\forall f \in \mathfrak{K}$. Then $\mathfrak{W}_{\mathbb{X},\mathbb{Y}} = \mathfrak{id}_{\mathfrak{K}}$. □

Remark 36. Given a Krein space $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ and consider a bounded linear operator $\psi : (\mathfrak{K},[\cdot,\cdot]) \to (\mathfrak{K},[\cdot,\cdot])$, then the following diagram commutes

$$\begin{array}{ccc} (\mathfrak{K},[\cdot,\cdot]) & \overset{\psi}{\to} & (\mathfrak{K},[\cdot,\cdot]) \\ \mathfrak{id}^{\mathfrak{J}}\downarrow & & \uparrow \mathfrak{id}_{\mathfrak{J}} \\ (\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}}) & \overset{\psi^\circ}{\to} & (\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}}) \end{array}$$

In fact, for any $k \in \mathfrak{K}$ we have that

$$\begin{aligned} \mathfrak{id}_{\mathfrak{J}}\psi^\circ\mathfrak{id}^{\mathfrak{J}}(k) &= \mathfrak{id}_{\mathfrak{J}}\psi^\circ\mathfrak{id}_{\mathfrak{K}}(k) = \mathfrak{id}_{\mathfrak{J}}\psi^\circ(k) \\ &= \mathfrak{id}_{\mathfrak{K}}(\psi(k)) = \psi(k). \end{aligned}$$

In addition, note that ψ is surjective if and only if ψ° es surjective.

Proposition 37. *Let $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ be a Krein space and $\psi : (\mathfrak{K},[\cdot,\cdot]) \to (\mathfrak{K},[\cdot,\cdot])$ a bounded and surjective linear operator. Then there exists $\psi^\dagger : (\mathfrak{K},[\cdot,\cdot]) \to (\mathfrak{K},[\cdot,\cdot])$ bounded linear such that $\psi\psi^\dagger = \mathfrak{id}_{\mathfrak{K}}$.*

Proof. From the above observation the operator $\psi^\circ : (\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}}) \to (\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}})$ is surjective and then there exists $(\psi^\circ)^\dagger : (\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}}) \to (\mathfrak{K},[\cdot,\cdot]_{\mathfrak{J}})$ such that $\psi^\circ(\psi^\circ)^\dagger = \mathfrak{id}^\circ_{\mathfrak{K}}$. So, if we consider $\psi^\dagger := \mathfrak{id}_{\mathfrak{J}}(\psi^\circ)^\dagger\mathfrak{id}^{\mathfrak{J}}$, note that $\psi^\dagger$ is a bounded linear operator of $(\mathfrak{K},[\cdot,\cdot])$ and also

$$\begin{aligned} \psi\psi^\dagger &= \mathfrak{id}_{\mathfrak{J}}\psi^\circ\mathfrak{id}^{\mathfrak{J}}\mathfrak{id}_{\mathfrak{J}}(\psi^\circ)^\dagger\mathfrak{id}^{\mathfrak{J}} = \mathfrak{id}_{\mathfrak{J}}\psi^\circ\mathfrak{id}^{\mathfrak{J}}(\mathfrak{id}^{\mathfrak{J}})^{-1}(\psi^\circ)^\dagger\mathfrak{id}^{\mathfrak{J}} = \mathfrak{id}_{\mathfrak{J}}\psi^\circ\mathfrak{id}^\circ_{\mathfrak{K}}(\psi^\circ)^\dagger\mathfrak{id}^{\mathfrak{J}} \\ &= \mathfrak{id}_{\mathfrak{J}}\psi^\circ(\psi^\circ)^\dagger\mathfrak{id}^{\mathfrak{J}} = \mathfrak{id}_{\mathfrak{J}}\mathfrak{id}^\circ_{\mathfrak{K}}\mathfrak{id}^{\mathfrak{J}} = \mathfrak{id}_{\mathfrak{J}}\mathfrak{id}^{\mathfrak{J}} = \mathfrak{id}_{\mathfrak{J}}(\mathfrak{id}_{\mathfrak{J}})^{-1} = \mathfrak{id}_{\mathfrak{K}}. \end{aligned}$$

□

Lemma 38. *Let $(\mathfrak{K},[\cdot,\cdot],\mathfrak{J})$ be a Krein space, $\mathbb{X}$ a frame in $(\mathfrak{K},[\cdot,\cdot])$ with bounds $B \geq A > 0$ and $\{e_n\}_{n\in\mathbb{N}}$ the canonical basis of $\ell_2(\mathbb{N})$. Then the dual frames to $\mathbb{X}$ are the families of the form:*

$$\mathbb{Y} = \{y_n\}_{n\in\mathbb{N}} := \{\psi(e_n)\}_{n\in\mathbb{N}},$$

where $\psi : (\mathfrak{R}_2(\mathbb{N}),[\cdot,\cdot]_{\mathfrak{R}_2}) \longrightarrow (\mathfrak{K},[\cdot,\cdot])$ is a bounded linear operator and left inverse of $\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}$.

Proof. Let ψ a left inverse of $\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}$, this is, $\psi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}} = \mathfrak{id}_{\mathfrak{K}}$. Therefore $\mathbb{Y}$ is a frame in $(\mathfrak{K},[\cdot,\cdot])$ with bounds $B\|\psi\|^2 \geq A\|\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}\|^{-2} > 0$.
Also for all $f \in \mathfrak{K}$,

$$\begin{aligned} \mathfrak{id}_{\mathfrak{K}}(f) &= \psi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}}(f) = \psi\mathfrak{J}_{\mathfrak{R}_2}\mathfrak{J}_{\mathfrak{R}_2}\left(\{[f,x_n]\}_{n\in\mathbb{N}}\right) = \psi\left(\{[f,x_n]\}_{n\in\mathbb{N}}\right) \\ &= \psi\left(\sum_{n\in\mathbb{N}}[f,x_n]e_n\right) = \sum_{n\in\mathbb{N}}[f,x_n]\psi(e_n) = \sum_{n\in\mathbb{N}}[f,x_n]y_n = \mathfrak{W}_{\mathbb{X},\mathbb{Y}}(f). \end{aligned}$$

Then $\mathfrak{W}_{\mathbb{X},\mathbb{Y}} = \mathfrak{id}_{\mathfrak{K}}$ and thus $\mathbb{Y}$ is a dual frame a $\mathbb{X}$ in $(\mathfrak{K},[\cdot,\cdot])$. On the other hand if $\mathbb{Y}$ is a dual frame a $\mathbb{X}$ in $(\mathfrak{K},[\cdot,\cdot])$ then it is enough to take $\psi := \mathcal{T}_{\mathbb{Y}}$ and observe that

$$\mathcal{T}_{\mathbb{Y}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}^*_{\mathbb{X}} = \mathfrak{W}_{\mathbb{X},\mathbb{Y}} = \mathfrak{id}_{\mathfrak{K}}. \tag{2.1}$$

Also,

$$\psi(e_n) = \mathcal{T}_{\mathbb{Y}}(e_n) = \sum_{k\in\mathbb{N}}\delta_{n,k}y_k = y_n,$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 July 2022 doi:10.20944/preprints202207.0175.v1

where

$$\delta_{n,k} := \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$

□

Lemma 39. *Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ be a Krein space and $\mathbb{X}$ a frame in $(\mathfrak{K}, [\cdot,\cdot])$. Then the bounded linear operators that are left inverse of $\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*$ are of the form:*

$$\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} + \varphi\left(\mathfrak{id}_{\mathfrak{K}} - \mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}}\right),$$

where $\varphi : (\mathfrak{R}_2(\mathbb{N}), [\cdot,\cdot]_{\mathfrak{R}_2}) \longrightarrow (\mathfrak{K}, [\cdot,\cdot])$ is a bounded linear operator.

Proof. Let $\psi := \mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} + \varphi\left(\mathfrak{id}_{\mathfrak{K}} - \mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}}\right)$ with $\varphi : (\mathfrak{R}_2(\mathbb{N}), [\cdot,\cdot]_{\mathfrak{R}_2}) \longrightarrow (\mathfrak{K}, [\cdot,\cdot])$ a bounded linear operator. Let us see that $\psi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^* = \mathfrak{id}_{\mathfrak{K}}$,

$$\begin{aligned} \psi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^* &= \left(\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} + \varphi\left(\mathfrak{id}_{\mathfrak{K}} - \mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}}\right)\right)\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^* \\ &= \mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^* + \varphi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^* - \varphi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*\mathcal{S}_{\mathbb{X}}^{-1}T_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^* \\ &= \mathcal{S}_{\mathbb{X}}^{-1}\mathcal{S}_{\mathbb{X}} + \varphi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^* - \varphi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{S}_{\mathbb{X}} \\ &= \mathfrak{id}_{\mathfrak{K}} + \varphi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^* - \varphi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^* = \mathfrak{id}_{\mathfrak{K}}. \end{aligned}$$

Thus ψ is a left inverse of $\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*$. For the other implication, let us assume that ψ is a left inverse of $\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*$. Take $\varphi := \psi$. then,

$$\begin{aligned} \mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} + \varphi\left(\mathfrak{id}_{\mathfrak{K}} - \mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}}\right) &= \mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} + \psi\left(\mathfrak{id}_{\mathfrak{K}} - \mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}}\right) = \mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} + \psi - \psi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} \\ &= \mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} + \psi - \mathfrak{id}_{\mathfrak{K}}\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} = \mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} - \mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}} + \psi = \mathbf{0} + \psi = \psi. \end{aligned}$$

□

Theorem 40. *Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ a Krein space and $\mathbb{X}$ a frame in $(\mathfrak{K}, [\cdot,\cdot])$. Then the dual frames to $\mathbb{X}$ have the form:*

$$\mathbb{Y} = \left\{\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}}(e_n) + \varphi(e_n) - \varphi\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{X}}^*\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}}(e_n)\right\}_{n\in\mathbb{N}},$$

where $\varphi : (\mathfrak{R}_2(\mathbb{N}), [\cdot,\cdot]_{\mathfrak{R}_2}) \longrightarrow (\mathfrak{K}, [\cdot,\cdot])$ is a bounded linear operator and $\{e_n\}_{n\in\mathbb{N}}$ the canonical basis of $\ell_2(\mathbb{N}$.

Proof. It is an immediate consequence of the lemma 38 and lemma 39. □

Proposition 41. *Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ a Krein space, $\mathbb{Y}$ a dual frame a $\mathbb{X}$ in $(\mathfrak{K}, [\cdot,\cdot])$ and $\varphi : (\mathfrak{R}_2(\mathbb{N}), [\cdot,\cdot]_{\mathfrak{R}_2}) \longrightarrow (\mathfrak{K}, [\cdot,\cdot])$ a bounded linear operator and sujertive, then $(\varphi^\dagger)^*(\mathbb{Y})$ is a dual frame a $\varphi(\mathbb{X})$.*

Proof. Note that,

$$\begin{aligned} \mathfrak{W}_{\varphi(\mathbb{X}),(\varphi^\dagger)^*(\mathbb{Y})} &= \mathcal{T}_{\varphi(\mathbb{X})}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{(\varphi^\dagger)^*(\mathbb{Y})}^* = \varphi\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}\mathcal{T}_{\mathbb{Y}}^*\left(\left(\varphi^\dagger\right)^*\right)^* = \varphi\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\mathfrak{R}_2}T_{\mathbb{Y}}^*\varphi^\dagger \\ &= \varphi\mathfrak{W}_{\mathbb{X},\mathbb{Y}}\varphi^\dagger = \varphi\mathfrak{id}_{\mathfrak{K}}\varphi^\dagger = \varphi\varphi^\dagger = \mathfrak{id}_{\mathfrak{K}}. \end{aligned}$$

Thus $(\varphi^\dagger)^*(\mathbb{Y})$ is a dual frame a $\varphi(\mathbb{X})$. □

Theorem 42. *Let $(\mathfrak{K}, [\cdot,\cdot], \mathfrak{J})$ a Krein space, $\mathcal{P} : (\mathfrak{K}, [\cdot,\cdot]) \longrightarrow (\mathfrak{K}, [\cdot,\cdot])$ an orthogonal projection that commutes with $\mathfrak{J}$ and $\mathbb{X}, \mathbb{Y}$ orthogonal frames in $(\mathfrak{K}, [\cdot,\cdot])$. Then*

i) *$\mathcal{P}\mathbb{X}$, $\mathcal{P}\mathbb{Y}$ define orthogonal frames in Krein space $(\mathcal{P}\mathfrak{K}, [\cdot,\cdot])$,*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 July 2022 doi:10.20944/preprints202207.0175.v1

ii) $(\mathfrak{id}_{\mathfrak{K}} - \mathcal{P})\mathbb{X}$, $(\mathfrak{id}_{\mathfrak{K}} - \mathcal{P})\mathbb{Y}$ *define orthogonal frames in the Krein space given by* $((\mathfrak{id}_{\mathfrak{K}} - \mathcal{P})\mathfrak{K}, [\cdot,\cdot])$.

Proof. i) $\mathfrak{W}_{\mathcal{P}\mathbb{X},\mathcal{P}\mathbb{Y}} = \mathcal{T}_{\mathcal{P}\mathbb{X}}\mathfrak{J}_{\Re_2}\mathcal{T}^*_{\mathcal{P}\mathbb{Y}} = \mathcal{P}\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\Re_2}\mathcal{T}^*_{\mathbb{Y}}\mathcal{P}^* = \mathcal{P}\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\Re_2}\mathcal{T}^*_{\mathbb{Y}}\mathcal{P} = \mathcal{P}\mathfrak{W}_{\mathbb{X},\mathbb{Y}}\mathcal{P} = \mathcal{P}\mathbf{0}\mathcal{P} \equiv \mathbf{0}$.
ii)

$$\begin{aligned}\mathfrak{W}_{(\mathfrak{id}_{\mathfrak{K}}-\mathcal{P})\mathbb{X},(\mathfrak{id}_{\mathfrak{K}}-\mathcal{P})\mathbb{Y}} &= \mathcal{T}_{(\mathfrak{id}_{\mathfrak{K}}-\mathcal{P})\mathbb{X}}\mathfrak{J}_{\Re_2}\mathcal{T}^*_{(\mathfrak{id}_{\mathfrak{K}}-\mathcal{P})\mathbb{Y}} = (\mathfrak{id}_{\mathfrak{K}}-\mathcal{P})\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\Re_2}\mathcal{T}^*_{\mathbb{Y}}(\mathfrak{id}_{\mathfrak{K}}-\mathcal{P})^* \\ &= (\mathfrak{id}_{\mathfrak{K}}-\mathcal{P})\mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\Re_2}\mathcal{T}^*_{\mathbb{Y}}(\mathfrak{id}_{\mathfrak{K}}-\mathcal{P}) = (\mathfrak{id}_{\mathfrak{K}}-\mathcal{P})\mathfrak{W}_{\mathbb{X},\mathbb{Y}}(\mathfrak{id}_{\mathfrak{K}}-\mathcal{P}) \\ &= (\mathfrak{id}_{\mathfrak{K}}-\mathcal{P})\mathbf{0}(\mathfrak{id}_{\mathfrak{K}}-\mathcal{P}) \equiv \mathbf{0}.\end{aligned}$$

□

Remark 43. In $\mathbb{C}^n$ we have the following indefinite inner product $[\cdot,\cdot]_{\mathbb{C}^n} : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$, given by

$$[\alpha,\beta] := \sum_{i=1}^{n}(-1)^i\alpha_i\overline{\beta_i}$$

for all $\alpha = \sum_{i=1}^n \alpha_i e_i$, $\beta = \sum_{i=1}^n \beta_i e_i \in \mathbb{C}^n$, where $\{e_i\}_{i=1}^n$ is the canonical orthonormal basis in $(\mathbb{C}^n, \langle\cdot,\cdot\rangle_{\mathbb{C}^n})$. Then $\mathbb{C}^n$ admits the fundamental decomposition

$$\mathbb{C}^n = (\mathbb{C}^n)^+ \oplus (\mathbb{C}^n)^-, \quad \begin{aligned}(\mathbb{C}^n)^+ &:= \operatorname{span}\{e_{2i} : 1 \le 2i \le n,\ i \in \mathbb{N}\} \\ (\mathbb{C}^n)^- &:= \operatorname{span}\{e_{2i-1} : 1 \le 2i-1 \le n,\ i \in \mathbb{N}\}\end{aligned}$$

with associated fundamental symmetry

$$\mathfrak{J}_{\mathbb{C}^n} : (\mathbb{C}^n, [\cdot,\cdot]_{\mathbb{C}^n}) \longrightarrow (\mathbb{C}^n, [\cdot,\cdot]_{\mathbb{C}^n})$$

given by $\mathfrak{J}_{\mathbb{C}^n}(e_j) = (-1)^j e_j$. Then $[\cdot,\cdot]_{\mathfrak{J}_{\mathbb{C}^n}} = \langle\cdot,\cdot\rangle_{\mathbb{C}^n}$. When $\mathbb{C}^n$ is viewed as a Krein space with this fundamental symmetry $\mathfrak{J}_{\mathbb{C}^n}$, we will write $\left(\Re(n), [\cdot,\cdot]_{\Re(n)}, \mathfrak{J}_{\Re(n)}\right)$.

Example 44. In $(\Re(2), [\cdot,\cdot]_{\Re(2)}, \mathfrak{J}_{\Re(2)})$ we consider the sequences $\mathbb{X} = \{e_1, e_1, e_1, e_2\}$ and $\mathbb{Y} = \{-e_1 - e_2, e_1, e_2, \mathbf{0}\}$ where $\{e_i\}_{i=1}^2$ is the canonical orthonormal basis in $(\mathbb{C}^2, \langle\cdot,\cdot\rangle_{\mathbb{C}^2})$. It is clear that both $\mathbb{X}$ and $\mathbb{Y}$ are frames in $(\Re(2), [\cdot,\cdot]_{\Re(2)})$ because the kernel of $\mathcal{T}_{\mathbb{X}}$ and $\mathcal{T}_{\mathbb{Y}}$ have dimension 2:

$$\ker(\mathcal{T}_{\mathbb{X}}) = \operatorname{span}\{(-1,1,0,0),(1,0,-1,0)\}\ y\ \ker(\mathcal{T}_{\mathbb{Y}}) = \operatorname{span}\{(1,1,1,0),(0,0,0,1)\}$$

this is, $\mathcal{T}_{\mathbb{X}}$, $\mathcal{T}_{\mathbb{Y}}$ are both surjective linear transformations. Moreover, for any $(\beta_1,\beta_2) \in \Re(2)$,

$$\begin{aligned}\mathfrak{W}_{\mathbb{X},\mathbb{Y}}(\beta_1,\beta_2) &= \mathcal{T}_{\mathbb{X}}\mathfrak{J}_{\Re(4)}\mathcal{T}^*_{\mathbb{Y}}(\beta_1,\beta_2) \\ &= \mathcal{T}_{\mathbb{X}}\mathfrak{J}^2_{\Re(4)}\left([(\beta_1,\beta_2),-e_1-e_2],[(\beta_1,\beta_2),e_1],[(\beta_1,\beta_2),e_2],(0,0)\right) \\ &= \mathcal{T}_{\mathbb{X}}\,\mathfrak{id}_{\Re(4)}\left([(\beta_1,\beta_2),-e_1-e_2],[(\beta_1,\beta_2),e_1],[(\beta_1,\beta_2),e_2],(0,0)\right) \\ &= [(\beta_1,\beta_2),-e_1-e_2]e_1 + [(\beta_1,\beta_2),e_1]e_1 + [(\beta_1,\beta_2),e_2]e_1 \\ &= (\beta_1-\beta_2-\beta_1+\beta_2)\,e_1 \\ &= 0\,e_1 = 0.\end{aligned}$$

Therefore, $\mathbb{X}$, $\mathbb{Y}$ are orthogonal frames in $(\Re(2), [\cdot,\cdot]_{\Re(2)})$.

Example 45. Let $\mathfrak{K}=\mathbb{C}^5$ be and consider the inner product $[\cdot,\cdot]_{\mathfrak{K}}:\mathfrak{K}\times\mathfrak{K}\longrightarrow\mathbb{C}$ given by

$$[\alpha,\beta]:=-\sum_{i=1}^{5}\alpha_i\overline{\beta_{6-i}}\ \text{ for all }\ \alpha=\sum_{i=1}^{5}\alpha_i e_i,\ \beta=\sum_{i=1}^{5}\beta_i e_i$$

in $\mathfrak{K}$, where $\{e_i\}_{i=1}^{5}$ is the canonical orthonormal basis in $(\mathbb{C}^5,\langle\cdot,\cdot\rangle_{\mathbb{C}^5})$,which is an indefinite inner product. Note that we can define the bijective linear transformation

$$\mathfrak{J}:(\mathfrak{K},[\cdot,\cdot])\longrightarrow(\mathfrak{K},[\cdot,\cdot]),\quad \mathfrak{J}(e_i)=-e_{6-i},\ i=1,2,3,4,5.$$

whose matrix associated with $\{e_i\}_{i=1}^{5}$ is

$$[\mathfrak{J}]=\begin{pmatrix}0&0&0&0&-1\\0&0&0&-1&0\\0&0&-1&0&0\\0&-1&0&0&0\\-1&0&0&0&0\end{pmatrix}$$

and $\mathfrak{J}$ is the fundamental symmetry associated with the fundamental decomposition $\mathfrak{K}=\mathfrak{K}^+\oplus\mathfrak{K}^-$, where

$$\begin{aligned}\mathfrak{K}^+ &:= \text{span}\,\{e_1-e_5,e_2-e_4\}\\ \mathfrak{K}^- &:= \text{span}\,\{e_1+e_5,e_2+e_4,e_3\}\end{aligned}$$

and also

$$\begin{aligned}[\alpha,\beta]_{\mathfrak{J}} &= [\mathfrak{J}(\alpha),\beta]_{\mathfrak{K}}=\left[\mathfrak{J}\left(\sum_{i=1}^{5}\alpha_i e_i\right),\sum_{j=1}^{5}\beta_j e_j\right]_{\mathfrak{K}}=\left[\sum_{i=1}^{5}\alpha_i\mathfrak{J}(e_i),\sum_{j=1}^{5}\beta_j e_j\right]_{\mathfrak{K}}\\ &=\left[\sum_{i=1}^{5}\alpha_i(-e_{6-i}),\sum_{j=1}^{5}\beta_j e_j\right]_{\mathfrak{K}}=\sum_{i=1}^{5}\alpha_i\sum_{j=1}^{5}\overline{\beta_j}\,[-e_{6-i},e_j]\\ &=\sum_{i=1}^{5}\alpha_i\sum_{j=1}^{5}\overline{\beta_j}\,\langle e_i,e_j\rangle_{\mathbb{C}^5}=\left\langle\sum_{i=1}^{5}\alpha_i e_i,\sum_{j=1}^{5}\beta_j e_j\right\rangle_{\mathbb{C}^5}=\langle\alpha,\beta\rangle_{\mathbb{C}^5}\,.\end{aligned}$$

Let us consider $\mathbb{X}=\{e_1+e_2,e_2+e_3,e_3+e_4,e_4+e_5,e_5\}$ in $(\mathfrak{K},[\cdot,\cdot]_{\mathfrak{K}})$. Note that

$$\begin{aligned}\mathcal{T}_{\mathbb{X}}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= \alpha_1(e_1+e_2)+\alpha_2(e_2+e_3)+\alpha_3(e_3+e_4)+\alpha_4(e_4+e_5)+\alpha_5e_5\\ &= (\alpha_1,\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_4,\alpha_4+\alpha_5)\end{aligned}$$

for all $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\in\mathfrak{K}$ and $\dim(\ker(\mathcal{T}_{\mathbb{X}}))=0$ which implies that $\mathbb{X}$ is a frame in $(\mathfrak{K},[\cdot,\cdot]_{\mathfrak{K}})$. Now, taking $\varphi:(\Re(5),[\cdot,\cdot]_{\Re(5)})\longrightarrow(\mathfrak{K},[\cdot,\cdot]_{\mathfrak{K}})$, $\varphi:=\mathfrak{id}_{\Re(5)}$, and as $\mathcal{S}_{\mathbb{X}}^{-1}(x_1)=-e_5$, $\mathcal{S}_{\mathbb{X}}^{-1}(x_2)=-e_4+e_5$, $\mathcal{S}_{\mathbb{X}}^{-1}(x_3)=-e_3+e_4-e_5$, $\mathcal{S}_{\mathbb{X}}^{-1}(x_4)=-e_2+e_3-e_4+e_5$ and $\mathcal{S}_{\mathbb{X}}^{-1}(x_5)=-e_1+e_2-e_3+e_4-e_5$, then

$$\begin{aligned}\varphi(\mathfrak{id}_{\Re(5)}-\mathfrak{J}_{\Re(5)}\mathcal{T}_{\mathbb{X}}^{*}\mathcal{S}_{\mathbb{X}}^{-1}\mathcal{T}_{\mathbb{X}})(e_i) &= e_i-\mathfrak{J}_{\Re(5)}\mathcal{T}_{\mathbb{X}}^{*}\mathcal{S}_{\mathbb{X}}^{-1}(x_i)\\ &= e_i-\big([\mathcal{S}_{\mathbb{X}}^{-1}(x_i),x_1]_{\mathfrak{K}},[\mathcal{S}_{\mathbb{X}}^{-1}(x_i),x_2]_{\mathfrak{K}},\\ [\mathcal{S}_{\mathbb{X}}^{-1} &\quad (x_i),x_3]_{\mathfrak{K}},[\mathcal{S}_{\mathbb{X}}^{-1}(x_i),x_4]_{\mathfrak{K}},[\mathcal{S}_{\mathbb{X}}^{-1}(x_i),x_5]_{\mathfrak{K}}\big)\\ &= \mathbf{0}\end{aligned}$$

for every i, $1 \leq i \leq 5$. Therefore $\mathbb{Y} = \left\{\mathcal{S}_{\mathbb{X}}^{-1}(x_1), \mathcal{S}_{\mathbb{X}}^{-1}(x_2), \mathcal{S}_{\mathbb{X}}^{-1}(x_3), \mathcal{S}_{\mathbb{X}}^{-1}(x_4), \mathcal{S}_{\mathbb{X}}^{-1}(x_5)\right\}$ is a dual frame a $\mathbb{X}$ in $(\mathfrak{K}, [\cdot,\cdot]_{\mathfrak{K}})$.

Conflict of interest

The authors declare that they have no conflict of interest in this work.

References

[1] Acosta, P.; Esmeral, K.; Ferrer, O. *Frames of subspaces in Hilbert spaces with W-metrics. An. St. Univ. "Ovidius" Constanta Ser. Mat.* **2015**, *23*, 5–22.
[2] Xiang, Z.-Q. *More on inequalities for weaving frames in Hilbert spaces. Mathematics.* **2019**, *7*, Article 141, 11 pages.
[3] Ferrer, O; Sierra, A; Sanabria, J. *Soft Frames in Soft Hilbert Spaces. Mathematics.* **2021**, *9*, Article 18, 2249 pages.
[4] O. Christensen, T. K. Jensen. *An introduction to the theory of bases, frames, and wavelets.* Technical University of Denmark, Department of Mathemtics, 1999.
[5] Esmeral, K., Ferrer O., and Wagner E., *Frames in Krein spaces Arising from a Non-regular W-metric*, *Banach J. Math. Anal.*, 9 (2015), No. 1 , 1–16.
[6] Escobar, G.; Esmeral, K.; Ferrer, O. *Extension and Coupling of Frames in Hilbert Spaces with Metrics.* En: Colombia Revista Integración ISSN: 0120-419X ed: Ediciones Universidad Industrial De Santander v.34 fasc.1 p.81 - 93 ,2016, DOI:
[7] Ferrer, O.; Esmeral, K.; Lora, B. *Dual and similar frames in Krein spaces.* En: Colombia International Journal Of Mathematical Analysis ISSN: 1312-8876 ed: v.2 fasc. p.1 - 15 ,2016.
[8] Duffin, R. J., and Schaeffer, A. C., *A class of nonharmonic Fourier series*, *Trans. Amer. Math. Soc.*, 72 (1952), 341–366.
[9] Daubechies, I., Grossmann, A., and Meyer, Y., *Painless nonorthogonal expansions*, *J. Math. Phys.*, 27 (1986), 1271–1283.
[10] Bhatt, G. *Sums of A Pair of Orthogonal Frames.* Mathematics, 2019, vol. 7, no 7, p. 582.
[11] Hong Oh Kim, RaeYoung Kim, Jae Kun Lim,, Zuowei Shen, *A pair of orthogonal frames.* Journal of Approximation Theory 147 (2007) 196-204
[12] Azizov, T. Ya., Iokhvidov, I. S., *Linear operator in spaces with an indefinite metric*, Wiley-Interscience, Chichester, 1989.
[13] Bognár, J., *Indefinite inner product spaces*, Springer Verlag, Berlin-Heidelberg, 1974.

[1] O. Ferrer, Department of Mathematics, Universidad de Sucre, Colombia
Email address: osmin.ferrer@unisucre.co

[2] A. Sierra, Department of Mathematics, Universidad de Sucre, Colombia
Email address: arley.sierra@unisucre.edu.co

[3] O. Polo, Department of Mathematics, Universidad de córdoba, Colombia
Email address: ojpoloflorez@correo.unicordoba.edu.co