 doi:10.20944/preprints202207.0166.v1

Vorticity and Micro-Rotation in Micropolar Flows

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Dedicated in loving memory of Bella (2003-2022) *and Sophie* (2007-2018)

Abstract

In this work the close relation between vorticity and micro-rotation in micropolar flows in $\mathbb{R}^n (n = 2, 3)$ is identified and used to explain the faster decay by $t^{-1/2}$ of the angular velocity of micro-rotation of the fluid particles, as well as establishing its optimality. For this purpose important upper and lower bounds for Leray solutions in homogeneous Sobolev spaces $\dot{H}^m(\mathbb{R}^n)$ are derived, using the monotonicity approach recently introduced by the authors for dissipative systems in general. Several related results of interest are also given along the discussion.

Porto Alegre, June 10, 2022

AMS Subject Classification Numbers: 35B40, 35K45, 35Q30

Keywords: incompressible micropolar flows; vorticity and micro-rotation; dissipative systems; monotonicity method; upper and lower estimates

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 July 2022 doi:10.20944/preprints202207.0166.v1

Summary and Conclusions

Section 1. Introduction
Monotonicity properties of some solution functionals are used to show that, in micropolar flows, the *angular velocity* of the micro-rotation of fluid particles, $\mathbf{w}(\cdot,t)$, is given very closely (at least for large t) by half of the local *flow vorticity*, $\frac{1}{2}\nabla\wedge\boldsymbol{u}(\cdot,t)$, which describes the apparent circulation of neighboring particles as seen by the particles themselves because of local differences in their translational motion. Although this synchronization effect was to be physically expected due to the action of micro-rotation viscosity, it remained elusive in 60 years of investigations of micropolar fluid flows.

Section 2. Mathematical preliminaries
Basic results regarding decay properties of $(\boldsymbol{u},\mathbf{w})(\cdot,t)$ are derived via the monotonicity approach introduced by the authors in [17], with an eye on the faster decay of $\mathbf{w}(\cdot,t)$.

Section 3. Proof of Theorems A and B
Examining the stronger decay behavior of the difference $\boldsymbol{\varepsilon}(\cdot,t) = \mathbf{w}(\cdot,t) - \frac{1}{2}\nabla\wedge\boldsymbol{u}(\cdot,t)$ allows the identification of the special role of the kinematic viscosity and the determination of improved upper estimates for the fields $\boldsymbol{u}(\cdot,t)$, $\mathbf{w}(\cdot,t)$ and ultimately $\boldsymbol{\varepsilon}(\cdot,t)$.

Section 4. Proof of Theorems C and D
Lower estimates for the translational velocity of fluid particles confirm the faster decay of the error term $\boldsymbol{\varepsilon}(\cdot,t)$ in comparison with the vorticity field, indicating that $\mathbf{w}(\cdot,t)$ and $\frac{1}{2}\nabla\wedge\boldsymbol{u}(\cdot,t)$ become very much the same for large t.

Appendix.
A simple proof is given for the existence of micropolar flows $(\boldsymbol{u},\mathbf{w})(\cdot,t)$ such that $c_0\,(1+t)^{-\alpha} \leq \|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq C_0\,(1+t)^{-\alpha}$ for all $t>0$ when $0<\alpha<\frac{1}{2}$ $(n=2,3)$.

1. Introduction

We begin with $n = 3$. Given arbitrary states $\boldsymbol{u}_0 = (u_0^1, u_0^2, u_0^3) \in L^2_\sigma(\mathbb{R}^3)$ and $\mathbf{w}_0 = (\mathrm{w}_0^1, \mathrm{w}_0^2, \mathrm{w}_0^3) \in L^2(\mathbb{R}^3)$, we consider solutions $(\boldsymbol{u}, \mathbf{w}) = (u_1, u_2, u_3, \mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3)$ in the Leray-Hopf sense of the incompressible micropolar equations [12, 13, 21, 28]

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = (\mu+\chi)\,\Delta\boldsymbol{u} + 2\chi\,\nabla\wedge\mathbf{w}, \tag{1.1a}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) = 0, \tag{1.1b}$$

$$\mathbf{w}_t + \boldsymbol{u}\cdot\nabla\mathbf{w} = \nu\,\Delta\mathbf{w} + \kappa\,\nabla(\nabla\cdot\mathbf{w}) - 4\chi\,\mathbf{w} + 2\chi\,\nabla\wedge\boldsymbol{u}, \tag{1.1c}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \qquad \mathbf{w}(\cdot,0) = \mathbf{w}_0, \tag{1.1d}$$

where the coefficients μ (kinematic viscosity), ν (angular viscosity) and χ (vortex or micro-rotation viscosity) are positive, and κ (gyroviscosity) is nonnegative, all assumed to be constant. Leray-Hopf (or simply Leray) solutions in $\mathbb{R}^3$ are global mappings $(\boldsymbol{u},\mathbf{w})(\cdot,t) \in C_{\mathrm{w}}([0,\infty), L^2_\sigma(\mathbb{R}^3)\times L^2(\mathbb{R}^3)) \cap L^2((0,\infty), \dot{H}^1(\mathbb{R}^3)\times\dot{H}^1(\mathbb{R}^3))$ with $(\boldsymbol{u},\mathbf{w})(\cdot,0) = (\boldsymbol{u}_0,\mathbf{w}_0)$ that satisfy the equations in weak sense for $t > 0$ and in addition the energy estimate

$$\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2}^2 + 2\int_s^t \bigl\{\mu\,\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2}^2 + \nu\,\|D\mathbf{w}(\cdot,\tau)\|_{L^2}^2\bigr\}\,d\tau \;\leq\; \|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^2}^2 \tag{1.2}$$

for all $t > s$, for $s = 0$ and almost all $s > 0$, where $\|\cdot\|_{L^2}$ denotes the norm in $L^2(\mathbb{R}^3)$ (see (1.16)-(1.18) below for notation). The existence of Leray solutions for the equations (1.1) and other similar systems is well known, but their uniqueness and exact regularity properties are still open, except for small initial data in suitable spaces [2, 13, 21, 25, 28]. This is the case, for example, of small data in $H^1_\sigma(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$: Serrin's regularity conditions [9, 27] and standard calculations show that $(\boldsymbol{u},\mathbf{w}) \in C^\infty(\mathbb{R}^3\times(0,\infty))$, with $(\boldsymbol{u},\mathbf{w})(\cdot,t) \in C((0,\infty), H^m(\mathbb{R}^3)\times H^m(\mathbb{R}^3))$ for all $m \geq 0$, if

$$\|(\boldsymbol{u}_0,\mathbf{w}_0)\|_{L^2(\mathbb{R}^3)}^{1/2}\,\|D(\boldsymbol{u}_0,\mathbf{w}_0)\|_{L^2(\mathbb{R}^3)}^{1/2} \leq 3.182\cdot\min\{\mu,\nu\}$$

(see (2.5), Section 2). From (1.2) it follows that, for *any* $(\boldsymbol{u}_0,\mathbf{w}_0) \in L^2_\sigma(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$, there will exist $t_* \geq 0$ sufficiently large so that $(\boldsymbol{u},\mathbf{w}) \in C^\infty(\mathbb{R}^3\times(t_*,\infty))$ and

$$(\boldsymbol{u},\mathbf{w})(\cdot,t) \in C((t_*,\infty), H^m(\mathbb{R}^3)\times H^m(\mathbb{R}^3)), \qquad \forall\; m \geq 0. \tag{1.3}$$

It also follows the existence of some $t_{**} \geq t_*$, with t_*, t_{**} satisfying

$$0 \leq t_* \leq t_{**} \leq K \cdot \big(\min\{\mu,\nu\}\big)^{-5} \,\|\,(\boldsymbol{u}_0, \mathbf{w}_0)\,\|_{L^2(\mathbb{R}^3)}^4, \qquad K < 0.005, \tag{1.4}$$

so that $\|\, D(\boldsymbol{u},\mathbf{w})(\cdot,t)\,\|_{L^2(\mathbb{R}^3)}$ is monotonically decreasing in (t_{**},∞), cf. THEOREM 2.1 of SECTION 2. From (1.2) and (1.4) we obtain

$$\lim_{t\to\infty} t^{1/2}\,\|\, D(\boldsymbol{u},\mathbf{w})(\cdot,t)\,\|_{L^2(\mathbb{R}^3)} = 0.$$

More is true: for arbitrary data $(\boldsymbol{u}_0,\mathbf{w}_0) \in L^2_\sigma(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, $n = 3$, it is known that

$$\lim_{t\to\infty} t^{m/2}\,\|\, D^m\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} = 0 \tag{1.5a}$$

and

$$\lim_{t\to\infty} t^{(m+1)/2}\,\|\, D^m\mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} = 0 \tag{1.5b}$$

for every $m \geq 0$, see e.g. [15, 16]. The extra factor $t^{1/2}$ in (1.5b) as compared to (1.5a) seems to have been first obtained in the incompressible case in [14, 23] (for compressible flows, see [20]), as a consequence of the damping term in the equation (1.1c). It will follow from our results below that the speed-up gain by $t^{1/2}$ observed in (1.5b) is indeed optimal and has its roots in an intimate relation between micro-rotation and flow vorticity that is an important effect of vortex viscosity.

Similar results can of course be obtained for two-dimensional flows. In this case, $\boldsymbol{u} = (u_1(x_1,x_2,t), u_2(x_1,x_2,t))$, $\mathbf{w} = (0,0,\mathrm{w}(x_1,x_2,t))$ and from (1.1) we get

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = (\mu+\chi)\,\Delta\boldsymbol{u} + 2\chi\,\nabla\wedge\mathrm{w}, \tag{1.6a}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) = 0, \tag{1.6b}$$

$$\mathrm{w}_t + \boldsymbol{u}\cdot\nabla\mathrm{w} = \nu\,\Delta\mathrm{w} - 4\chi\,\mathrm{w} + 2\chi\,\nabla\wedge\boldsymbol{u}, \tag{1.6c}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \in L^2_\sigma(\mathbb{R}^2), \qquad \mathrm{w}(\cdot,0) = \mathrm{w}_0 \in L^2(\mathbb{R}^2), \tag{1.6d}$$

where $\nabla\wedge\mathrm{w} = (D_2\mathrm{w}, -D_1\mathrm{w})$ and $\nabla\wedge\boldsymbol{u} = D_1u_2 - D_2u_1$ (notice that in (1.6) above the term associated with the gyroviscosity κ now drops out, because $\nabla\cdot\mathbf{w} = 0$). As in the 2D Navier-Stokes equations, Leray solutions to (1.6) are regular for $t > 0$, that is, $(\boldsymbol{u},\mathrm{w}) \in C^\infty(\mathbb{R}^2\times(0,\infty))$ and $(\boldsymbol{u},\mathrm{w})(\cdot,t) \in C((0,\infty), H^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2))$ for every $m \geq 0$, so that in 2D we have $t_* = 0$ [10, 11, 21]. Moreover, (1.5) remains valid in the two-dimensional case, as can be shown by a completely similar derivation.

 doi:10.20944/preprints202207.0166.v1

Before continuing our discussion, let us give some physical insight for (1.5b) above. Micropolar fluids are an important model of non-Newtonian fluids in which the particles are endowed with a rigid structure, so that they can rotate about their center of mass (micro-rotation), whose *angular velocity* is given by $\mathbf{w}$. Conservation of linear and angular momentum, along with some natural assumptions on the stress tensor, lead to the equations (1.1a)/(1.6a) and (1.1c)/(1.6c), respectively, in the absence of external forces [12, 21]. On the other hand, the *flow vorticity* at time t (given by the curl of the translational velocity field, $\nabla\wedge\boldsymbol{u}$) is a measure of the local flow rotation, as we now briefly review. Consider, for example, at some time t the two-dimensional flow depicted in Figure 1a, with positive and negative vorticity, say, at the points P and Q shown, lying on some streamline A. In this situation, a fluid particle located at P would be moving faster (at least momentaneously) than neighboring particles on the streamline B shown there and slower than those on C, so that to the particle P it would seem that neighboring particles were moving in opposite directions on his left and right sides. Thus, from its point of view, the particles would appear to be circulating counterclockwise about it (Figure 1b). A similar perception would have an observer sitting on the particle Q shown downstream, except that this time the circulation of neighboring particles would appear to be clockwise. That is, flow vorticity at any given time is related to local flow circulation as seen by the particles at that time. Local approximation of the velocity field by Taylor expansion shows that this apparent rotation happens with angular velocity given, to first approximation, by half of the local vorticity, that is, $\frac{1}{2}\nabla\wedge\boldsymbol{u}$ (see e.g. [7], pp. 19-21). In micropolar fluids, where the particles can also rotate themselves, local circulation can be transmitted to their micro-rotation, and vice-versa, through the mechanism of vortex or micro-rotation viscosity. It thus seems physically plausible that, in time, flow circulation and micro-rotation tend to synchronize (Figure 1c), so that we should expect

$$\mathbf{w}(x,t) \;\approx\; \frac{1}{2}\nabla\wedge\boldsymbol{u}(x,t)$$

for $t \gg 1$. In particular, as the flow develops, $\|\,\mathbf{w}(\cdot,t)\,\|_{L^2}$ should not really be much different from $\frac{1}{2}\,\|\,\nabla\wedge\boldsymbol{u}(\cdot,t)\,\|_{L^2}$, which accounts for the extra factor $t^{1/2}$ obtained in (1.5b), at least for $m = 0$. This spontaneous synchronization between micro-rotation and local flow circulation has apparently not been observed in the literature of micropolar fluids before, and it is at the heart of the mathematical results given in this paper, all related in one way or another to estimating the difference

$$\boldsymbol{\varepsilon}(x,t) \;=\; \mathbf{w}(x,t) \;-\; \frac{1}{2}\nabla\wedge\boldsymbol{u}(x,t). \tag{1.7}$$

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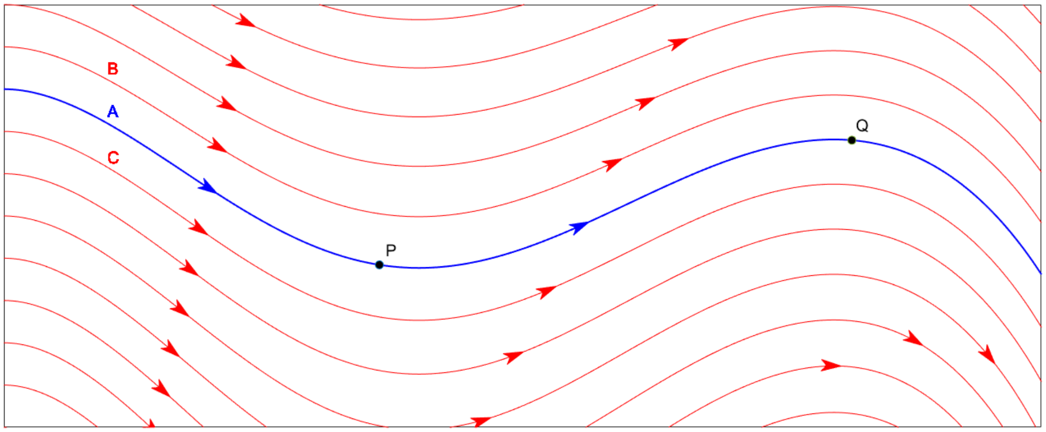

Fig. 1*a*: A snapshot at time t of some given two-dimensional flow, showing two points, P and Q, with positive and negative vorticity (blue streamline).

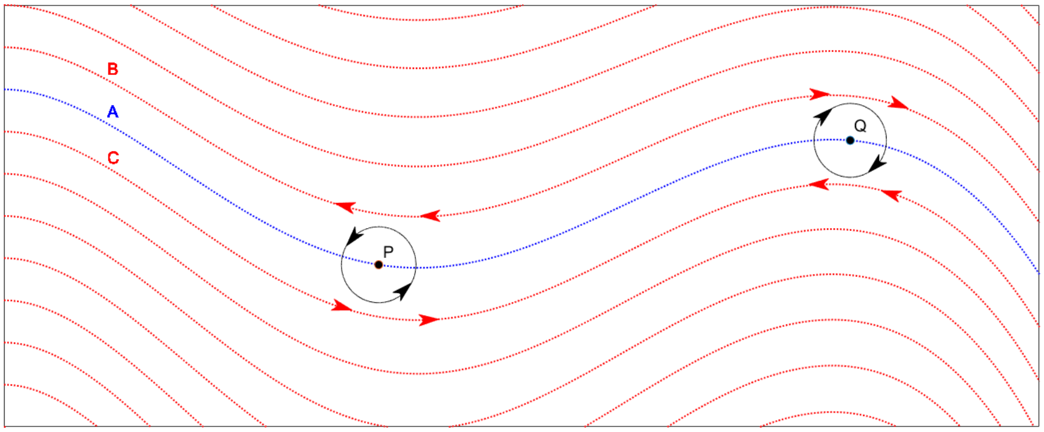

Fig. 1*b*: An observer sitting on the particle P would see neighboring particles circulating counterclockwise (black circle). At Q, local circulation would appear to be clockwise.

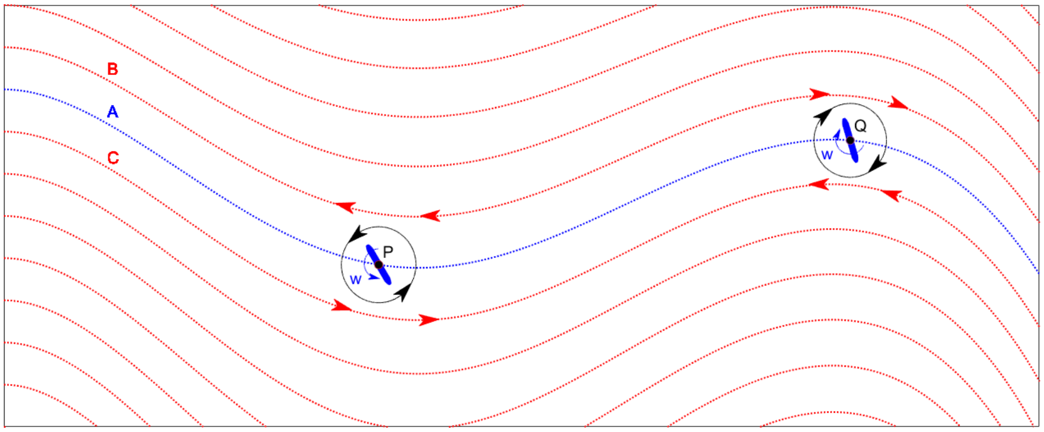

Fig. 1*c*: In micropolar flows, local circulation and micro-rotation tend to synchronize because of the action of vortex or micro-rotation viscosity, so that $\mathbf{w} \approx \frac{1}{2}\nabla\wedge\boldsymbol{u}$ for $t \gg 1$.

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We now describe our main results. We are concerned throughout the paper with Leray solutions $\boldsymbol{z}(\cdot,t) = (\boldsymbol{u}, \mathbf{w})(\cdot,t)$ in $\mathbb{R}^n$ $(n = 2, 3)$, whose velocity $\boldsymbol{u}(\cdot,t)$ satisfies

$$\| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \leq C_0 \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)} t^{-\alpha} \qquad \forall\ t > T_0 \tag{1.8}$$

for some constants $C_0, \alpha, T_0 \geq 0$, where $\boldsymbol{z}_0 = (\boldsymbol{u}_0, \mathbf{w}_0)$. For the existence of solutions satisfying (1.8) with $\alpha > 0$, see e.g. [3, 5, 6, 8, 22] and the Appendix to this paper.

Theorem A. *If* (1.8) *is valid for some* $\alpha \geq 0$, *then we have, for each* $m \geq 1$:

$$\| D^m \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \leq K_m(\alpha)\, C_0\, \mu^{-\frac{m}{2}} \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)} t^{-\alpha - \frac{m}{2}} \qquad \forall\ t > T_m \tag{1.9a}$$

where $K_m(\alpha)$ *depends solely on* (m, α), *and not on the solution or other parameters, and* T_m *only on* $(m, \alpha, \mu, \nu, \chi, C_0, T_0, \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)})$ *and, if* $n = 2$ *and* $\alpha = 0$, *also on how fast* $\| \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^2)}$ *vanishes as* $t \to \infty$. *Moreover, we have, for every* $m \geq 0$:

$$\| D^m \mathbf{w}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \leq K_{m+1}(\alpha)\, C_0\, \mu^{-\frac{m+1}{2}} \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)} t^{-\alpha - \frac{m+1}{2}} \qquad \forall\ t > T_m' \tag{1.9b}$$

with $K_{m+1}(\alpha)$ *given in* (1.9a), *and* T_m' *dependent on* $(m, \alpha, \mu, \nu, \chi, C_0, T_0, \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)})$ *and, if* $n = 2$ *and* $\alpha = 0$, *also on the vanishing speed of* $\| \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^2)}$ *as* $t \to \infty$.

Theorem B. *If* (1.8) *is valid for some* $\alpha \geq 0$, *then we have, for every* $m \geq 0$:

$$\begin{aligned} \| D^m \boldsymbol{\varepsilon}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \leq\ & K_m'(\alpha)\, C_0\, \frac{|\mu - \nu|}{\chi}\, \mu^{-\frac{m+3}{2}} \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)} t^{-\alpha - \frac{m+3}{2}} \\ & + K_m''(\alpha)\, C_0^2\, \chi^{-1} \mu^{-\frac{m+3}{2} - p_n} \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)}^2 t^{-2\alpha - \frac{m+3}{2} - p_n} \end{aligned} \tag{1.10}$$

for all $t > T_m''$, *with* $T_m'' > t_*$ *depending only on* $(m, \alpha, \mu, \nu, \chi, C_0, T_0, \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)})$ *and, in the case* $n = 2$ *and* $\alpha = 0$, *also on how fast* $\| \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^2)}$ *vanishes as* $t \to \infty$. *The constants* $K_m'(\alpha), K_m''(\alpha)$ *depend on* (m, α) *only, and* $p_n = (n-2)/4$.

Remark 1.1. In dimension 2, $\mathbf{w}(x_1, x_2, t) = (0, 0, \mathrm{w}(x_1, x_2, t))$, and so $\nabla \cdot \mathbf{w}(\cdot,t) = 0$. When $n = 3$, $\mathbf{w}(\cdot,t)$ is not solenoidal but (1.7) and Theorem B show that $\nabla \cdot \mathbf{w}(\cdot,t)$ decays very fast: one has $\| D^m [\nabla \cdot \mathbf{w}](\cdot,t) \|_{L^2(\mathbb{R}^3)} = O(t^{-\alpha - 2 - m/2})$ if $\mu \neq \nu$, and $\| D^m [\nabla \cdot \mathbf{w}](\cdot,t) \|_{L^2(\mathbb{R}^3)} = O(t^{-2\alpha - 2 - m/2 - 1/4})$ when $\mu = \nu$, for every $m \geq 0$, while, from Theorem A, $\| D^m [\nabla \wedge \mathbf{w}](\cdot,t) \|_{L^2(\mathbb{R}^3)} = O(t^{-\alpha - 1 - m/2})$ whether $\mu = \nu$ or not. A lower bound of $t^{-\alpha - 1 - m/2}$ for $\| D^m [\nabla \wedge \mathbf{w}](\cdot,t) \|_{L^2(\mathbb{R}^3)}$ follows from Theorem C, see (1.13) below. Other similar results can be obtained from Theorem D as well.

The estimates (1.9) and (1.10) are suggestive of a significantly faster decay of $\boldsymbol{\varepsilon}(\cdot,t)$ in comparison to $\mathbf{w}(\cdot,t)$ or $\nabla\wedge\boldsymbol{u}(\cdot,t)$. This can be rigorously established under some extra assumptions on the solution; for instance, if we assume, in addition to (1.8), a *lower* bound of $t^{-\alpha}$ for the solution $\boldsymbol{z}(\cdot,t) = (\boldsymbol{u},\mathbf{w})(\cdot,t)$. By THEOREM A, this is equivalent to having

$$\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \geq\; c_0\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\, t^{-\alpha} \qquad \forall\; t > t_0 \tag{1.11}$$

for some positive c_0, α, t_0, where again $\boldsymbol{z}_0 = (\boldsymbol{u}_0, \mathbf{w}_0)$. The existence of solutions satisfying (1.11) is obtained in [8] when $n = 2$. For the case $0 < \alpha < 1/2$ and $n = 2, 3$, it is a direct consequence of THEOREM B above, see the APPENDIX for details.

Theorem C. *If* (1.8) *and* (1.11) *hold for some* $\alpha > 0$*, then we have, for each* $m \geq 1$:

$$\|\,D^m\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \geq \Gamma_m\, c_0\, \mu^{-\,\frac{m}{2}}\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\, t^{-\alpha\,-\,\frac{m}{2}} \qquad \forall\; t > t_m \tag{1.12}$$

where $\Gamma_m = \Gamma_m(\alpha, c_0, C_0) > 0$ *depends on* (m, α, c_0, C_0) *only, and* t_m *depends solely on* $(m, \alpha, \mu, \nu, \chi, c_0, C_0, t_0, T_0, \|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$.

Remark 1.2. Recalling that $\|\,D^m[\,\nabla\wedge\boldsymbol{u}\,](\cdot,t)\,\|_{L^2(\mathbb{R}^n)} = \|\,D^{m+1}\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$, it follows from (1.7), (1.10) and (1.12) that, for every $m \geq 0$:

$$\|\,D^m\mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \geq \frac{1}{2}\,\Gamma_{m+1}\, c_0\, \mu^{-\,\frac{m+1}{2}}\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\, t^{-\alpha\,-\,\frac{m+1}{2}} + O(t^{-\alpha\,-\,\frac{m+3}{2}}) \tag{1.13}$$

for $t \gg 1$, so that (1.8) and (1.11) yield lower (and upper) bounds for $\mathbf{w}(\cdot,t)$ as well.

Further results can be obtained replacing (1.11) by the more general assumption

$$\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \geq\; c_0\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\, t^{-\eta} \qquad \forall\; t > t_0 \tag{1.14}$$

for some positive c_0, η, t_0, with $\eta \geq \alpha$ not too much larger than α (see below).

Theorem D. *If* (1.8) *and* (1.14) *are valid for some given* $\alpha > 0$ *and* $\eta \geq \alpha$ *such that* $\eta < \alpha\,(\alpha + 1 + m/2)/(\alpha + m/2)$ *for some* $m \geq 1$*, then: for every* $1 \leq \ell \leq m$*, we have*

$$\|\,D^\ell\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \geq\; c_\ell\, \mu^{-\,\frac{\ell}{2}}\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\, t^{-\eta\,-\,\frac{\ell}{2}q} \qquad \forall\; t > t'_\ell \tag{1.15}$$

for some $t'_\ell \gg 1$ *depending on* $(\ell, \alpha, \eta, \mu, \nu, \chi, c_0, C_0, t_0, T_0, \|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ *only, where* $q = \eta/\alpha$ *and* $c_\ell > 0$ *depends on* (ℓ, α, c_0, C_0) *only.*

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Remark 1.3. For (α, η) satisfying the condidions of THEOREM D, (1.10) and (1.15) imply that, for some $\theta > 0$ fixed: $\| D^k \boldsymbol{\varepsilon}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\theta}) \| D^k [\nabla \wedge \boldsymbol{u}](\cdot,t) \|_{L^2(\mathbb{R}^n)}$ for every $0 \leq k \leq m-1$. Recalling (1.7), we thus have for large t that the angular velocity of micro-rotation is essentially half of the vorticity, as physically expected.

Remark 1.4. When $\mu = \nu$, it follows from the proof of THEOREM D (see SECTION 4) that (1.15) is valid more generally if $0 < \alpha \leq \eta < \alpha(2\alpha + 1 + m/2 + p_n)/(\alpha + m/2)$, where $p_n = (n-2)/4$, and again $\| D^k \boldsymbol{\varepsilon}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\theta}) \| D^k [\nabla \wedge \boldsymbol{u}](\cdot,t) \|_{L^2(\mathbb{R}^n)}$, $0 \leq k \leq m-1$, for some $\theta > 0$, as in the previous case (cf. REMARK 1.3).

Here is a summary of the next sections. In SECTION 2 some preliminary results are provided to prepare the way for the derivation of the theorems above. Although this material is basically known, some proofs are new and a few improvements are offered. In particular, we use the monotonicity approach developed by the authors in [17] to obtain new upper or lower bounds for the solutions or their derivatives (of arbitrary order) out of previously known estimates. For the most part of SECTION 2, the special structure of the micropolar equations, dissipating the particles' micro-rotation more effectively than their translational velocity, is not taken into account, with solutions $\boldsymbol{z} = (\boldsymbol{u}, \mathbf{w})$ treated as a whole. In SECTION 3 the distinction between the fields $\boldsymbol{u}, \mathbf{w}$ is stressed from the start, leading to a series of results which are of interest on their own and which ultimately lead to THEOREMS A and B. The argument is once again based on monotonicity properties, which is also the main tool in SECTION 4. The latter focus instead on *lower* bounds, deriving THEOREMS C and D by a similar approach. An APPENDIX supplements the discussion providing proof for the existence of Leray solutions satisfying (1.8) and (1.11) in the case $0 < \alpha < 1/2$.

Notation. Throughout the text vector quantities are denoted by boldface letters, $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ indicates the standard inner product of two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, and $|\cdot|$ is used to denote ABSOLUTE VALUE (for scalars) or the EUCLIDEAN NORM (for vectors). $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, are the usual Lebesgue spaces; $\boldsymbol{v} = (v_1, v_2, ..., v_k) \in L^p(\mathbb{R}^n)$ means that $v_i \in L^p(\mathbb{R}^n)$ for every $1 \leq i \leq k$. By $\|\cdot\|_{L^p(\mathbb{R}^n)}$ or simply $\|\cdot\|_{L^p}$ are meant the usual L^p norms in $\mathbb{R}^n$; for vector functions,

$$\| \boldsymbol{v} \|_{L^p}^p \;\equiv\; \| (v_1, v_2, ..., v_k) \|_{L^p}^p \;=\; \sum_{i=1}^{k} \int_{\mathbb{R}^n} | v_i(x) |^p \, dx \tag{1.16}$$

if $1 \leq p < \infty$, and $\| \boldsymbol{v} \|_{L^\infty} = \operatorname{ess\,sup} \{ |(v_1(x), v_2(x), ..., v_k(x))| \colon x \in \mathbb{R}^n \}$ if $p = \infty$. Similarly, if $1 \leq p < \infty$ and $\boldsymbol{v} = (v_1, v_2, ..., v_k)$, we have

$$\| D\boldsymbol{v} \|_{L^p}^p \equiv \| D\boldsymbol{v} \|_{L^p(\mathbb{R}^n)}^p = \sum_{i=1}^{k} \sum_{j=1}^{n} \int_{\mathbb{R}^n} | D_j v_i(x) |^p \, dx, \tag{1.17a}$$

$$\| D^2\boldsymbol{v} \|_{L^p(\mathbb{R}^n)}^p = \sum_{i=1}^{k} \sum_{j=1}^{n} \sum_{\ell=1}^{n} \int_{\mathbb{R}^n} | D_j D_\ell v_i(x) |^p \, dx, \tag{1.17b}$$

and, for general $m \geq 1$,

$$\| D^m\boldsymbol{v} \|_{L^p(\mathbb{R}^n)}^p = \sum_{i=1}^{k} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_m=1}^{n} \int_{\mathbb{R}^n} | D_{j_1} D_{j_2} \cdots D_{j_m} v_i(x) |^p \, dx, \tag{1.17c}$$

where $D_j = \partial/\partial x_j$, $D_j D_\ell = \partial^2/\partial x_j \partial x_\ell$, and so forth, while

$$\| D\boldsymbol{v} \|_{L^\infty(\mathbb{R}^n)} = \max \Big\{ \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} | D_j v_i(x) | \colon 1 \leq i \leq k, \ 1 \leq j \leq n \Big\} \tag{1.18a}$$

and, more generally,

$$\| D^m\boldsymbol{v} \|_{L^\infty(\mathbb{R}^n)} = \max \Big\{ \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} | D_{j_1} \cdots D_{j_m} v_i(x) | \colon 1 \leq i \leq k, \ 1 \leq j_1, ..., j_m \leq n \Big\} \tag{1.18b}$$

(for $m \geq 1$) if $p = \infty$. In the text, we will only use $p = 2$ or $p = \infty$. We will also be using the Sobolev space $H^m(\mathbb{R}^n)$, i.e., the space of all those functions in $L^2(\mathbb{R}^n)$ whose derivatives of order m are again in $L^2(\mathbb{R}^n)$. If $\boldsymbol{v} = (v_1, v_2, ..., v_k)$, we write $\boldsymbol{v} \in H^m(\mathbb{R}^n)$ when $v_i \in H^m(\mathbb{R}^n)$ for all $1 \leq i \leq k$. By $L^2_\sigma(\mathbb{R}^n)$ we denote the space of vector functions $\boldsymbol{v} = (v_1, v_2, ..., v_n) \in L^2(\mathbb{R}^n)$ with $\nabla \cdot \boldsymbol{v} = 0$ in distributional sense, that is, $\nabla \cdot \boldsymbol{v} = 0$ in $\mathcal{D}'(\mathbb{R}^n)$, and, similarly, $H^m_\sigma(\mathbb{R}^n) = \{ \boldsymbol{v} \in L^2_\sigma(\mathbb{R}^n) \colon \boldsymbol{v} \in H^m(\mathbb{R}^n) \}$. Here, $\nabla \cdot$ denotes the DIVERGENCE operator; the CURL and GRADIENT are denoted by $\nabla \wedge$ and ∇, respectively, as in the equations (1.1a), (1.1c) above. Finally, we will also occasionally mention the homogeneous Sobolev space $\dot{H}^m(\mathbb{R}^n)$, $m \geq 1$, that is, the space of tempered distributions with locally integrable Fourier transforms and whose distributional derivatives of order m are all in $L^2(\mathbb{R}^n)$.

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2. Mathematical preliminaries

In this section we present a brief review of several basic results of (global) Leray solutions to the micropolar systems (1.1) or (1.6) which will be needed later. The discussion below is adapted from [4, 14, 17, 18, 23] and considers general initial data $(\boldsymbol{u}_0, \mathrm{w}_0) \in L^2_\sigma(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, where $n = 2$ or 3. It will prove convenient to introduce

$$\gamma \;=\; \min\{\,\mu, \nu\,\} \tag{2.1}$$

where $\mu, \nu > 0$ are given in the equations (1.1) or (1.6) above, and recall the general estimates

$$\|\, D^\ell v(\cdot,t)\,\|_{L^\infty(\mathbb{R}^2)} \|\, D^{m-\ell} v(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} \leq \|\, v(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} \|\, D^{m+1} v(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} \tag{2.2a}$$

$$\|\, D^\ell v(\cdot,t)\,\|_{L^\infty(\mathbb{R}^3)} \|\, D^{m-\ell} v(\cdot,t)\,\|_{L^2(\mathbb{R}^3)} \leq \|\, v(\cdot,t)\,\|_{L^2}^{1/2} \|\, Dv(\cdot,t)\,\|_{L^2}^{1/2} \|\, D^{m+1} v(\cdot,t)\,\|_{L^2(\mathbb{R}^3)} \tag{2.2b}$$

for every $m \geq 1$, $0 \leq \ell \leq m-1$ and $v \in H^{m+1}(\mathbb{R}^n)$, $n = 2, 3$, see ([4], Lemma 3.1).

In dimension $n = 2$, solutions are known to be smooth ($C^\infty(\mathbb{R}^2 \times (0,\infty))$) and uniquely defined by $(\boldsymbol{u}_0, \mathrm{w}_0)$, with $(\boldsymbol{u}, \mathrm{w})(\cdot,t) \in C((0,\infty), H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2))$ for every $m \geq 0$, see e.g. [13, 21]. If $n = 3$, uniqueness and exact regularity properties are not known for arbitrary data, but the following result is available.

Theorem 2.1. *Given $(\boldsymbol{u}_0, \mathbf{w}_0) \in L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and any Leray solution $(\boldsymbol{u}, \mathbf{w})(\cdot,t)$ to the equations* (1.1)*, there exists $t_{**} \geq 0$ satisfying*

$$t_{**} \;\leq\; 0.005 \cdot \gamma^{-5}\, \|\, (\boldsymbol{u}_0, \mathbf{w}_0)\,\|_{L^2(\mathbb{R}^3)}^4 \tag{2.3}$$

such that $(\boldsymbol{u}, \mathbf{w}) \in C^\infty(\mathbb{R}^3 \times (t_{**}, \infty))$, $(\boldsymbol{u}, \mathbf{w})(\cdot,t) \in C((t_{**},\infty), H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3))$ for every $m \geq 0$, and $\|\, D(\boldsymbol{u}, \mathbf{w})(\cdot,t)\,\|_{L^2(\mathbb{R}^3)}$ is monotonically decreasing in (t_{**}, ∞).

Proof. If $(\boldsymbol{u}, \mathbf{w})(\cdot,t) \in C([t_0, T], H^1(\mathbb{R}^3))$ for some $0 \leq t_0 < T$, the solution is smooth in (t_0, T) and we obtain, differentiating the equations (1.1), the energy estimate

$$\begin{aligned} \|\, D\boldsymbol{z}(\cdot,t)\,\|_{L^2(\mathbb{R}^3)}^2 + 2\gamma \int_{t_0}^{t} \|\, D^2\boldsymbol{z}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^3)}^2 d\tau \;\leq\; \|\, D\boldsymbol{z}(\cdot,t_0)\,\|_{L^2(\mathbb{R}^3)}^2 \,+ \\ +\; 2\int_{t_0}^{t} \|\, D^2\boldsymbol{z}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^3)} \|\, \boldsymbol{z}(\cdot,\tau)\,\|_{L^\infty(\mathbb{R}^3)} \|\, D\boldsymbol{z}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^3)} d\tau \end{aligned} \tag{2.4}$$

for all $t \in (t_0, T)$, where $\boldsymbol{z} = (\boldsymbol{u}, \mathbf{w})$ and $\gamma > 0$ is given in (2.1). Observing that

$$\begin{aligned} \| \boldsymbol{z}(\cdot,\tau) \|_{L^\infty(\mathbb{R}^3)} \| D\boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^3)} &\leq K \, \| \boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^3)}^{1/4} \| D\boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^3)} \| D^2\boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^3)}^{3/4} \\ &\leq K \, \| \boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^3)}^{1/2} \| D\boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^3)}^{1/2} \| D^2\boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^3)} \end{aligned}$$

(by Fourier transform), where $K = \sqrt[8]{12}\,/\sqrt{6\pi}$ ([24], Theorem 2.2), it follows from (2.4) that $\| D\boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^3)}$ is monotonically decreasing for $t \geq t_0$ if we have

$$K \, \| \boldsymbol{z}(\cdot,t_0) \|_{L^2(\mathbb{R}^3)}^{1/2} \| D\boldsymbol{z}(\cdot,t_0) \|_{L^2(\mathbb{R}^3)}^{1/2} < \gamma \tag{2.5}$$

with, in particular, $\boldsymbol{z} \in C^\infty(\mathbb{R}^3 \times (t_0,\infty))$ and $\boldsymbol{z}(\cdot,t) \in C((t_0,\infty), H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3))$ for each m (see e.g. [9, 12, 13]). Taking $\hat{t} > \frac{1}{2} K^4 \gamma^{-5} \| (\boldsymbol{u}_0, \mathrm{w}_0) \|_{L^2(\mathbb{R}^3)}^4$, we have from (1.2) that

$$2\gamma \int_0^{\hat{t}} \| D\boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau \;\leq\; \| (\boldsymbol{u}_0, \mathrm{w}_0) \|_{L^2(\mathbb{R}^3)}^2$$

and so there esists $E \subseteq (0, \hat{t})$ with positive Lebesgue measure such that

$$\| D\boldsymbol{z}(\cdot,t') \|_{L^2(\mathbb{R}^3)}^2 d\tau \;\leq\; \frac{1}{2\gamma} \, \hat{t}^{\,-1} \| (\boldsymbol{u}_0, \mathrm{w}_0) \|_{L^2(\mathbb{R}^3)}^2$$

for all $t' \in E$. Therefore, for each $t' \in E$:

$$\begin{aligned} \| \boldsymbol{z}(\cdot,t') \|_{L^2(\mathbb{R}^3)}^2 \| D\boldsymbol{z}(\cdot,t') \|_{L^2(\mathbb{R}^3)}^2 &\leq \| \boldsymbol{z}(\cdot,t') \|_{L^2(\mathbb{R}^3)}^2 \frac{1}{2\gamma} \, \hat{t}^{\,-1} \| (\boldsymbol{u}_0, \mathrm{w}_0) \|_{L^2(\mathbb{R}^3)}^2 \\ &\leq \frac{1}{2\gamma} \, \hat{t}^{\,-1} \| (\boldsymbol{u}_0, \mathrm{w}_0) \|_{L^2(\mathbb{R}^3)}^4 < K^{-4} \gamma^4 \end{aligned}$$

where the last estimate follows from the choice of $\hat{t}$. By (2.5), this gives the result. □

Remark 2.1. In dimension $n = 2$, we have, for $\boldsymbol{z} = (\boldsymbol{u}, \mathrm{w})$,

$$\begin{aligned} \| \boldsymbol{z}(\cdot,\tau) \|_{L^\infty(\mathbb{R}^2)} \| D\boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^2)} &\leq \frac{1}{2} \| \boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^2)}^{1/2} \| D\boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^2)} \| D^2\boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^2)}^{1/2} \\ &\leq \frac{1}{2} \| \boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^2)} \| D^2\boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^2)} \end{aligned}$$

(see e.g. [24]). Since (2.4) is also valid for $n = 2$, it follows that $\| D(\boldsymbol{u}, \mathrm{w})(\cdot,t) \|_{L^2(\mathbb{R}^2)}$ is monotonically decreasing in the interval (t_{**}, ∞) if $t_{**} \geq 0$ is such that

$$\| (\boldsymbol{u}, \mathrm{w})(\cdot, t_{**}) \|_{L^2(\mathbb{R}^2)} \leq 2\gamma. \tag{2.6}$$

Because $\| \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \to 0$ as $t \to \infty$ (see e.g. [14, 16, 23]), this condition is satisfied for suitable t_{**}, which depends on how fast $\| \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^2)}$ becomes small for $t \gg 1$.

The monotonicity of $\| D\boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$ has important consequences, as illustrated by the next results. More examples are given in Section 3 (see also [17]).

Theorem 2.2. *For any Leray solution* $\boldsymbol{z} = (\boldsymbol{u}, \mathbf{w})$ *to the micropolar equations* (1.1) *or* (1.6) *above, we have*

$$\| D\boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \leq \gamma^{-1/2} \| (\boldsymbol{u}_0, \mathbf{w}_0) \|_{L^2(\mathbb{R}^n)} t^{-1/2} \qquad \forall\ t > 2t_{**} \tag{2.7}$$

where $t_{**} \geq t_*$ *denotes the monotonicity time of* $\| D\boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$, *cf.* (2.3) *or* (2.6).

Proof. From the energy inequality for $\| \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$ (see (1.2)), we have

$$\| \boldsymbol{z}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} + 2\gamma \int_{t_0}^{t} \| D\boldsymbol{z}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^n)} d\tau \leq \| \boldsymbol{z}(\cdot,t_0) \|^2_{L^2(\mathbb{R}^n)} \tag{2.8}$$

for every $t > t_0 > t_*$ (the solution's regularity time), so that, for $t > 2t_{**}$:

$$\gamma\, t \| D\boldsymbol{z}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} \leq 2\gamma \int_{t/2}^{t} \| D\boldsymbol{z}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^n)} d\tau \leq \| (\boldsymbol{u}_0, \mathbf{w}_0) \|^2_{L^2(\mathbb{R}^n)}. \qquad \square$$

Remark 2.2. In a similar way, we obtain that $\| D\boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = o(t^{-\frac{1}{2}})$ as $t \to \infty$, since (for $t > t_{**}$): $t \| D\boldsymbol{z}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} \leq \int_t^{2t} \| D\boldsymbol{z}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^n)} d\tau \to 0$. See e.g. [19, 29].

Theorem 2.3. (*i*) *If* $n = 3$, *for each* $m \geq 0$ *there exists an absolute constant* $K_m > 0$ (*i.e., depending only on* m) *such that, given any Leray solution* $\boldsymbol{z} = (\boldsymbol{u}, \mathbf{w})$ *to the equations* (1.1), *we have* $\| D^m \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^3)}$ *monotonically decreasing in* $(t_{**}^{(m)}, \infty)$ *for some* $t_{**}^{(m)} \geq t_*$ *satisfying*

$$t_{**}^{(m)} \leq K_m \gamma^{-5} \| (\boldsymbol{u}_0, \mathbf{w}_0) \|^4_{L^2(\mathbb{R}^3)}. \tag{2.9a}$$

(*ii*) *If* $n = 2$, *for each* $m \geq 1$ *there exists a constant* $k_m > 0$ (*depending only on* m) *so that, for any solution* $\boldsymbol{z} = (\boldsymbol{u}, \mathrm{w})$ *of the system* (1.6), *we have* $\| D^m \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^2)}$ *monotonically decreasing in the interval* $(t_{**}^{(m)}, \infty)$, *for any* $t_{**}^{(m)} \geq 0$ *satisfying*

$$\| (\boldsymbol{u}, \mathrm{w})(\cdot, t_{**}^{(m)}) \|_{L^2(\mathbb{R}^2)} \leq k_m \gamma. \tag{2.9b}$$

Proof. The case $m \leq 1$ has already been considered (we may take $K_0 = K_1 = 0.005$ and $k_1 = 2$, cf. (2.3) and (2.6) above). Now, given $m \geq 1$, we obtain, from (1.1) or (1.6),

$$\| D^m \boldsymbol{z}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} + 2\gamma \int_{t_0}^{t} \| D^{m+1} \boldsymbol{z}(\cdot,\tau) \|^2_{L^2(\mathbb{R}^n)} d\tau \leq \| D^m \boldsymbol{z}(\cdot,t_0) \|^2_{L^2(\mathbb{R}^n)} +$$
$$H'_{m,n} \sum_{\ell=0}^{[m/2]} \int_{t_0}^{t} \| D^{m+1} \boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^n)} \| D^{\ell} \boldsymbol{z}(\cdot,\tau) \|_{L^\infty(\mathbb{R}^n)} \| D^{m-\ell} \boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^n)} d\tau$$

for all $t > t_0 > t_*$, where $[\,r\,]$ denotes the integer part of $r \in \mathbb{R}$. This expression gives

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$$\begin{aligned} \| D^m \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)}^2 + 2\gamma \int_{t_0}^{t} \| D^{m+1} \boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^n)}^2 \, d\tau \;\leq\; \| D^m \boldsymbol{z}(\cdot,t_0) \|_{L^2(\mathbb{R}^n)}^2 + \\ +\; H_{m,n} \int_{t_0}^{t} g_n(\tau) \, \| D^{m+1} \boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^n)}^2 \, d\tau \end{aligned} \tag{2.10}$$

by (2.2), where $g_2(t) = \| \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^2)}$ and $g_3(t) = \| \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^3)}^{1/2} \| D\boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^3)}^{1/2}$. Here, $H'_{m,n}$, $H_{m,n}$ are positive constants that depend on (m,n) only, both increasing with $m \geq 1$ (for example, $H'_{1,n} = 1$, $H_{1,2} = 1/2$, $H_{1,3} = \sqrt[8]{12}/\sqrt{6\pi}$, and so forth). Let $m \geq 2$: from (2.10), monotonicity of $\| D^m \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$ is achieved if $H_{m,n}\, g_n(t) \leq 2\gamma$ when $t > t_{**}^{(m)}$. For $n = 2$, this is (2.9b) with $k_m = 2/H_{m,2}$. For $n = 3$, recalling (2.3) and (2.7) we see that this condition is guaranteed by (2.9a) if we take $K_m = H_{m,3}^4/16$. □

Remark 2.3. From the proof of Theorem 2.3 we see that *more* can be achieved by redefining slightly the values K_m, k_m (and $t_{**}^{(m)}$) in (2.9) for $m \geq 1$, as follows. Setting $k_m = 1/H_{m,2}$ and $K_m = H_{m,3}^4$, $m \geq 1$, we obtain $H_{m,n}\, g_n(t) \leq \gamma$ for $t > t_{**}^{(m)}$, with $t_{**}^{(m)}$ now defined by

$$t_{**}^{(m)} \;=\; \inf \left\{ t \geq 0 \colon \| (\boldsymbol{u}, \mathrm{w})(\cdot,t) \|_{L^2(\mathbb{R}^2)} \leq \gamma / H_{m,2} \right\} \qquad \text{if } n = 2, \tag{2.11a}$$

$$t_{**}^{(m)} \;=\; H_{m,3}^4 \, \gamma^{-5} \, \| (\boldsymbol{u}_0, \mathbf{w}_0) \|_{L^2(\mathbb{R}^3)}^4 \qquad \text{if } n = 3, \tag{2.11b}$$

for $m \geq 1$, where $H_{m,2}, H_{m,3} > 0$ are given in (2.10). These definitions assure us that

$$H_{m,n}\, g_n(t) \;\leq\; \gamma \qquad \forall\; t > t_{**}^{(m)} \tag{2.11c}$$

and thus produce, from (2.10) above, the energy estimate

$$\| D^m \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)}^2 + \gamma \int_{t_0}^{t} \| D^{m+1} \boldsymbol{z}(\cdot,\tau) \|_{L^2(\mathbb{R}^n)}^2 \, d\tau \;\leq\; \| D^m \boldsymbol{z}(\cdot,t_0) \|_{L^2(\mathbb{R}^n)}^2 \tag{2.12}$$

for all $t > t_0 > t_{**}^{(m)}$, and every $m \geq 1$, with $t_{**}^{(m)}$ defined in (2.11a) and (2.11b) above. This estimate allows a quick generalization of Theorem 2.2, given next.

Theorem 2.4. *For any Leray solution* $\boldsymbol{z} = (\boldsymbol{u}, \mathbf{w})$ *to the micropolar equations* (1.1) *or* (1.6) *above, we have, for every* $m \geq 1$:

$$\| D^m \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \;\leq\; K_m \, \gamma^{-m/2} \, \| (\boldsymbol{u}_0, \mathbf{w}_0) \|_{L^2(\mathbb{R}^n)} \, t^{-m/2} \qquad \forall\; t > 2^m \, t_{**}^{(m)} \tag{2.13}$$

where the constant K_m *depends only on* m *(and not on the solution), and* $t_{**}^{(m)}$ *is the monotonicity time for* $\| D^m \boldsymbol{z}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$ *introduced in* (2.11a) *and* (2.11b) *above.*

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Proof. For $m = 1$, (2.13) is obvious from (2.7) of Theorem 2.2, with $K_1 = 1$, observing that $t_{**}^{(1)} \geq t_{**}$. For $m = 2$, (2.12) with $m = 1$ gives, for $t > 2t_{**}^{(2)}$:

$$\gamma\,\frac{t}{2}\,\|\,D^2\boldsymbol{z}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} \leq \gamma\int_{t/2}^{t}\|\,D^2\boldsymbol{z}(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^n)}\,d\tau \;\leq\; \|\,D\boldsymbol{z}(\cdot,\frac{t}{2})\,\|^2_{L^2(\mathbb{R}^n)},$$

where the first inequality is due to the monotonicity of $\|\,D^2\boldsymbol{z}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^n)}$, as $t/2 > t_{**}^{(2)}$. For $t > 4\,t_{**}^{(2)}$, we have $t/2 > 2\,t_{**}^{(1)}$ (because $t_{**}^{(2)} > t_{**}^{(1)}$), so that we have

$$\|\,D\boldsymbol{z}(\cdot,\frac{t}{2})\,\|^2_{L^2(\mathbb{R}^n)} \leq K_1^2\,\gamma^{-1}\,\|\,(\boldsymbol{u}_0,\mathbf{w}_0)\,\|^2_{L^2(\mathbb{R}^n)}\left\{\frac{t}{2}\right\}^{-1}$$

from the previous case ($m = 1$). This shows the result for $m = 2$ as well. Proceeding by induction in a similar way, (2.12) is obtained for every $m \geq 1$, as claimed. □

We conclude this section with an important improvement for $\|\,D^m\mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$.

Theorem 2.5. *For any Leray solution* $\boldsymbol{z} = (\boldsymbol{u},\mathbf{w})$ *to the equations* (1.1) *or* (1.6), (*i*) *we have*

$$\|\,\mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq \gamma^{-1/2}\,\|\,(\boldsymbol{u}_0,\mathbf{w}_0)\,\|_{L^2(\mathbb{R}^n)}\,t^{-1/2} \qquad \forall\; t > \tau_0 \tag{2.14a}$$

for some $\tau_0 > 0$ *depending only on* γ, χ *and the monotonicity time* t_{**} *given in* (2.7); (*ii*) *for every* $m \geq 1$, *we have*

$$\|\,D^m\mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq C_m\,\gamma^{-\,\frac{m+1}{2}}\,\|\,(\boldsymbol{u}_0,\mathbf{w}_0)\,\|_{L^2(\mathbb{R}^n)}\,t^{-\,\frac{m+1}{2}} \qquad \forall\; t > \tau_m \tag{2.14b}$$

for some constant C_m *which depends only on* m, *and where* $\tau_m > 0$ *depends only on* m, $\|\,(\boldsymbol{u}_0,\mathbf{w}_0)\,\|_{L^2(\mathbb{R}^n)}$, γ, χ *and* $t_{**}^{(m+2)}$ *given in* (2.11).

Proof. (*i*) From the equation for $\mathbf{w}(\cdot,t)$ given in (1.1) or (1.6) we get, for every $t > t_*$ (the solution's regularity time), that

$$\begin{aligned}\frac{d}{dt}\,\|\,\mathbf{w}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} + 2\nu\,\|\,D\mathbf{w}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} + 8\chi\,\|\,\mathbf{w}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} \\ \leq\; 4\chi\,\|\,\mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}\,\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \\ \leq\; 4\chi\,\|\,\mathbf{w}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} + \chi\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

so that we have

$$\frac{d}{dt}\,\|\,\mathbf{w}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} + 4\chi\,\|\,\mathbf{w}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} \leq\; \chi\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)}.$$

Recalling (2.7) of Theorem 2.2, this gives, for $t > 4\,t_{**}$, that

$$\begin{aligned}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\leq e^{-2\chi t}\,\|(\boldsymbol{u}_0,\mathbf{w}_0)\|^2_{L^2(\mathbb{R}^n)} + \chi\int_{t/2}^{t} e^{-4\chi(t-\tau)}\,\|D\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau\\ &\leq \|(\boldsymbol{u}_0,\mathbf{w}_0)\|^2_{L^2(\mathbb{R}^n)}\left\{e^{-2\chi t} + \frac{1}{2}\gamma^{-1}t^{-1}\right\}, \qquad [\text{by } (2.7)]\end{aligned}$$

from which we get (2.14a) for all $t > 4\,t_{**}$ with $t\,e^{-2\chi t} \leq \gamma^{-1}/2$, finishing the proof of (i).

(ii) Given $t > t_*$, we have, differentiating m times the equation for $\mathbf{w}(\cdot,t)$ in (1.1) or (1.6),

$$\begin{aligned}&\frac{d}{dt}\,\|D^m\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\nu\,\|D^{m+1}\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 8\chi\,\|D^m\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\\ &\qquad\leq 4\chi\,\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)}\,\|D^{m+1}\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} + K'_m\,\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)}G_m(t)\end{aligned}$$

where $G_m(t) = \sum_{\ell=0}^{m}\|D^\ell\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\|D^{m+1-\ell}\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)}$, for some constant K'_m depending only on m. This gives

$$\frac{d}{dt}\,\|D^m\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\chi\,\|D^m\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq \chi\,\|D^{m+1}\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + K''_m\,\chi^{-1}G_m(t)^2$$

where $K''_m = (K'_m/\sqrt{8}\,)^2$, so that we have, for $t > 2\,t_*$:

$$\begin{aligned}\|D^m\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\leq e^{-\chi t}\,\|D^m\mathbf{w}(\cdot,\tfrac{t}{2})\|^2_{L^2(\mathbb{R}^n)} + \chi\int_{t/2}^{t} e^{-2\chi(t-\tau)}\,\|D^{m+1}\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau\\ &\quad+ K''_m\,\chi^{-1}\int_{t/2}^{t} e^{-2\chi(t-\tau)}\,G_m(\tau)^2\,d\tau.\end{aligned}$$

Taking $t/2 > 2^{(m+2)}\,t_{**}^{(m+2)}$ and using (2.13) of Theorem 2.4 to estimate the three terms on the righthand side of the expression above, we obtain (2.14b), increasing t further if necessary (depending on the values of m, $\|(\boldsymbol{u}_0,\mathbf{w}_0)\|_{L^2(\mathbb{R}^n)}$, γ and χ), as claimed. □

Remark 2.4. In an entirely similar way to the derivation of (2.7) and (2.13) above, we obtain from (2.7) and (2.12) that

$$\|D\boldsymbol{z}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \gamma^{-1/2}\,\|\boldsymbol{z}(\cdot,t/2)\|_{L^2(\mathbb{R}^n)}\,t^{-1/2} \qquad \forall\; t > 2\,t_{**} \tag{2.15a}$$

(where t_{**} is the monotonicity time given in (2.3) and (2.6)) and, for every $m > 1$,

$$\|D^m\boldsymbol{z}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \tilde{K}_m\,\gamma^{-m/2}\,\|\boldsymbol{z}(\cdot,t/2)\|_{L^2(\mathbb{R}^n)}\,t^{-m/2} \qquad \forall\; t > 2^m\,t_{**}^{(m)} \tag{2.15b}$$

for some constant $\tilde{K}_m$ which depends *only* on m, with $t_{**}^{(m)}$ given in (2.11), as before. Using (2.15) we can obtain, similarly to the derivation of (2.14), for every $m \geq 0$:

$$\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \tilde{C}_m\,\gamma^{-\frac{m+1}{2}}\,\|\boldsymbol{z}(\cdot,t/2)\|_{L^2(\mathbb{R}^n)}\,t^{-\frac{m+1}{2}} \qquad \forall\; t > \tilde{\tau}_m \tag{2.16}$$

with $\tilde{C}_0 = 1$, $\tilde{\tau}_0 = \tau_0$, and $\tilde{C}_m \geq C_m$, $\tilde{\tau}_m \geq \tau_m$, $m \geq 1$, behaving as C_m, τ_m in (2.14b).

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3. Proof of Theorems A and B

In this section we derive THEOREM A and THEOREM B stated in the INTRODUCTION, dividing their proofs into several individual results of interest on their own. Throughout the section $\boldsymbol{z}(\cdot,t) = (\boldsymbol{u},\mathbf{w})(\cdot,t)$ denotes an (arbitrary) Leray solution to the equations (1.1) or (1.6), with initial data $\boldsymbol{z}_0 = (\boldsymbol{u}_0,\mathbf{w}_0) \in L^2_\sigma(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, satisfying

$$\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq C_0\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\, t^{-\alpha} \qquad \forall\; t > T_0 \tag{3.1}$$

for given nonnegative constants C_0, α, T_0. We recall that $\gamma = \min\{\mu,\nu\}$, see (2.1), and, by (2.11a), it will be convenient if ($n=2$, $\alpha=0$) to introduce $g(\cdot)$ such that

$$\|\,\boldsymbol{z}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} \leq\; g(t) \qquad \forall\; t > 0. \tag{3.2}$$

Theorem 3.1. *Assuming* (3.1)*, then we have, for each* $m \geq 0$:

$$\|\,D^m\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq K_m(\alpha)\,C_0\,\gamma^{-\frac{m}{2}}\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\,t^{-\alpha-\frac{m}{2}} \qquad \forall\; t > T_m \tag{3.3a}$$

and

$$\|\,D^m\mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq K'_m(\alpha)\,C_0\,\gamma^{-\frac{m+1}{2}}\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\,t^{-\alpha-\frac{m+1}{2}} \qquad \forall\; t > T'_m \tag{3.3b}$$

for some constants $K_m(\alpha), K'_m(\alpha)$ *which depend only on* (m,α)*, and not on the solution or any other parameters, and where the time instants* T_m, T'_m *depend only on* $(m,\alpha,\gamma,\chi,C_0,T_0,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ *and, if* $n=2$ *and* $\alpha=0$*, also on the function* $g(\cdot)$ *considered in* (3.2) *above.*

Proof. In view of (2.15) and (2.16), it is sufficient to show that we have, from (3.1),

$$\|\,\mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq K(\alpha)\,C_0\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\,t^{-\alpha} \qquad \forall\; t > T'_0 \tag{3.4}$$

for some constant $K(\alpha) > 0$ dependent only on α, and some $T'_0 > 0$ depending only on α, γ,χ,C_0,T_0 and $\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}$. From the equation for $\mathbf{w}(\cdot,t)$ in (1.1c) or (1.6c) we get

$$\frac{d}{dt}\,\|\,\mathbf{w}(\cdot,t)\,\|^2_{L^2} + 2\,\nu\,\|\,D\mathbf{w}(\cdot,t)\,\|^2_{L^2} + 8\,\chi\,\|\,\mathbf{w}(\cdot,t)\,\|^2_{L^2} \leq\; 4\,\chi\,\|\,D\mathbf{w}(\cdot,t)\,\|_{L^2}\,\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2}$$

for $t > t_*$ (the solution's regularity time), so that we have

$$\frac{d}{dt}\,\|\,\mathbf{w}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} + 8\,\chi\,\|\,\mathbf{w}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} \leq\; 2\,\nu^{-1}\chi^2\,\|\,\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)}$$

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for $t > t_*$. Therefore, if $t/2 > \max\{t_*, T_0\}$ we get, by (3.1),

$$\begin{aligned} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)}^2 &\leq \|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}^2 e^{-4\chi t} + 2\nu^{-1}\chi^2 \int_{t/2}^{t} e^{-8\chi(t-\tau)} \|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \\ &\leq \|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}^2 \left\{ e^{-4\chi t} + 2^{2\alpha}\nu^{-1}\chi\, C_0^2\, t^{-2\alpha} \right\}, \end{aligned}$$

so that

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \left(1 + 2^{\alpha}\right) C_0\, \gamma^{-\frac{1}{2}} \|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)} \chi^{\frac{1}{2}}\, t^{-\alpha} \qquad \forall\ t > t_1 \tag{3.5}$$

for some $t_1 = t_1(\alpha, \gamma, \chi, t_*, C_0, T_0) > 1$ large enough. This has the form (3.4) except for the factor $\gamma^{-1/2}\chi^{1/2}$, which can be dismissed by redefining t_1 as follows. If $0 < \alpha \leq 1$, increasing t_1 if necessary so that $\gamma^{-2}\chi^2 \leq t_1^{\alpha}$, then (3.5) gives

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \left(1 + 2^{\alpha}\right) C_0 \|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)} t^{-\frac{3\alpha}{4}}$$

for $t > t_1$. By (3.1), it follows that $\|\boldsymbol{z}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \left(2 + 2^{\alpha}\right) C_0 \|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)} t^{-\frac{3\alpha}{4}}$ (for $t > t_1$) and then, by (2.16), increasing t_1 if necessary so that $\gamma^{-2} \leq t_1$,

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq 2^{\alpha}\left(2 + 2^{\alpha}\right) C_0 \|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)} t^{-\alpha} \tag{3.6}$$

for $t > 2t_1$. This is (3.4), as we proposed to show, in the case $0 < \alpha \leq 1$. (Having (3.6) gives us that $\|\boldsymbol{z}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq 2^{\alpha}\left(3 + 2^{\alpha}\right) C_0 \|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)} t^{-\alpha}$ for all such t, because of (3.1), and (3.3) results then directly from (2.15) and (2.16).) For $\alpha > 1$, we proceed similarly: increasing t_1 in (3.5), if necessary, so that $\gamma^{-2}\chi^2 \leq t_1$, we obtain from (3.5) that

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \left(1 + 2^{\alpha}\right) C_0 \|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)} t^{-\alpha + \frac{1}{4}}$$

for $t > t_1$. Hence, from (3.1): $\|\boldsymbol{z}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \left(2 + 2^{\alpha}\right) C_0 \|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)} t^{-\alpha+\frac{1}{4}}$ (for $t > t_1$) and so, by (2.16), increasing t_1 if necessary so that $\gamma^{-2} \leq t_1$, we obtain (3.6) for $t > 2t_1$, thus showing (3.4) if $\alpha > 1$ as well. By (3.1), $\|\boldsymbol{z}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq 2^{\alpha}\left(3 + 2^{\alpha}\right) C_0 \|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)} t^{-\alpha}$ for all such t, so that we can apply (2.15) and (2.16) once again and obtain (3.3). □

Recalling (1.7), we now begin estimating the remainder $\boldsymbol{\varepsilon}(\cdot,t)$ in the expression

$$\mathbf{w}(x,t) = \frac{1}{2}\nabla \wedge \boldsymbol{u}(x,t) + \boldsymbol{\varepsilon}(x,t) \tag{3.7}$$

for $t > t_*$ (the solution's regularity time). From the equations for $\boldsymbol{u}(\cdot,t)$ and $\mathbf{w}(\cdot,t)$ we get

$$\boldsymbol{\varepsilon}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\varepsilon} + 4\chi\boldsymbol{\varepsilon} = \mathbb{L}\boldsymbol{\varepsilon} + (\nu - \mu)\Delta\mathbf{w} + \frac{1}{2}\sum_{j=1}^{n}(\nabla u_j)\wedge(D_j\boldsymbol{u}) \tag{3.8a}$$

 doi:10.20944/preprints202207.0166.v1

where $\mathbb{L}$ denotes the elliptic operator

$$\mathbb{L}\mathbf{v} \;=\; \mu\,\Delta\mathbf{v} \,+\, \kappa\nabla(\nabla\cdot\mathbf{v}) \,-\, \chi\nabla\wedge(\nabla\wedge\mathbf{v}). \tag{3.8b}$$

The equations (3.8) allow us to obtain the following preliminary (but very useful) estimate for $\|\,\varepsilon(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$, which will play an important role for the next results, leading to the improved estimate given in (3.15). We recall that $\gamma = \min\{\mu,\nu\}$.

Theorem 3.2. *Assuming* (3.1)*, then we have, for each* $m \geq 0$:

$$\|\,D^m\boldsymbol{\varepsilon}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq K''_m(\alpha)\,C_0\,\frac{\mu+\nu}{\chi}\,\gamma^{-\,\frac{m+3}{2}}\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\,t^{-\alpha-\,\frac{m+3}{2}} \qquad \forall\; t > T''_m \tag{3.9}$$

for some constant $K''_m(\alpha)$ *which depends only on* (m,α)*, and not on the solution or other parameters, with* T''_m *depending only on* $(m,\alpha,\mu,\nu,\chi,C_0,T_0,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ *and, if* $n = 2$ *and* $\alpha = 0$*, also on the function* $g(\cdot)$ *considered in* (3.2) *above.*

Remark 3.1. Similarly to (2.15) and (2.16), we also have, for each $m \geq 0$:

$$\|\,D^m\boldsymbol{\varepsilon}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq S_m\,\frac{\mu+\nu}{\chi}\,\gamma^{-\,\frac{m+3}{2}}\,\|\,\boldsymbol{z}(\cdot,t/2)\,\|_{L^2(\mathbb{R}^n)}\,t^{-\,\frac{m+3}{2}} \qquad \forall\; t > \sigma_m \tag{3.10}$$

for some constant S_m that depends only on m, and some instant σ_m depending only on $(m,\alpha,\mu,\nu,\chi,C_0,T_0,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ and, if $n = 2$ and $\alpha = 0$, also on $g(\cdot)$.

Proof of (3.9): By (3.1) and (3.6), it suffices to show (3.10). Given $m \geq 0$, we have

$$\begin{aligned}\frac{d}{dt}\,\|\,D^m\boldsymbol{\varepsilon}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} &+ 2\mu\,\|\,D^{m+1}\boldsymbol{\varepsilon}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} + 8\chi\,\|\,D^m\boldsymbol{\varepsilon}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} \\ &\leq\; |\,\mu-\nu\,|\,\|\,D^m\boldsymbol{\varepsilon}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}\,\|\,D^{m+2}\mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} + K_m\,\|\,D^m\boldsymbol{\varepsilon}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}H_m(t)\end{aligned}$$

for $t > t_*$, by (3.8), where

$$\begin{aligned}H_m(t) \;=\; &\sum_{\ell=0}^{m}\|\,D^{\ell}\boldsymbol{u}(\cdot,t)\,\|_{L^\infty(\mathbb{R}^n)}\|\,D^{m+1-\ell}\boldsymbol{\varepsilon}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \,+ \\ &+\sum_{\ell=0}^{m}\|\,D^{\ell+1}\boldsymbol{u}(\cdot,t)\,\|_{L^\infty(\mathbb{R}^n)}\|\,D^{m+1-\ell}\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

and K_m is some constant dependent only on m. This gives

$$\frac{d}{dt}\,\|\,D^m\boldsymbol{\varepsilon}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} + 4\chi\,\|\,D^m\boldsymbol{\varepsilon}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} \leq \frac{(\mu+\nu)^2}{8\chi}\,\|\,D^{m+2}\mathbf{w}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^n)} + \frac{K_m^2}{8\chi}\,H_m(t)^2$$

from which (3.10) follows, recalling (3.7) and the estimates (2.15) and (2.16). □

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In terms of $\boldsymbol{\varepsilon}(\cdot,t)$, the equation for $\boldsymbol{u}(\cdot,t)$, $\mathbf{w}(\cdot,t)$ can be rewritten in the form

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \mu\Delta\boldsymbol{u} + 2\chi\nabla\wedge\boldsymbol{\varepsilon}, \qquad \nabla\cdot\boldsymbol{u}(\cdot,t) = 0, \tag{3.11a}$$

$$\mathbf{w}_t + \boldsymbol{u}\cdot\nabla\mathbf{w} = \nu\Delta\mathbf{w} + \kappa\nabla(\nabla\cdot\mathbf{w}) - 4\chi\,\boldsymbol{\varepsilon}, \tag{3.11b}$$

for $t > t_*$, which will be used in the rest of this section.

Theorem 3.3. *Assuming* (3.1), *then*: *for each* $m \geq 1$, *the function* $Z_m(\cdot)$ *given by*

$$Z_m(t) = \| D^m\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 4\chi^2\mu^{-1}\int_t^\infty \| D^m\boldsymbol{\varepsilon}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \tag{3.12}$$

(for $t > t_*$) is monotonically decreasing in the interval (ζ_m,∞), where $\zeta_m > t_*$ depends only on $(m,\alpha,\mu,\nu,\chi,C_0,T_0,\|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ and, if $n=2$ and $\alpha=0$, also on the function $g(\cdot)$ considered in (3.2) *above.*

Proof. From (3.11a) we obtain, using the estimates (2.2),

$$\frac{d}{dt}\| D^m\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \mu\,\| D^{m+1}\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq 4\chi\,\| D^{m+1}\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}\| D^m\boldsymbol{\varepsilon}(\cdot,t)\|_{L^2(\mathbb{R}^n)}$$

for all $t > \zeta_m$, with ζ_m appropriately large dependent on $(m,\alpha,\mu,\nu,\chi,C_0,T_0,\|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ and, if $n=2$ and $\alpha=0$, on $g(\cdot)$ as well. This gives

$$\frac{d}{dt}\| D^m\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq 4\chi^2\mu^{-1}\| D^m\boldsymbol{\varepsilon}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \qquad \forall\; t > \zeta_m$$

from which (3.12) immediately follows, completing the proof of Theorem 3.3. □

Remark 3.2. By a (simpler) similar argument, if follows directly from (3.11a) that

$$Z_0(t) = \|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\chi^2\mu^{-1}\int_t^\infty \|\boldsymbol{\varepsilon}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \tag{3.13}$$

decreases monotonically in (t_*,∞), where t_* denotes the solution's regularity time. Although (3.13) will not be used, monotonicity properties such as (3.12) and (3.13) are very important, see the general discussion in [17].

Theorem 3.4*a*. *Assuming* (3.1), *then we have, for each* $m \geq 1$:

$$\| D^m\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq K_m(\alpha)\,C_0\,\mu^{-\frac{m}{2}}\,\|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\,t^{-\alpha-\frac{m}{2}} \qquad \forall\; t > T_m \tag{3.14a}$$

where $K_m(\alpha)$ depends only on (m,α), and T_m only on $(m,\alpha,\mu,\nu,\chi,C_0,T_0,\|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ and, if $n=2$ and $\alpha=0$, also on the function $g(\cdot)$ given in (3.2) *above.*

Proof. For $m = 1$, we obtain, from (3.11a),

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2}^2 + 2\mu\int_{t/2}^{t}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2}^2\,d\tau \;\leq\; \|\boldsymbol{u}(\cdot,t/2)\|_{L^2}^2 + 4\chi\int_{t/2}^{t}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2}\|\boldsymbol{\varepsilon}(\cdot,\tau)\|_{L^2}\,d\tau$$

for $t > 2t_*$, so that we have

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2}^2 + \mu\int_{t/2}^{t}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2}^2\,d\tau \;\leq\; \|\boldsymbol{u}(\cdot,t/2)\|_{L^2}^2 + 4\chi^2\mu^{-1}\int_{t/2}^{t}\|\boldsymbol{\varepsilon}(\cdot,\tau)\|_{L^2}^2\,d\tau$$

for such t. Recalling (3.12), this gives

$$\mu\,\frac{t}{2}\,\|D\boldsymbol{u}(\cdot,t)\|_{L^2}^2 \;\leq\; \mu\int_{t/2}^{t}Z_1(\tau)\,d\tau \;\leq\; \|\boldsymbol{u}(\cdot,t/2)\|_{L^2}^2 + 4\chi^2\mu^{-1}\int_{t/2}^{t}\big\{\,\|\boldsymbol{\varepsilon}(\cdot,\tau)\|_{L^2}^2 + E_1(\tau)\,\big\}\,d\tau$$

if $t > 2\,\zeta_1$, where $E_1(\tau) = \mu\int_\tau^\infty\|D\boldsymbol{\varepsilon}(\cdot,s)\|_{L^2(\mathbb{R}^n)}^2\,ds$. By (3.1) and (3.9), this shows (3.14) if $m = 1$. The general case $m > 1$ can be obtained by induction with a similar argument, observing that, from (3.11a), we have, using (2.2) above,

$$\mu\int_{t/2}^{t}\|D^m\boldsymbol{u}(\cdot,\tau)\|_{L^2}^2\,d\tau \;\leq\; \|D^{m-1}\boldsymbol{u}(\cdot,t/2)\|_{L^2}^2 + 8\chi^2\mu^{-1}\int_{t/2}^{t}\|D^{m-1}\boldsymbol{\varepsilon}(\cdot,\tau)\|_{L^2}^2\,d\tau$$

for t sufficiently large (dependent on $(m,\alpha,\mu,\nu,\chi,C_0,T_0,\|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ and, if $n = 2$ and $\alpha = 0$, also on $g(\cdot)$ given in (3.2)). Recalling (3.12) and increasing t (if necessary) so that we also have $t > 2\zeta_m$, then we get

$$\mu\,\frac{t}{2}\,\|D^m\boldsymbol{u}(\cdot,t)\|_{L^2}^2 \;\leq\; \|D^{m-1}\boldsymbol{u}(\cdot,t/2)\|_{L^2}^2 + 8\chi^2\mu^{-1}\int_{t/2}^{t}\big\{\,\|D^{m-1}\boldsymbol{\varepsilon}(\cdot,\tau)\|_{L^2}^2 + E_m(\tau)\,\big\}\,d\tau$$

where $E_m(\tau) = \mu\int_\tau^\infty\|D^m\boldsymbol{\varepsilon}(\cdot,s)\|_{L^2(\mathbb{R}^n)}^2\,ds$. Using (3.9), c.f. Theorem 3.2, and the induction hypothesis for $\|D^{m-1}\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}^2$, this shows the result for m, completing the proof. □

Theorem 3.4*b*. *Assuming* (3.1)*, then we have, for each* $m \geq 0$:

$$\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \;\leq\; K_{m+1}(\alpha)\,C_0\,\mu^{-\frac{m+1}{2}}\,\|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)}\,t^{-\alpha-\frac{m+1}{2}} \qquad \forall\; t > T'_m \qquad (3.14b)$$

with $K_{m+1}(\alpha)$ *given in* (3.14a)*, and* T'_m *dependent on* $(m,\alpha,\mu,\nu,\chi,C_0,T_0,\|\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ *and, if* $n = 2$ *and* $\alpha = 0$*, also on the function* $g(\cdot)$ *given in* (3.2)*, and nothing else.*

Proof. Recalling (3.7), we have

$$\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \;\leq\; \frac{1}{2}\,\|D^{m+1}\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} + \|D^m\boldsymbol{\varepsilon}(\cdot,t)\|_{L^2(\mathbb{R}^n)}$$

for all $t > t_*$ (the solution's regularity time), so that (3.14b) is an immediate consequence of the estimates (3.9) and (3.14a) obtained in Theorems 3.2 and 3.4a above. □

Theorem 3.5. *Assuming* (3.1), *then we have, for each* $m \geq 0$:

$$\| D^m \boldsymbol{\varepsilon}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \leq K'_m(\alpha)\, C_0\, \frac{|\,\mu-\nu\,|}{\chi}\, \mu^{-\frac{m+3}{2}} \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)} t^{-\alpha - \frac{m+3}{2}} + K''_m(\alpha)\, C_0^2\, \chi^{-1} \mu^{-\frac{m+3}{2} - p_n} \| \boldsymbol{z}_0 \|^2_{L^2(\mathbb{R}^n)} t^{-2\alpha - \frac{m+3}{2} - p_n} \tag{3.15}$$

for all $t > T''_m$, *with* $T''_m > t_*$ *depending only on* $(m, \alpha, \mu, \nu, \chi, C_0, T_0, \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)})$ *and, if* $n = 2$ *and* $\alpha = 0$, *also on the function* $g(\cdot)$ *considered in* (3.2). *In addition, the constants* $K'_m(\alpha)$, $K''_m(\alpha)$ *depend on* (m, α) *only, and* $p_n = (n-2)/4$.

Proof. Let $t_* \geq 0$ be the solution's regularity time. From (3.8) we obtain

$$\frac{d}{dt} \| D^m \boldsymbol{\varepsilon}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} + 2\,\mu \, \| D^{m+1} \boldsymbol{\varepsilon}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} + 8\,\chi \, \| D^m \boldsymbol{\varepsilon}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)}$$
$$\leq \| D^m \boldsymbol{\varepsilon}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \Big\{ 2\,|\,\mu-\nu\,|\, \| D^{m+2} \mathbf{w}(\cdot,t) \|_{L^2(\mathbb{R}^n)} + A_m F_m(t) + B_m G_m(t) \Big\}$$

for $t > t_*$, where $F_m(t) = \sum_{\ell=0}^{m} \| D^{\ell+1} \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} \| D^{m-\ell+1} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$, the values A_m, B_m depend on m only, and $G_m(t) = \sum_{\ell=0}^{m} \| D^{\ell} \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} \| D^{m-\ell+1} \boldsymbol{\varepsilon}(\cdot,t) \|_{L^2(\mathbb{R}^n)}$. This gives

$$\frac{d}{dt} \| D^m \boldsymbol{\varepsilon}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} + 4\,\chi \, \| D^m \boldsymbol{\varepsilon}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} \leq \frac{1}{2}\, \chi^{-1} |\,\mu-\nu\,|^2 \, \| D^{m+2} \mathbf{w}(\cdot,t) \|^2_{L^2(\mathbb{R}^n)} + \frac{1}{4}\, \chi^{-1} A_m^2 F_m(t)^2 + \frac{1}{4}\, \chi^{-1} B_m^2 G_m(t)^2$$

from which (3.15) follows, in view of the estimates (3.9), (3.14*a*) and (3.14*b*) above. □

An immediate consequence is the fast decay of $\nabla \cdot \mathbf{w}(\cdot,t)$, i.e., the divergence of the micro-rotational angular velocity, in comparison with $\nabla \wedge \mathbf{w}(\cdot,t)$:

Theorem 3.6. *Assuming* (3.1), *then we have, for each* $m \geq 0$:

$$\| D^m [\, \nabla \cdot \mathbf{w}\,](\cdot,t) \|_{L^2(\mathbb{R}^n)} \leq K'_{m+1}(\alpha)\, C_0\, \frac{|\,\mu-\nu\,|}{\chi}\, \mu^{-\frac{m+4}{2}} \| \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)} t^{-\alpha - \frac{m+4}{2}} + K''_{m+1}(\alpha)\, C_0^2\, \chi^{-1} \mu^{-\frac{m+4}{2} - p_n} \| \boldsymbol{z}_0 \|^2_{L^2(\mathbb{R}^n)} t^{-2\alpha - \frac{m+4}{2} - p_n} \tag{3.16}$$

for all $t > T''_{m+1}$, *where* $K'_{m+1}(\alpha)$, $K''_{m+1}(\alpha)$ *and* T''_{m+1} *are given in* Theorem 3.5 *above, and* $p_n = (n-2)/4$, *as before.*

Proof. This follows directly from (3.15), as, by (3.7), we have $\nabla \cdot \mathbf{w}(\cdot,t) = \nabla \cdot \boldsymbol{\varepsilon}(\cdot,t)$. □

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4. Proof of Theorems C and D

In this section we consider a solution $\boldsymbol{z} = (\boldsymbol{u}, \mathbf{w})$ to the equations (1.1) or (1.6), with initial state $\boldsymbol{z}_0 = (\boldsymbol{u}_0, \mathbf{w}_0) \in L^2_\sigma(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, which, in addition to satisfying the upper bound (3.1) for some $\alpha > 0$, it also admits a lower bound of the form $t^{-\eta}$ for some $\eta \geq \alpha$. Beginning with the case $\eta = \alpha$, we therefore assume that

$$\|\, \boldsymbol{u}(\cdot,t) \,\|_{L^2(\mathbb{R}^n)} \geq c_0 \,\|\, \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)} t^{-\alpha} \qquad \forall\; t > t_0 \tag{4.1}$$

for some $c_0 > 0$, $t_0 \geq 0$, $\alpha > 0$. Note that, by Theorem 3.4*b*, the condition (4.1) is equivalent to having a lower bound $t^{-\alpha}$ of similar type for the whole solution $\boldsymbol{z}(\cdot,t)$.

Theorem 4.1. *If* (3.1) *and* (4.1) *hold for some $\alpha > 0$, then we have, for each $m \geq 1$:*

$$\|\, D^m \boldsymbol{u}(\cdot,t) \,\|_{L^2(\mathbb{R}^n)} \geq \Gamma_m(\alpha, c_0, C_0)\, c_0\, \mu^{-\frac{m}{2}} \,\|\, \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)} t^{-\alpha - \frac{m}{2}} \qquad \forall\; t > t_m \tag{4.2}$$

where $\Gamma_m(\alpha, c_0, C_0) > 0$ depends solely on (m, α, c_0, C_0), and t_m depends solely on $(m, \alpha, \mu, \nu, \chi, c_0, C_0, t_0, T_0, \|\, \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)})$.

Proof. Beginning with $m = 1$, we have, from the equation (3.11*a*),

$$\|\, \boldsymbol{u}(\cdot,T) \,\|^2_{L^2(\mathbb{R}^n)} + 2\mu \int_t^T \|\, D\boldsymbol{u}(\cdot,\tau) \,\|^2_{L^2(\mathbb{R}^n)} d\tau \;=\; \|\, \boldsymbol{u}(\cdot,t) \,\|^2_{L^2(\mathbb{R}^n)} + 4\chi \int_t^T \!\! \int_{\mathbb{R}^n} \langle\, \nabla \wedge \boldsymbol{u}, \boldsymbol{\varepsilon} \,\rangle \, dx\, d\tau$$

for all $T > t > \max\{ t_0, T_0, t_* \}$ (t_* being the solution's regularity time), so that we get

$$\begin{aligned} 4\mu \int_t^T \|\, D\boldsymbol{u}(\cdot,\tau) \,\|^2_{L^2(\mathbb{R}^n)} d\tau &\geq \|\, \boldsymbol{u}(\cdot,t) \,\|^2_{L^2(\mathbb{R}^n)} - \|\, \boldsymbol{u}(\cdot,T) \,\|^2_{L^2(\mathbb{R}^n)} - 2\chi^2 \mu^{-1} \int_t^T \|\, \boldsymbol{\varepsilon}(\cdot,\tau) \,\|^2_{L^2(\mathbb{R}^n)} d\tau \\ &\geq c_0^2 \,\|\, \boldsymbol{z}_0 \|^2_{L^2(\mathbb{R}^n)} \big(t^{-2\alpha} - \lambda_1^2\, T^{-2\alpha} \big) - 2\chi^2 \mu^{-1} \int_t^T \|\, \boldsymbol{\varepsilon}(\cdot,\tau) \,\|^2_{L^2(\mathbb{R}^n)} d\tau \end{aligned} \tag{4.3a}$$

by (3.1) and (4.1), where $\lambda_1 = C_0 / c_0 \geq 1$. This gives, choosing $T = (2\lambda_1)^{1/\alpha} t$,

$$4\mu\, (2\lambda_1)^{1/\alpha}\, t\, Z_1(t) \;\geq\; \frac{1}{2}\, c_0^2 \,\|\, \boldsymbol{z}_0 \|^2_{L^2(\mathbb{R}^n)} t^{-2\alpha} - 2\chi^2 \mu^{-1} \int_t^\infty \|\, \boldsymbol{\varepsilon}(\cdot,\tau) \,\|^2_{L^2(\mathbb{R}^n)} d\tau$$

by (3.12), assuming that $t > \zeta_1$ also, cf. Theorem 3.3, so that we have

$$4\mu\, (2\lambda_1)^{1/\alpha}\, t \,\|\, D\boldsymbol{u}(\cdot,t) \,\|^2_{L^2(\mathbb{R}^n)} \;\geq\; \frac{1}{2}\, c_0^2 \,\|\, \boldsymbol{z}_0 \|^2_{L^2(\mathbb{R}^n)} t^{-2\alpha} - 2\chi^2 \mu^{-1} \int_t^\infty \|\, \boldsymbol{\varepsilon}(\cdot,\tau) \,\|^2_{L^2(\mathbb{R}^n)} d\tau - F_1(t)$$

where $F_1(t) = 16\, (2\lambda_1)^{1/\alpha} \chi^2\, t \int_t^\infty \|\, D\boldsymbol{\varepsilon}(\cdot,\tau) \,\|^2_{L^2(\mathbb{R}^n)} d\tau$. By (3.9), this shows (4.2) if $m = 1$.

For the general case $m > 1$ we proceed by induction. From (3.11a) we get, using (2.2),

$$\begin{aligned} 4\,\mu \int_t^T \|\, D^m \boldsymbol{u}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^n)}^2 d\tau \;\geq\; \|\, D^{m-1}\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}^2 - \;\|\, D^{m-1}\boldsymbol{u}(\cdot,T)\,\|_{L^2(\mathbb{R}^n)}^2 \\ -\; 4\,\chi^2 \mu^{-1} \int_t^T \|\, D^{m-1}\boldsymbol{\varepsilon}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^n)}^2 d\tau \end{aligned} \tag{4.3b}$$

for all $T > t$ with $t > t_*$ large (dependent on $(m, \alpha, \mu, \nu, \chi, C_0, T_0, \|\, \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)})$). Assuming also $t > \zeta_m$ (ζ_m given in Theorem 3.3), $t > t_{m-1}$ (found in the previous step, cf. (4.2)) and $t > T_{m-1}$ (given in (3.14a), Theorem 3.4a), we then obtain, recalling (3.12),

$$\begin{aligned} 4\,\mu\, T\, \|\, D^m \boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}^2 \;\geq\; c_{m-1}^2 \,\|\, \boldsymbol{z}_0 \,\|_{L^2(\mathbb{R}^n)}^2 \,\mu^{-m+1}\big(t^{-2\alpha-m+1} - \lambda_m^2 T^{-2\alpha-m+1}\big) \;+ \\ -\; 4\,\chi^2\mu^{-1}\int_t^\infty \|D^{m-1}\boldsymbol{\varepsilon}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^n)}^2 d\tau \;-\; 16\,\chi^2\, T \int_t^\infty \|\, D^m \boldsymbol{\varepsilon}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^n)}^2 d\tau \end{aligned}$$

where $c_{m-1} = \Gamma_{m-1}(\alpha, c_0, C_0)\, c_0$ is the coefficient in (4.2) obtained in the previous step (i.e., step $m-1$) and $\lambda_m = C_{m-1}/c_{m-1} \geq 1$, where $C_{m-1} = K_{m-1}(\alpha)\, C_0$, with $K_{m-1}(\alpha)$ given in (3.14a), Theorem 3.4a. Choosing $T = (2\lambda_m)^{1/\alpha}\, t$ gives the result (4.2) for m too, in view of the estimates (3.9) of Theorem 3.2, and the proof is now complete. □

Remark 4.1. If the conditions (3.1) and (4.1) are valid for some $\alpha > 0$, it follows from (3.9), (3.11a), (3.14a) and (4.2) that, for every $m \geq 0$, there will exist $\zeta_m' > t_*$ sufficiently large (dependent on $(m, \alpha, \mu, \nu, \chi, c_0, C_0, t_0, T_0, \|\, \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)})$ only) so that $\|\, D^m \boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$ is monotonically decreasing on (ζ_m', ∞), thus improving (3.12) and (3.13) in this case (cf. Theorem 3.3 and Remark 3.2).

Remark 4.2. In addition, when (3.1) and (4.1) hold for some $\alpha > 0$, it follows from (3.7), (3.9) and (4.2) above that, for each $m \geq 0$, we have

$$\|\, D^m \mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \geq \frac{1}{4}\,\Gamma_{m+1}(\alpha, c_0, C_0)\, c_0\, \mu^{-\frac{m+1}{2}}\, \|\, \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)}\, t^{-\alpha - \frac{m+1}{2}} \tag{4.4}$$

for all $t > t_m'$, for some t_m' depending only on $(m, \alpha, \mu, \nu, \chi, c_0, C_0, t_0, T_0, \|\, \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)})$, where the coefficient $\Gamma_{m+1}(\alpha, c_0, C_0)$ is given in (4.2). Moreover, by (3.11b), (3.14), (3.15) and (4.4), we obtain when $\mu = \nu$ that, for every $m \geq 0$, there exists $\zeta_m'' > t_*$ sufficiently large (dependent on $(m, \alpha, \mu, \chi, c_0, C_0, t_0, T_0, \|\, \boldsymbol{z}_0 \|_{L^2(\mathbb{R}^n)})$ only) such that $\|\, D^m \mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$ is monotonic on (ζ_m'', ∞). Thus, if $\mu = \nu$ and (3.1), (4.1) are valid for some $\alpha > 0$, then $\|\, D^m \mathbf{w}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$ also becomes monotonic at large times, for every $m \geq 0$, and not just $\|\, D^m \boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$. This complements Remark 4.1 above.

From Theorem 4.1 we see in particular that, for solutions $\boldsymbol{z} = (\boldsymbol{u}, \mathbf{w})$ satisfying the conditions (3.1) and (4.1) for some $\alpha > 0$, the term $\frac{1}{2}\nabla\wedge\boldsymbol{u}(\cdot,t)$ in the decomposition (3.7) decays slower (as $t \to \infty$) in $\dot{H}^m(\mathbb{R}^n)$, for every m, than the term $\boldsymbol{\varepsilon}(\cdot,t)$, thereby eventually dictating the main behavior of the micro-rotation of fluid particles. This will also be the case under the more general condition

$$\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \geq c_0\,\|\,\boldsymbol{z}_0\,\|_{L^2(\mathbb{R}^n)} t^{-\eta} \qquad \forall\; t > t_0 \tag{4.5}$$

for $\eta \geq \alpha$ not too much larger than α, as shown in Theorem 4.2 below.

Theorem 4.2. *If* (3.1) *and* (4.5) *hold for some* $\alpha > 0$, $\alpha \leq \eta < \alpha\,(\alpha{+}3/2)/(\alpha{+}1/2)$, *then*

$$\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \geq c_1\,\mu^{-\frac{1}{2}}\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)} t^{-\eta-\frac{1}{2}q} \qquad \forall\; t > t_1' \tag{4.6a}$$

for some $t_1' \gg 1$*, which depends on* $(\alpha, \eta, \mu, \nu, \chi, c_0, C_0, t_0, T_0, \|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ *only, where* $q = \eta/\alpha$ *and* $c_1 = c_1(\alpha, c_0, C_0) > 0$ *depends solely on* (α, c_0, C_0). *More generally: if* $\alpha \leq \eta < \alpha\,(\alpha + m/2 + 1)/(\alpha + m/2)$ *for some* $m \geq 1$*, then, for every* $1 \leq \ell \leq m$:

$$\|\,D^\ell\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \geq c_\ell\,\mu^{-\frac{\ell}{2}}\,\|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)} t^{-\eta-\frac{\ell}{2}q} \qquad \forall\; t > t_\ell' \tag{4.6b}$$

for some $t_\ell' \gg 1$ *that is dependent on* $(\ell, \alpha, \eta, \mu, \nu, \chi, c_0, C_0, t_0, T_0, \|\,\boldsymbol{z}_0\|_{L^2(\mathbb{R}^n)})$ *solely, with the coefficient* $c_\ell > 0$ *depending on* (ℓ, α, c_0, C_0) *only.*

Proof. Considering (4.6a) first, we have from (3.1), (4.3a) and (4.5) that

$$4\,\mu\!\int_t^T\!\|\,D\boldsymbol{u}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^n)}^2 d\tau \geq c_0^2\,\|\,\boldsymbol{z}_0\,\|_{L^2(\mathbb{R}^n)}^2\big(t^{-2\eta} - \lambda_1^2\,T^{-2\alpha}\big) - 2\,\chi^2\mu^{-1}\!\int_t^T\!\|\,\boldsymbol{\varepsilon}(\cdot,\tau)\,\|_{L^2(\mathbb{R}^n)}^2 d\tau$$

for all $T > t > \max\{t_0, T_0, t_*\}$, where $\lambda_1 = C_0/c_0$. Taking $T = (2\lambda_1)^{1/\alpha}\,t^q$ and using (3.9) and (3.12), as in the proof of Theorem 4.1, we get (4.6a) if $\eta < \alpha\,(\alpha + 3/2)/(\alpha + 1/2)$. Finally, (4.6$b$) follows by induction, using (3.9), (3.12) and (4.3b) in a similar way. ☐

Remark 4.3. When $\alpha \leq \eta < \alpha\,(\alpha + m/2 + 1)/(\alpha + m/2)$ for some $m \geq 1$, it follows from (3.9) and (4.6b) that $\|\,D^\ell\boldsymbol{\varepsilon}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$ decays faster than $\|\,D^\ell\,[\nabla\wedge\boldsymbol{u}](\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$ for every $0 \leq \ell < m$, showing the dominant influence of vorticity on micro-rotation, cf. (3.7). In the special case $\mu = \nu$, by repeating the proof of Theorem 4.2 but using (3.15) instead of (3.9), the estimate (4.6b) can be obtained for $\eta \geq \alpha > 0$ satisfying

$$\frac{\eta}{\alpha} < \frac{2\alpha + m/2 + 1 + (n-2)/4}{\alpha + m/2}$$

and again $\|\,D^\ell\boldsymbol{\varepsilon}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$ will decay faster than $\|\,D^\ell\,[\nabla\wedge\boldsymbol{u}](\cdot,t)\,\|_{L^2(\mathbb{R}^n)}$, $0 \leq \ell < m$.

Appendix

In this appendix we show the existence of solutions $\boldsymbol{z} = (\boldsymbol{u}, \mathbf{w})$ to the micropolar equations (1.1) or (1.6) satisfying the condition

$$c_0\,(1+t)^{-\alpha} \;\leq\; \|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq C_0\,(1+t)^{-\alpha} \tag{A.1}$$

for all $t > 0$, and some constants $0 < c_0 < C_0$ (which depend on the solution and various parameters) when $0 < \alpha < 1/2$. This is an easy consequence of Theorem B, with the help of some well known results in the literature [1, 3, 22, 26].

From Schonbek's theory of decay characters [3, 22] we can find $\mathbf{v}_0 \in L^2_\sigma(\mathbb{R}^n)$ with $\mathbf{v}_0 \in H^m(\mathbb{R}^n)$ for all m and such that $\mathbf{v}(\cdot,t) := e^{\mu\Delta t}\mathbf{v}_0$ satisfies, for all $t > 0$,

$$c_0'\,(1+t)^{-\alpha} \;\leq\; \|\,\mathbf{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq C_0'\,(1+t)^{-\alpha} \tag{A.2}$$

(for some constants $0 < c_0' < C_0'$), where $e^{\mu\Delta t}$ denotes the heat semigroup, that is, convolution with the heat kernel. Let then $\boldsymbol{z}(\cdot,t) = (\boldsymbol{u}, \mathbf{w})(\cdot,t)$ be the (unique, global, smooth) solution of the equations (1.1) or (1.6) with initial data $\boldsymbol{z}_0 = (\mathbf{v}_0, \mathbf{0})$. (In dimension $n = 3$, this may require replacing $\mathbf{v}_0$ by a smaller multiple $\lambda\mathbf{v}_0$, with $\lambda > 0$ chosen so that $\lambda\,\|\,\mathbf{v}_0\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,D\mathbf{v}_0\|^{1/2}_{L^2(\mathbb{R}^3)} \leq 3\cdot\min\{\mu,\nu\}$, cf. Section 1.) Given this solution $\boldsymbol{z}(\cdot,t)$, let $\boldsymbol{\varepsilon}(\cdot,t)$ be defined by (1.7), so that we have, by Theorem B,

$$\|\,\boldsymbol{\varepsilon}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq\; E_0\,(1+t)^{-\frac{3}{2}}, \qquad \|\,D\boldsymbol{\varepsilon}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq\; E_1\,(1+t)^{-2} \tag{A.3}$$

for all $t > 0$, and some constants E_0, E_1 (independent of t). Recalling the equation (3.11a), we then consider the solution $\boldsymbol{v}(\cdot,t) \in C([0,\infty), L^2_\sigma(\mathbb{R}^n))$ of the problem

$$\boldsymbol{v}_t \;=\; \mu\Delta\boldsymbol{v}(\cdot,t) \;+\; 2\,\chi\,\nabla\wedge\boldsymbol{\varepsilon}(\cdot,t), \qquad \boldsymbol{v}(\cdot,0) = \mathbf{v}_0. \tag{A.4}$$

By (A.3) and standard heat kernel estimates, we get $\|\,\boldsymbol{v}(\cdot,t) - \mathbf{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} = O(t^{-\frac{1}{2}})$. Since $\alpha < 1/2$, we then obtain from (A.2) that

$$c_0''\,(1+t)^{-\alpha} \;\leq\; \|\,\boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq C_0''\,(1+t)^{-\alpha} \tag{A.5}$$

for all $t > 0$, and some constants $0 < c_0'' < C_0''$. By (3.11a) and Wiegner's theorem ([26], p. 305), we have, for all $t > 0$,

$$\|\,\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq K\,(1+t)^{-\,2\alpha\,-\,(n\,-\,2)/4} \tag{A.6}$$

for some constant $K > 0$. Together with (A.5), this gives (A.1), as claimed. □

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