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Upper and lower $\dot{H}^m$ estimates for solutions to dissipative systems

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Dedicated in loving memory of Bella (2003-2022) *and Sophie* (2007-2018)

Abstract

In this work we introduce a novel approach to generate lower and upper L^2 estimates for solution derivatives of arbitrary order to a general class of dissipative systems in the case that such estimates are available for the solutions themselves. Our method also works in reverse order: from the L^2 estimates of solution derivatives of some (arbitrary) order we can derive lower and upper L^2 estimates for the solutions and then to their derivatives of any order. This procedure is based on very simple monotonicity properties combined with standard energy estimates in physical space, following previous ideas of Kreiss, Hagstrom, Lorenz and Zingano. For simplicity, it is applied here in the context of algebraic rates, but the method can be used in other contexts as well (exponential, logarithm, and so forth).

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Summary and Conclusions

Section 0. Upper and lower estimates for dissipative systems
Monotonicity properties of $\dot{H}^m$ norms can be used to very easily produce new upper and lower bounds for solutions of dissipative systems out of previous estimates.

Example 1. Heat equation
Illustration of the method begins for simplicity with the familiar linear heat equation.

Example 2. Advection-diffusion equations $(n = 1)$
Advection-diffusion equations on $\mathbb{R}$ are not so easy due to the slow solution decay.

Example 3. Advection-diffusion equations in higher dimensions
Advection-diffusion equations in $\mathbb{R}^n$ $(n \geq 2)$ are easier due to the faster solution decay.

Example 4. Incompressible Navier-Stokes equations $(2 \leq n \leq 4)$
Old and new results are obtained for Leray solutions of the Navier-Stokes equations.

Example 5. Incompressible MHD equations $(2 \leq n \leq 4)$
Old and new results are obtained for Leray solutions of the MHD equations in $\mathbb{R}^n$.

Example 6. Incompressible micropolar flows $(n = 2, 3)$
Old and new results are obtained (or announced) for micropolar fluid flows in $\mathbb{R}^n$.

Example 7. Inverse Wiegner's theorem for the Navier-Stokes equations
A simple proof of Wiegner's inverse theorem for the NS equations in $\mathbb{R}^n$ $(n \geq 2)$.

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Upper and lower $\dot{H}^m$ estimates for solutions of parabolic equations

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Consider the equation

$$\boldsymbol{u}_t + \mathbb{G}(\boldsymbol{u}) = \nu\,\Delta \boldsymbol{u} + \boldsymbol{f}, \qquad x \in \mathbb{R}^n,\; t > 0 \tag{0.1}$$

(where $\nu > 0$ is constant, and $\boldsymbol{u} = \boldsymbol{u}(x,t)$, $\boldsymbol{u} = (u_1, u_2, ..., u_N)$, $\boldsymbol{f} = (f_1, f_2, ..., f_N)$, etc), with some global weak solution $\boldsymbol{u}(\cdot,t) \in C_{\text{w}}([0,\infty), L^2(\mathbb{R}^n))$ which, together with $\boldsymbol{f}$, becomes eventually smooth:

$$\boldsymbol{f}, \boldsymbol{u} \in C^\infty(\mathbb{R}^n \times (t_*, \infty)), \tag{0.2a}$$

$$\boldsymbol{f}(\cdot,t), \boldsymbol{u}(\cdot,t) \in C^0((t_*,\infty), H^m(\mathbb{R}^n)^N), \qquad \forall\; m \geq 0, \tag{0.2b}$$

for some $t_* > 0$. Let $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^n)}$. Assume that we have

$$\|\,\boldsymbol{u}(\cdot,t)\,\| \leq C_0\, t^{-\alpha} \qquad \forall\; t > T_0 \tag{H1}$$

for some constants $C_0, T_0, \alpha > 0$. We also assume that we have, for some $\hat{m} \geq 1$:

$$\left| \sum_{\ell_1,\dots,\ell_m} \int_{\mathbb{R}^n} \langle D_{\ell_1}\!\cdots D_{\ell_m}\boldsymbol{u}(x,t), D_{\ell_1}\!\cdots D_{\ell_m}\mathbb{G}(\boldsymbol{u}(x,t))\,\rangle\, dx \right| \leq g_m(t)\,\|\, D^{m+1}\boldsymbol{u}(\cdot,t)\,\|^2 \qquad \forall\; t > \tau_m \tag{H2}$$

for some $\tau_m > t_*$, for every $0 \leq m \leq \hat{m}$, where the sum is over all indices $1 \leq \ell_1, ..., \ell_m \leq n$ (no sum implied if $m = 0$), where $g_m(t) \to 0$ as $t \to \infty$, and where $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{R}^N$. Finally, when $\boldsymbol{f}$ is present (i.e., $\boldsymbol{f} \not\equiv \boldsymbol{0}$), we must additionally assume that

$$\|\, D^m \boldsymbol{f}(\cdot,t)\,\| \leq F_m\, t^{-\beta - m/2} \qquad \forall\; t > \sigma_m \tag{H3}$$

for some $\beta > 0$ (specified in the results below) and some $F_m, \sigma_m > 0$, for each $0 \leq m \leq \hat{m}$. For the main results (Theorem B and Theorem C), it will be also necessary to assume

$$\|\,\boldsymbol{u}(\cdot,t)\,\| \geq c(0)\, t^{-\eta} \qquad \forall\; t > t_0 \tag{H4}$$

for some positive constants $c(0), t_0, \eta$ given (with η satisfying, by (H1) above: $\eta \geq \alpha$).

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Notation. $\| \cdot \|$ denotes L^2 norm, so that

$$\| \boldsymbol{u}(t) \|^2 \equiv \| \boldsymbol{u}(\cdot, t) \|^2 = \sum_{i=1}^{N} \int_{\mathbb{R}^n} | u_i(x,t) |^2 \, dx. \tag{0.3a}$$

Similarly,

$$\| D\boldsymbol{u}(t) \|^2 \equiv \| D\boldsymbol{u}(\cdot, t) \|^2 = \sum_{i=1}^{N} \sum_{j=1}^{n} \int_{\mathbb{R}^n} | D_j u_i(x,t) |^2 \, dx, \tag{0.3b}$$

$$\| D^2\boldsymbol{u}(t) \|^2 \equiv \| D^2\boldsymbol{u}(\cdot, t) \|^2 = \sum_{i=1}^{N} \sum_{j=1}^{n} \sum_{\ell=1}^{n} \int_{\mathbb{R}^n} | D_j D_\ell u_i(x,t) |^2 \, dx, \tag{0.3c}$$

and so forth, where $D_j = \partial / \partial x_j$, $D_j D_\ell = \partial^2 / \partial x_j \partial x_\ell$, etc.

Here is a quick overview of the basic properties shown in the text (the main results being Theorem B and Theorem C):

Theorem A (*upper estimates for derivatives*).
Assume (H1), (H2) and (H3) above, with $\beta \geq \alpha + 1$. Then we have, for every $1 \leq m \leq \hat{m}$:

$$\| D^m \boldsymbol{u}(\cdot, t) \| \leq C_m \, \nu^{-m/2} \, t^{-\alpha - m/2} \qquad \forall \; t > T_m \tag{0.4}$$

for some constants $C_m > 0$, $T_m > t_*$. Moreover, C_m can be chosen to depend *only* on m, α and C_0 if $\beta > \alpha + 1$, and on m, α, ν, C_0 and $\{ F_\ell \colon 0 \leq \ell < m \}$ if $\beta = \alpha + 1$; T_m depends on all these and $\beta, F_m, \{ \tau_\ell, \sigma_\ell \colon 0 \leq \ell \leq m \}$, as well as on the functions g_ℓ, $0 \leq \ell \leq m$.

Theorem B (*lower estimates for derivatives*: *the case* $\eta = \alpha$).
Assume (H1), (H2), (H3) and (H4), with $\eta = \alpha$ and $\beta > \alpha + 1$. Then, for every $1 \leq m \leq \hat{m}$:

$$\| D^m \boldsymbol{u}(\cdot, t) \| \geq c(m) \, \nu^{-m/2} \, t^{-\alpha - m/2} \qquad \forall \; t > t_m \tag{0.5}$$

for some constants $c(m) > 0$, $t_m > t_*$. Moreover, $c(m)$ can be chosen to depend *only* on m, α, $c(0)$ and C_0, while t_m depends on $m, \alpha, \beta, \nu, c(0), C_0, t_*, t_0, T_0, \{ F_\ell, \tau_\ell, \sigma_\ell \colon 0 \leq \ell \leq m \}$ and the functions g_ℓ, $0 \leq \ell \leq m$.

Remark 0.1: Theorem A is basically known (see [4, 9, 15, 22, 23]), although not generally stated in the present form. It is included for completeness, because of its role in the derivation of the main results (Theorems B and C) and also for the similarities in their proofs. In fact, Theorem A can be shown in the same way as Theorem B if $\boldsymbol{f} = \boldsymbol{0}$ or if Lemma 0.1 is available. As this is not the case, we followed an alternative route (adapted from [9]).

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Remark 0.2: THEOREM A is also valid in the case $\alpha = 0$, as it will become clear from its derivation. Other generalizations are clearly possible: for example, if instead of (H1) it is assumed that $\| \boldsymbol{u}(t) \| = o(t^{-\alpha})$ as $t \to \infty$ (for some $\alpha \geq 0$), then repeating the proof below it will be obtained that, as $t \to \infty$: $\| D^m \boldsymbol{u}(\cdot, t) \| = o(t^{-\alpha - m/2})$ for every $1 \leq m \leq \hat{m}$.

Remark 0.3: THEOREM B has the following generalization, which seems particularly useful in the case $\boldsymbol{f} = \boldsymbol{0}$ or if $\hat{m}$ is not too large (as, for example: $\hat{m} = 1$ or $\hat{m} = 2$):

Theorem C (*lower estimates for derivatives*: *the case* $\eta > \alpha$).
Assume (H1), (H2), (H3) and (H4), with $\eta > \alpha$ and β given in (H3) satisfying

$$\beta \,>\, 2\eta \,-\, \alpha \,+\, \Big(\frac{\eta}{\alpha} \,-\, 1\Big)\,\hat{m} \,+\, 1. \tag{0.6a}$$

Then, setting $q = \eta/\alpha$, we have, for every $1 \leq m \leq \hat{m}$:

$$\| D^m \boldsymbol{u}(\cdot, t) \| \,\geq\, c(m)\, \nu^{-m/2}\, t^{-\eta - m q/2} \qquad \forall\; t > t_m \tag{0.6b}$$

for some $c(m) > 0$, $t_m > t_*$. Moreover, $c(m)$ can be chosen to depend *only* on m, α, $c(0)$ and C_0, while t_m depends on $m, \alpha, \eta, \beta, \nu, c(0), C_0, t_*, t_0, T_0, \{ F_\ell, \tau_\ell, \sigma_\ell \colon 0 \leq \ell \leq m \}$ and the functions g_ℓ, $0 \leq \ell \leq m$.

Remark 0.4: There are three basic ingredients in the proof of THEOREMS B and C: (*i*) use of energy estimates for $\| D^{m-1} \boldsymbol{u}(\cdot, t) \|$, (*ii*) availability of lower and upper estimates for $\| D^{m-1} \boldsymbol{u}(\cdot, t) \|$, and (*iii*) monotonicity results for $\| D^m \boldsymbol{u}(\cdot, t) \|$ (LEMMA 0.1).

Remark 0.5: The estimates (0.4), (0.5) above show that we gain an extra factor $(\nu t)^{-1/2}$ each time the derivative order is increased by one unit. In THEOREM A and THEOREM B the starting point was an initial estimate for $\| \boldsymbol{u}(\cdot, t) \|$, given in (H1) or (H4), but a similar result would have been obtained if we had begun with some higher derivative instead. For example, knowing that $\| D^k \boldsymbol{u}(\cdot, t) \| \leq C_k\, t^{-\alpha}$ if $t > T_k$, say, for some $k \geq 1$, it would have followed that $\| D^m \boldsymbol{u}(\cdot, t) \| \leq C_m\, \nu^{-(m-k)/2}\, t^{-\alpha - (m-k)/2}$ for every $k \leq m \leq \hat{m}$, $t \gg 1$. Under appropriate conditions, we can also go *backwards*, as the following results illustrate.

Theorem D (*upper estimates from higher derivatives*).
Let (H2) be valid with $m = 0$, and assume that the solution of (0.1) considered satisfies $\lim_{t \to \infty} \| \boldsymbol{u}(\cdot, t) \| = 0$ and

$$\| D \boldsymbol{u}(\cdot, t) \| \,\leq\, C_1 \nu^{-1/2}\, t^{-\alpha_1} \qquad \forall\; t > T_1 \tag{0.7a}$$

for some $C_1, \alpha_1, T_1 > 0$. Then we have:

(i) If $\alpha_1 > 1/2$ and (H3) holds with $m = 0$ for some $\beta > \alpha_1 + 1/2$, then

$$\| \boldsymbol{u}(\cdot, t) \| \;\leq\; C_0 \, t^{-\alpha_1 + 1/2} \qquad \forall \; t > T_0 \tag{0.7b}$$

where $C_0 > 0$ depends on $C_1, \alpha_1, \beta, \sigma_0, F_0$ and $\| \boldsymbol{u}(a_*) \|$ only, with a_* given in (0.9) below, and T_0 depends on $t_*, T_1, \alpha_1, \beta, \tau_0, \sigma_0, F_0, \nu, a_*$ and the function g_0.

(ii) If $\alpha_1 > 1/2$ and (H3) holds with $m = 0$ for $\beta = \alpha_1 + 1/2$, then, for every $\epsilon > 0$:

$$\| \boldsymbol{u}(\cdot, t) \| \;\leq\; C_0(\epsilon) \, t^{-\alpha_1 + 1/2 + \epsilon} \qquad \forall \; t > T_0 \tag{0.7c}$$

where $C_0(\epsilon) > 0$ depends on $\epsilon, C_1, \alpha_1, \beta, \sigma_0, F_0$ and $\| \boldsymbol{u}(a_*) \|$ only, with a_* given in (0.9), and T_0 depends on $t_*, T_1, \alpha_1, \beta, \tau_0, \sigma_0, F_0, \nu, a_*$ and the function g_0, but not on ϵ.

Theorem E (*lower estimates from higher derivatives*).
Let (H2) be valid with $m = 0$, and assume that the solution of (0.1) considered satisfies (0.7a) for some $\alpha_1 > 1/2$. Assume also that $\lim_{t \to \infty} \| \boldsymbol{u}(\cdot, t) \| = 0$ and

$$\| D\boldsymbol{u}(\cdot, t) \| \;\geq\; c(1) \, \nu^{-1/2} \, t^{-\alpha_1} \qquad \forall \; t > t_1 \tag{0.8a}$$

for some $c(1), t_1 > 0$. Then, if (H3) holds with $m = 0$ and some $\beta > \alpha_1 + 1/2$, we have

$$\| \boldsymbol{u}(\cdot, t) \| \;\geq\; c(0) \, t^{-\alpha_1 + 1/2} \qquad \forall \; t > t_0 \tag{0.8b}$$

with $c(0) = c(1)/(2\sqrt{\alpha_1 - 1/2}\,)$, where t_0 depends on $t_*, c(1), C_1, t_1, T_1, \tau_0, \sigma_0, F_0, \alpha_1, \beta$ $\nu, a_*, \| \boldsymbol{u}(a_*) \|$ and the function g_0 given in (H2), with a_* defined in (0.9) below.

Remark 0.6: Regarding the condition on $\| \boldsymbol{u}(\cdot, t) \|$ that appears in THEOREMS D and E, it will become clear from the derivation of (0.7b), (0.7c) and (0.8b) given later that only the fact that $\liminf_{t \to \infty} \| \boldsymbol{u}(t) \| = 0$ is actually needed (and used) there. However, using a Gronwall-type argument (see REMARK 0.11 below) it can be shown that, for some $a_* \gg 1$ (depending in general on t_*, τ_0, ν and the function g_0), we have

$$\| \boldsymbol{u}(t) \| \;\leq\; \| \boldsymbol{u}(a) \| \,+\, \int_a^t \| \boldsymbol{f}(\tau) \| \, d\tau \qquad \forall \; t > a \geq a_*. \tag{0.9}$$

Since $\beta > 1$, it follows that $\| \boldsymbol{f}(\cdot) \| \in L^1(a, \infty)$; this then gives, in view of (0.9) above, that $\limsup_{t \to \infty} \| \boldsymbol{u}(t) \| \leq \liminf_{t \to \infty} \| \boldsymbol{u}(t) \|$. That is, the limit $\lim_{t \to \infty} \| \boldsymbol{u}(t) \|$ does exist.

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Remark 0.7: THEOREMS A-E above have, of course, simpler statements when $\boldsymbol{f} = \mathbf{0}$, i.e., in the case of the equation

$$\boldsymbol{u}_t \,+\, \mathbb{G}(\boldsymbol{u}) \;=\; \nu\,\Delta\boldsymbol{u}, \qquad x \in \mathbb{R}^n, \;\; t > 0, \tag{0.10}$$

with solutions $\boldsymbol{u}(\cdot,t)$ satisfying (0.2). Here, as before, the term $\mathbb{G}(\boldsymbol{u})$ is assumed to satisfy the condition (H2), for some $\hat{m} \geq 1$. For convenience, the corresponding statements are reproduced below (THEOREMS A′- E′). Results for the solutions of (0.10) can be obtained, of course, as direct corollaries of the theorems above, but sometimes it will be more advantageous to derive them from the *proofs*. This was the case, for example, of THEOREM D′ ann THEOREM E′ (see below). In the latter, the condition that $\|\,\boldsymbol{u}(t)\,\| \to 0$ (as $t \to \infty$) was dropped because it is simply not needed when $\boldsymbol{f} = \mathbf{0}$. (The condition on $\|\,\boldsymbol{u}(t)\,\|$ is necessary in THEOREM E because the upper estimate (0.7b) is needed in the proof there.) The same goes when applying the results above to particular equations: it may be better sometimes to obtain the results from the *proofs* and not from the statements given above. For example, consider the situation of obtaining (0.7b) from the estimate (0.7a) when we have, say, $\beta > 2\,\alpha_1$. This stronger assumption eliminates the need to bootstrap on the estimate (0.19a), leading to a neater expression for the constant C_0 in this case (namely, $C_0 = 2\,C_1/\sqrt{\alpha_1 - 1/2}\,$). Additional examples are given in EXAMPLES 1-4 at the end of the text.

Theorem A′ (*upper estimates for derivatives*: $\boldsymbol{f} = \mathbf{0}$).
Let $\boldsymbol{u}(\cdot,t)$ be a solution to (0.10). If (H1) and (H2) are valid, then, for every $1 \leq m \leq \hat{m}$:

$$\|\,D^m\boldsymbol{u}(\cdot,t)\,\| \;\leq\; C_m\,\nu^{-m/2}\,t^{-\alpha\,-\,m/2} \qquad \forall\; t > T_m \tag{0.11}$$

for some constants $C_m > 0$, $T_m > t_*$. Moreover, C_m can be chosen to depend only on m, α, and C_0, while T_m depends on m, α, C_0 and also on ν, $\{\,\tau_\ell \colon\; 0 \leq \ell \leq m\,\}$ and the functions g_ℓ, $0 \leq \ell \leq m$, given in (H2).

Theorem B′ (*lower estimates for derivatives*: *the case* $\eta = \alpha$, $\boldsymbol{f} = \mathbf{0}$).
Let $\boldsymbol{u}(\cdot,t)$ be a solution to (0.10). Assuming (H1), (H2) and (H4), with $\eta = \alpha$, then we have, for every $1 \leq m \leq \hat{m}$:

$$\|\,D^m\boldsymbol{u}(\cdot,t)\,\| \;\geq\; c(m)\,\nu^{-m/2}\,t^{-\alpha\,-\,m/2} \qquad \forall\; t > t_m \tag{0.12}$$

for some constants $c(m) > 0$, $t_m > t_*$. Moreover, $c(m)$ can be chosen to depend only on m, α, $c(0)$ and C_0, while t_m depends on m, α, $c(0)$, C_0, t_*, t_0, T_0, ν, $\{\,\tau_\ell \colon\; 0 \leq \ell \leq m\,\}$ and the functions g_ℓ, $0 \leq \ell \leq m$.

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Theorem C′ (*lower estimates for derivatives*: *the case* $\eta > \alpha$, $\boldsymbol{f} = \mathbf{0}$).
Let $\boldsymbol{u}(\cdot, t)$ be a solution to (0.10). Assuming (H1), (H2) and (H4) with $\eta > \alpha$, and letting $q = \eta/\alpha$, we have, for every $1 \leq m \leq \hat{m}$:

$$\| D^m \boldsymbol{u}(\cdot, t) \| \geq c(m)\, \nu^{-m/2}\, t^{-\eta - m q/2} \qquad \forall\; t > t_m \tag{0.13}$$

for some constants $c(m) > 0$, $t_m > t_*$. Moreover, $c(m)$ can be chosen to depend only on m, α, $c(0)$ and C_0, while t_m depends on m, α, $c(0)$, C_0, t_*, t_0, T_0, ν, η, $\{\tau_\ell \colon 0 \leq \ell \leq m\}$ and the functions g_ℓ, $0 \leq \ell \leq m$.

Theorem D′ (*upper estimates from higher derivatives*: $\boldsymbol{f} = \mathbf{0}$).
Let $\boldsymbol{u}(\cdot, t)$ be a solution to (0.10). Assuming (H2) with $m = 0$, and that $\lim_{t \to \infty} \| \boldsymbol{u}(\cdot, t) \| = 0$ and

$$\| D\boldsymbol{u}(\cdot, t) \| \leq C_1 \nu^{-1/2}\, t^{-\alpha_1} \qquad \forall\; t > T_1 \tag{0.14a}$$

for some constants $C_1, T_1 > 0$ and $\alpha_1 > 1/2$, then

$$\| \boldsymbol{u}(\cdot, t) \| \leq C_0\, t^{-\alpha_1 + 1/2} \qquad \forall\; t > T_0 \tag{0.14b}$$

with $C_0 = \sqrt{2}\, C_1/(\alpha_1 - 1/2)^{1/2}$ and T_0 depending on $t_*, T_1, \tau_0, \sigma_0, \nu$ and the function g_0.

Theorem E′ (*lower estimates from higher derivatives*: $\boldsymbol{f} = \mathbf{0}$).
Let $\boldsymbol{u}(\cdot, t)$ be a solution to (0.10). Let (H2) with $m = 0$ and (0.14a) above be both valid, where $\alpha_1 > 1/2$. If

$$\| D\boldsymbol{u}(\cdot, t) \| \geq c(1)\, \nu^{-1/2}\, t^{-\alpha_1} \qquad \forall\; t > t_1 \tag{0.15a}$$

for some $c(1), t_1 > 0$, then

$$\| \boldsymbol{u}(\cdot, t) \| \geq c(0)\, t^{-\alpha_1 + 1/2} \qquad \forall\; t > t_0 \tag{0.15b}$$

with $c(0) = c(1)/\sqrt{2\alpha_1 - 1}$ and t_0 depending on t_*, t_1, τ_0, ν and the function g_0.

Remark 0.8: As will be seem from the proofs of Theorem D and Theorem E (and also of Theorem D′ and Theorem E′), when trying to obtain estimates proceeding from *higher* derivatives to *lower* derivatives it is in general easier to do it for the case of *lower* estimates than it is for *upper* estimates.

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Proof of Theorem A (*adapted from* [9]):

Let $\gamma > 2\alpha$ be given (fixed from now on), and let $a \geq \max\{1, t_*, T_0, \tau_0, \sigma_0\}$. We get, taking the dot product of the equation (0.1) with $2(t-a)^{\gamma}\boldsymbol{u}(x,t)$ and integrating on $\mathbb{R}^n \times (a,t)$,

$$(t-a)^{\gamma}\,\|\,\boldsymbol{u}(t)\,\|^2 \;+\; 2\nu\!\int_a^t (\tau-a)^{\gamma}\,\|\,D\boldsymbol{u}(\tau)\,\|^2\,d\tau \;=$$

$$=\;\gamma\!\int_a^t (\tau-a)^{\gamma-1}\,\|\,\boldsymbol{u}(\tau)\,\|^2\,d\tau \;+\; 2\!\int_a^t (\tau-a)^{\gamma}\!\int_{\mathbb{R}^n}\langle\,\boldsymbol{u}(x,\tau),\,\boldsymbol{f}(x,\tau)-\mathbb{G}(\boldsymbol{u})\,\rangle\,dx\,d\tau$$

$$\leq\;\gamma\!\int_a^t (\tau-a)^{\gamma-1}\,\|\,\boldsymbol{u}(\tau)\,\|^2\,d\tau \;+\; 2\!\int_a^t (\tau-a)^{\gamma}\left\{\,\|\,\boldsymbol{u}(\tau)\,\|\,\|\,\boldsymbol{f}(\tau)\,\| \;+\; g_0(\tau)\,\|\,D\boldsymbol{u}(\tau)\,\|^2\right\}d\tau$$

$$\leq\; C_0\,(\,\gamma\,C_0 \;+\; 2F_0\,a^{-\delta})\!\int_a^t (\tau-a)^{\gamma-2\alpha-1}\,d\tau \;+\; 2\!\int_a^t (\tau-a)^{\gamma}\,g_0(\tau)\,\|\,D\boldsymbol{u}(\tau)\,\|^2\,d\tau$$

for all $t > a$, using (H1)-(H3), where $\delta = \beta - (\alpha+1)$. Because $g_0(\infty) = 0$, we then obtain, increasing a if necessary,

$$(t-a)^{\gamma}\,\|\,\boldsymbol{u}(t)\,\|^2 \;+\; \nu\!\int_a^t (\tau-a)^{\gamma}\,\|\,D\boldsymbol{u}(\tau)\,\|^2\,d\tau \;\leq\; E_0\,(t-a)^{\gamma-2\alpha}$$

for all $t > a$, where $E_0 = C_0\,(\gamma\,C_0 + 2F_0\,a^{-\delta})/(\gamma - 2\alpha)$. This gives, in particular,

$$\int_a^t (\tau-a)^{\gamma}\,\|\,D\boldsymbol{u}(\tau)\,\|^2\,d\tau \;\leq\; E_0\,\nu^{-1}\,(t-a)^{\gamma-2\alpha} \tag{0.16a}$$

for all $t > a$, and any $a \geq \max\{1, t_*, T_0, \tau_0, \sigma_0\}$ sufficiently large (depending on g_0, ν). Now, differentiating the equation (0.1) with respect to x_ℓ, multiplying (dot product) the result by $2\,(t-a)^{\gamma+1}D_\ell\,\boldsymbol{u}(x,t)$ and integrating on $\mathbb{R}^n \times (a,t)$, we similarly obtain, summing over ℓ and increasing a if necessary (so that, in particular, $a \geq \max\{1, t_*, T_0, \tau_0, \tau_1, \sigma_0, \sigma_1\}$),

$$(t-a)^{\gamma+1}\,\|\,D\boldsymbol{u}(t)\,\|^2 \;+\; \nu\!\int_a^t (\tau-a)^{\gamma+1}\,\|\,D^2\boldsymbol{u}(\tau)\,\|^2\,d\tau \;\leq$$

$$\leq\;(\gamma+1)\!\int_a^t (\tau-a)^{\gamma}\,\|\,D\boldsymbol{u}(\tau)\,\|^2\,d\tau \;+\; 2\!\int_a^t (\tau-a)^{\gamma+1}\,\|\,D\boldsymbol{u}(\tau)\,\|\,\|\,D\boldsymbol{f}(\tau)\,\|\,d\tau \tag{0.16b}$$

for all $t > a$. If $\beta = \alpha + 1$, we then have

$$(t-a)^{\gamma+1}\,\|\,D\boldsymbol{u}(t)\,\|^2 \;+\; \nu\!\int_a^t (\tau-a)^{\gamma+1}\,\|\,D^2\boldsymbol{u}(\tau)\,\|^2\,d\tau \;\leq$$

$$\leq\;(\gamma+1+\nu)\!\int_a^t (\tau-a)^{\gamma}\,\|\,D\boldsymbol{u}(\tau)\,\|^2\,d\tau \;+\; \nu^{-1}\!\int_a^t (\tau-a)^{\gamma+2}\,\|\,D\boldsymbol{f}(\tau)\,\|^2\,d\tau \tag{0.16c}$$

while, if $\beta > \alpha + 1$, we obtain from (0.16b) that

$$(t-a)^{\gamma+1}\,\|\,D\boldsymbol{u}(t)\,\|^2 \;+\; \nu\!\int_a^t (\tau-a)^{\gamma+1}\,\|\,D^2\boldsymbol{u}(\tau)\,\|^2\,d\tau \;\leq$$
$$\leq\; (\gamma+2)\!\int_a^t (\tau-a)^{\gamma}\,\|\,D\boldsymbol{u}(\tau)\,\|^2\,d\tau \;+\; \int_a^t (\tau-a)^{\gamma+2}\,\|\,D\boldsymbol{f}(\tau)\,\|^2\,d\tau$$
$$\leq\; (\gamma+2)\!\int_a^t (\tau-a)^{\gamma}\,\|\,D\boldsymbol{u}(\tau)\,\|^2\,d\tau \;+\; F_1^2\,a^{-2\delta}\!\int_a^t (\tau-a)^{\gamma-2\alpha-1}\,d\tau$$

by (H3) with $m = 1$, where $\delta = \beta - (\alpha+1)$. Therefore, in the case $\beta > \alpha + 1$, we obtain, using (0.16a) above and increasing a if necessary,

$$(t-a)^{\gamma+1}\,\|\,D\boldsymbol{u}(t)\,\|^2 \;+\; \nu\!\int_a^t (\tau-a)^{\gamma+1}\,\|\,D^2\boldsymbol{u}(\tau)\,\|^2\,d\tau \;\leq\; E_1\,\nu^{-1}\,(t-a)^{\gamma-2\alpha} \tag{0.16d}$$

for all $t > a$, with E_1 depending only on C_0, say: $E_1 = (\gamma+2)^2\,C_0^2$. This shows (0.4) with $m = 1$ and also gives that

$$\int_a^t (\tau-a)^{\gamma+1}\,\|\,D^2\boldsymbol{u}(\tau)\,\|^2\,d\tau \;\leq\; E_1\,\nu^{-2}\,(t-a)^{\gamma-2\alpha} \tag{0.16e}$$

for all $t > a$, from which we can go to the next level ($m = 2$), repeating the analysis, etc. If $\beta = \alpha + 1$, we proceed similarly from (0.16c) to obtain (0.4) for $m = 1$, then moving to the next level, and so on. Keep going this way, we prove (0.4) for all $m \leq \hat{m}$, as claimed. □

The proof of (0.5) and (0.6b) requires the upper estimates given in THEOREM A *and* the following monotonicity property, which extends a similar result in [3] (see [3], THEOREM B):

Lemma 0.1 (*monotonicity lemma*).
Assume (H1), (H2) and (H3) above, with $\beta \geq \alpha + 1$. Then we have, for every $0 \leq m \leq \hat{m}$:

$$\frac{d}{dt}\left\{\,\|\,D^m\boldsymbol{u}(\cdot,t)\,\|^2 \;+\; K_m(\alpha,\beta)\;t^{-\alpha-\beta-m+1}\right\} \;\leq\; 0 \qquad \forall\; t > a_m \tag{0.17}$$

where $K_m(\alpha,\beta) = 2\,C_m F_m/(\alpha+\beta+m-1)$, with $a_m > t_*$ depending only on the values of $m, T_m, \tau_m, \sigma_m, \nu$ and the funtion g_m given in (H2). (The constants C_m, F_m referred to here are those given in (H1), (H3) and THEOREM A above.)

Remark 0.9: If $\boldsymbol{f} = \boldsymbol{0}$, it follows from (0.17) above that $\|\,D^m\,\boldsymbol{u}(\cdot,t)\,\|$ is monotonically decreasing in the interval (a_m, ∞), since in this case we have $K_m = 0$ (because $F_m = 0$). This property also follows very easily from the proof of LEMMA 0.1 given next.

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Proof of Lemma 0.1:

Let $a = \max\{t_*, T_0, \tau_0, \sigma_0\}$, and let $t > a$. From the energy identity

$$\|\boldsymbol{u}(t)\|^2 + 2\nu\int_a^t \|D\boldsymbol{u}(\tau)\|^2\,d\tau = \|\boldsymbol{u}(a)\|^2 + 2\int_a^t\int_{\mathbb{R}^n}\langle\,\boldsymbol{u}(x,\tau),\, \boldsymbol{f}(x,\tau) - \mathbb{G}(\boldsymbol{u})\,\rangle\,dx\,d\tau$$

we obtain, by (H2),

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{u}(t)\|^2 + 2\nu\|D\boldsymbol{u}(t)\|^2 &= 2\int_{\mathbb{R}^n}\langle\,\boldsymbol{u}(x,t),\,\boldsymbol{f}(x,t) - \mathbb{G}(\boldsymbol{u})\,\rangle\,dx\\ &\le 2\|\boldsymbol{u}(t)\|\,\|\boldsymbol{f}(t)\| + 2\,g_0(t)\,\|D\boldsymbol{u}(t)\|^2\end{aligned}$$

so that we have, increasing a if necessary,

$$\frac{d}{dt}\|\boldsymbol{u}(t)\|^2 + \nu\|D\boldsymbol{u}(t)\|^2 \le 2C_0F_0\,t^{-\alpha-\beta}$$

for all $t > a$, by (H1) and (H3). This gives (0.17) when $m = 0$. For general $1 \le m \le \hat{m}$, we proceed in a similar way: taking $a = \max\{t_*, T_m, \tau_m, \sigma_m\}$, we have, by (H2),

$$\frac{d}{dt}\|D^m\boldsymbol{u}(t)\|^2 + 2\nu\|D^{m+1}\boldsymbol{u}(t)\|^2 \le 2\|D^m\boldsymbol{u}(t)\|\,\|D^m\boldsymbol{f}(t)\| + 2\,g_m(t)\,\|D^{m+1}\boldsymbol{u}(t)\|^2$$

for all $t > a$. Increasing a if necessary, we then obtain

$$\frac{d}{dt}\|D^m\boldsymbol{u}(t)\|^2 \le 2\|D^m\boldsymbol{u}(t)\|\,\|D^m\boldsymbol{f}(t)\| \le 2\,C_m F_m\,t^{-\alpha-\beta-m} \qquad \forall\; t > a,$$

by (H3) and Theorem A. This estimate gives (0.17), and the proof is now complete. □

Remark 0.10: From the proof of Lemma 0.1 above, we see that: in the case $m = 0$ it is sufficient that α, β be nonnegative reals satisfying $\alpha + \beta > 1$. In any case, Lemma 0.1 will only be needed (in the proof of Theorem B and Theorem C) for $1 \le m \le \hat{m}$.

Observing the expression (0.17), it will be convenient in the sequel to introduce the function $z_m(t)$ defined by

$$z_m(t) = \|D^m\boldsymbol{u}(\cdot,t)\|^2 + K_m(\alpha,\beta)\,t^{-\alpha-\beta-m+1}, \qquad t > a_m \tag{0.18}$$

where the constant $K_m(\alpha,\beta)$ is given in Lemma 0.1. According to this lemma, if $\beta \ge \alpha+1$ the function z_m is smooth and monotonically decreasing in the interval (a_m, ∞).

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Proof of Theorem B:

Let $m = 1$ first. Recalling (0.2), (0.17) and (H1)-(H4), let $t_1 = \max\{t_*, t_0, \tau_0, \sigma_0, a_1, T_0\}$ and let $t > t_1$. Given $T = Mt$, where $M > 1$ will be chosen later, we obtain, from (0.1),

$$\|\boldsymbol{u}(T)\|^2 + 2\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau = \|\boldsymbol{u}(t)\|^2 + 2\int_t^T\!\!\int_{\mathbb{R}^n}\langle\,\boldsymbol{u}(x,\tau),\,\boldsymbol{f}(x,\tau) - \mathbb{G}(\boldsymbol{u})\,\rangle\,dx\,d\tau.$$

By (H2) with $m = 0$, we have

$$\|\boldsymbol{u}(T)\|^2 + 2\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau \ge \|\boldsymbol{u}(t)\|^2 - 2\int_t^T\big\{\|\boldsymbol{u}(\tau)\|\,\|\boldsymbol{f}(\tau)\| + g_0(\tau)\,\|D\boldsymbol{u}(\tau)\|^2\big\}\,d\tau.$$

Increasing t_1 (if needed) so that $g_0(\tau) < \nu$ for all $\tau > t_1$, we get, by (H1), (H3) and (H4),

$$\begin{aligned} 4\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau &\ge \|\boldsymbol{u}(t)\|^2 - \|\boldsymbol{u}(T)\|^2 - 2\int_t^T \|\boldsymbol{u}(\tau)\|\,\|\boldsymbol{f}(\tau)\|\,d\tau \\ &\ge \Big\{c(0)^2 - C_0^2\,M^{-2\alpha} - \kappa_0\,t^{-\delta}\Big\}\,t^{-2\alpha} \end{aligned}$$

where $\kappa_0 = 2\,C_0\,F_0/(\alpha+\beta-1)$ and $\delta = \beta - \alpha - 1 > 0$. Choosing $M = \big(2\,C_0/c(0)\big)^{1/\alpha}$ and increasing t_1 (if necessary) so that $\kappa_0\,t_1^{-\delta} \le c(0)^2/4$, this gives, by Lemma 0.1,

$$4\,\nu\,T\,z_1(t) \ge 4\nu\int_t^T z_1(\tau)\,d\tau \ge 4\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau \ge c^2\,t^{-2\alpha}$$

where $c^2 = c(0)^2/2$. Therefore, $z_1(t) \ge c^2/(4\nu)\,T^{-1}\,t^{-2\alpha} = c^2/(4M\nu)\,t^{-2\alpha-1}$ if $t > t_1$, or

$$\|D\boldsymbol{u}(t)\|^2 \ge \Big\{\frac{c^2}{4M} - K_1(\alpha,\beta)\,\nu\,t^{-\delta}\Big\}\,\nu^{-1}\,t^{-2\alpha-1}$$

for all $t > t_1$. Increasing t_1 if needed, this gives $\|D\boldsymbol{u}(t)\| \ge c(1)\,\nu^{-1/2}\,t^{-\alpha-1/2}$ for $t > t_1$, where, say, $c(1) = c(0)\,M^{-1/2}/3$. This shows the result for $m = 1$. If $\hat{m} = 1$, we are done; otherwise, we proceed with $m = 2$ in a similar way. Setting $t_2 = \max\{t_1, \tau_1, \sigma_1, a_2, T_1\}$, let $t > t_2$ and $T = Mt$, where $M = (2\,C_1/c(1))^{1/\alpha}$. Increasing t_2 if necessary, we then get, from the equation (0.1) and the assumption (H2),

$$\begin{aligned} 4\nu\int_t^T \|D^2\boldsymbol{u}(\tau)\|^2\,d\tau &\ge \|D\boldsymbol{u}(t)\|^2 - \|D\boldsymbol{u}(T)\|^2 - 2\int_t^T \|D\boldsymbol{u}(\tau)\|\,\|D\boldsymbol{f}(\tau)\|\,d\tau \\ &\ge \Big\{c(1)^2 - C_1^2\,M^{-2\alpha-1} - \kappa_1\,\nu^{1/2}\,t^{-\delta}\Big\}\,\nu^{-1}\,t^{-2\alpha-1} \ge \frac{c(1)^2}{2}\,\nu^{-1}\,t^{-2\alpha-1} \end{aligned}$$

by (H3), Theorem A and the previous case, where $\kappa_1 = 2\,C_1 F_1/(\alpha+\beta)$ and $\delta = \beta-\alpha-1$. Now, introducing $z_2(t) \ge \|D^2\boldsymbol{u}(t)\|^2$ given in Lemma 0.1 and repeating the steps above, we obtain (0.5) for $m = 2$ as well. We then keep going in this way until $\hat{m}$ is reached. □

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Proof of Theorem C:

Let $q = \eta/\alpha$. Recalling (0.2), (0.17) and (H1)-(H4), let $t_1 = \max\{t_*, t_0, \tau_0, \sigma_0, a_1, T_0\}$, and let $t > t_1$. Given $T = M\,t^q$, where $M > 1$ will be chosen later, we have, as before,

$$\|\boldsymbol{u}(T)\|^2 + 2\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau = \|\boldsymbol{u}(t)\|^2 + 2\int_t^T\int_{\mathbb{R}^n}\langle\,\boldsymbol{u}(x,\tau),\,\boldsymbol{f}(x,\tau) - \mathbb{G}(\boldsymbol{u})\,\rangle\,dx\,d\tau.$$

By (H2) with $m = 0$, we obtain

$$\|\boldsymbol{u}(T)\|^2 + 2\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau \geq \|\boldsymbol{u}(t)\|^2 - 2\int_t^T\{\,\|\boldsymbol{u}(\tau)\|\,\|\boldsymbol{f}(\tau)\| + g_0(\tau)\,\|D\boldsymbol{u}(\tau)\|^2\,\}\,d\tau.$$

Increasing t_1 (if needed) so that $g_0(\tau) < \nu$ for all $\tau > t_1$, we have, by (H1), (H3) and (H4),

$$\begin{aligned} 4\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau &\geq \|\boldsymbol{u}(t)\|^2 - \|\boldsymbol{u}(T)\|^2 - 2\int_t^T \|\boldsymbol{u}(\tau)\|\,\|\boldsymbol{f}(\tau)\|\,d\tau \\ &\geq \left\{c(0)^2 - C_0^2\,M^{-2\alpha} - \kappa_0\,t^{-\delta_1}\right\}t^{-2\eta} \end{aligned}$$

where $\kappa_0 = 2\,C_0\,F_0/(\alpha+\beta-1)$ and $\delta_1 = \beta-(2\eta-\alpha+1) > 0$. Choosing $M = \big(2\,C_0/c(0)\big)^{1/\alpha}$ and increasing t_1 (if necessary) so that $\kappa_0\,t_1^{-\delta_1} \leq c(0)^2/4$, this gives, by Lemma 0.1,

$$4\nu\,T\,z_1(t) \geq 4\nu\int_t^T z_1(\tau)\,d\tau \geq 4\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau \geq c^2\,t^{-2\eta}$$

where $c^2 = c(0)^2/2$. Therefore, $z_1(t) \geq c^2/(4\nu)\,T^{-1}\,t^{-2\eta} = c^2/(4M\nu)\,t^{-2\eta-q}$ if $t > t_1$, or

$$\|D\boldsymbol{u}(t)\|^2 \geq \left\{\frac{c^2}{4M} - K_1(\alpha,\beta)\,\nu\,t^{-\varepsilon_1}\right\}\nu^{-1}\,t^{-2\eta-q}$$

for all $t > t_1$, where $\varepsilon_1 = \beta - (\,2\eta - \alpha + q\,) > 0$. Hence, increasing t_1 if necessary, we have $\|D\boldsymbol{u}(t)\| \geq c(1)\,\nu^{-1/2}\,t^{-\eta-q/2}$ for $t > t_1$, with $c(1) = c(0)\,M^{-1/2}/3$. This shows the result for $m = 1$. If $\hat{m} = 1$, the proof of Theorem C is complete; otherwise, we proceed with $m = 2$ in a similar way. Setting $t_2 = \max\{t_1, \tau_1, \sigma_1, a_2, T_1\}$, let $t > t_2$ and $T = M\,t^q$, where $M = (2\,C_1/c(1))^{1/\alpha}$. As before, increasing t_2 if necessary, we then obtain, from the equation (0.1) and the assumption (H2) with $m = 1$,

$$\begin{aligned} 4\nu\int_t^T \|D^2\boldsymbol{u}(\tau)\|^2\,d\tau &\geq \|D\boldsymbol{u}(t)\|^2 - \|D\boldsymbol{u}(T)\|^2 - 2\int_t^T \|D\boldsymbol{u}(\tau)\|\,\|D\boldsymbol{f}(\tau)\|\,d\tau \\ &\geq \left\{c(1)^2 - C_1^2\,M^{-2\alpha-1} - \kappa_1\,\nu^{1/2}\,t^{-\delta_2}\right\}\nu^{-1}\,t^{-2\eta-q} \end{aligned}$$

by (H3) and Theorem A (both applied with $m = 1$) and the previous case above, where

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$\kappa_1 = 2\,C_1\,F_1/(\alpha+\beta)$ and $\delta_2 = \beta - (2\,\eta - \alpha + q) > 0$. Increasing t_2 if necessary so that we have $\kappa_1\,\nu^{1/2}\,t_2^{-\delta_2} < c(1)^2/4$, this gives, recalling Lemma 0.1,

$$4\,\nu\,T\,z_2(t) \;\geq\; 4\nu\int_t^T z_2(\tau)\,d\tau \;\geq\; 4\nu\int_t^T \|\,D^2\boldsymbol{u}(\tau)\,\|^2\,d\tau \;\geq\; c^2\,\nu^{-1}\,t^{-2\eta-q}$$

for all $t > t_2$, where $c^2 = c(1)^2/2$. Therefore, $z_2(t) \geq c^2/(4\nu^2)\,T^{-1}\,t^{-2\eta-q}$, or

$$\|\,D^2\boldsymbol{u}(t)\,\|^2 \;\geq\; \left\{\frac{c^2}{4M} \,-\, K_2(\alpha,\beta)\,\nu^2\,t^{-\varepsilon_2}\right\}\nu^{-2}\,t^{-2\eta-2q}$$

for all $t > t_2$, where $\varepsilon_2 = \beta - (2\eta - \alpha + 2\,(q-1) + 1) > 0$. Thus, increasing t_2 if needed, we have $\|\,D^2\boldsymbol{u}(t)\,\| \geq c(2)\,\nu^{-1}\,t^{-\eta-q}$ for all $t > t_2$, with $c(2) = c(1)\,M^{-1/2}/3$. This completes the derivation of Theorem C if $\hat{m} = 2$. Otherwise, with $\hat{m} \geq 3$, we continue to the next level by considering the energy estimate for $\|\,D^2\boldsymbol{u}(t)\,\|$,

$$\|\,D^2\boldsymbol{u}(T)\,\|^2 \,+\, 2\nu\int_t^T \|\,D^3\boldsymbol{u}(\tau)\,\|^2\,d\tau \;=\; \|\,D^2\boldsymbol{u}(t)\,\|^2 \,+$$
$$+\; 2\sum_{j=1}^{n}\sum_{\ell=1}^{n}\int_t^T\!\!\int_{\mathbb{R}^n}\langle\, D_jD_\ell\,\boldsymbol{u}(x,\tau),\, D_jD_\ell\,\boldsymbol{f}(x,\tau) - D_jD_\ell\,\mathbb{G}(\boldsymbol{u})\,\rangle\,dx\,d\tau$$

for $t > t_3 = \max\{\,t_2,\,\tau_2,\,\sigma_2,\,a_3,\,T_2\}$, where $T = M\,t^q$, $M = (2\,C_2/\,c(2))^{1/\alpha}$, repeating the steps above to obtain (0.6b) for $m = 3$ as well. Because of the condition (0.6a) upon β, we can proceed in this way up to a last level, given by the energy estimate for $m = \hat{m}-1$,

$$\|\,D^m\boldsymbol{u}(T)\,\|^2 \,+\, 2\nu\int_t^T \|\,D^{m+1}\boldsymbol{u}(\tau)\,\|^2\,d\tau \;=\; \|\,D^m\boldsymbol{u}(t)\,\|^2 \,+$$
$$+\; 2\sum_{\ell_1=1}^{n}\sum_{\ell_2=1}^{n}\cdots\sum_{\ell_m=1}^{n}\int_t^T\!\!\int_{\mathbb{R}^n}\langle\, D_{\ell_1}D_{\ell_2}\cdots D_{\ell_m}\boldsymbol{u}(x,\tau),\, D_{\ell_1}D_{\ell_2}\cdots D_{\ell_m}\{\,\boldsymbol{f}(x,\tau) - \mathbb{G}(\boldsymbol{u})\,\}\,\rangle\,dx\,d\tau$$

for $t > t_{\hat{m}} = t_{m+1} = \max\{\,t_m,\,\tau_m,\,\sigma_m,\,a_{m+1},\,T_m\}$, where $T = M\,t^q$, $M = (2\,C_m/\,c(m))^{1/\alpha}$: using (H2), (H3), Theorem A and Lemma 0.1 as before, we then obtain

$$4\,\nu\,T\,z_{\hat{m}}(t) \;\geq\; 4\nu\int_t^T \|\,D^{\hat{m}}\boldsymbol{u}(\tau)\,\|^2\,d\tau \;\geq\; \left\{c(m)^2 - C_m^2\,M^{-2\alpha} - \kappa_m\,\nu^{m/2}\,t^{-\delta_{\hat{m}}}\right\}\nu^{-m}\,t^{-2\eta-mq}$$

where $\delta_{\hat{m}} = \beta - \big(2\eta - \alpha + (q-1)\,m + 1\big) > 0$. Increasing $t_{\hat{m}}$ if necessary, this gives that $z_{\hat{m}}(t) \geq c(m)^2/(8M)\,\nu^{-\hat{m}}\,t^{-2\eta-\hat{m}q}$ for all $t > t_{\hat{m}}$, or, in terms of $\|\,D^{\hat{m}}\boldsymbol{u}(t)\,\|$, by (0.17):

$$\|\,D^{\hat{m}}\boldsymbol{u}(t)\,\|^2 \;\geq\; \left\{c(m)^2/(8M) \,-\, K_{\hat{m}}(\alpha,\beta)\,t^{-\varepsilon_{\hat{m}}}\right\}\nu^{-\hat{m}}\,t^{-2\eta-\hat{m}q}$$

for all $t > t_{\hat{m}}$, where $\varepsilon_{\hat{m}} = \beta - (2\eta - \alpha + (q-1)\,\hat{m} + 1)$. Since, by (0.6$a$), we have $\varepsilon_{\hat{m}} > 0$, this shows the estimate (0.6b) for $\hat{m}$ as well, which completes the proof of Theorem C. $\square$

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Proof of Theorem D:

Let $T_0 = \max\{1, t_*, \tau_0, \sigma_0, T_1\}$, and let $T > t > T_0$. From the equation (0.1), we obtain

$$\|\boldsymbol{u}(T)\|^2 + 2\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau = \|\boldsymbol{u}(t)\|^2 + 2\int_t^T \langle\,\boldsymbol{u}(x,\tau), \boldsymbol{f}(x,\tau) - \mathbb{G}(\boldsymbol{u})\,\rangle\,dx\,d\tau,$$

so that, by (H2), we have (increasing T_0 if necessary, depending on ν and the function g_0)

$$\|\boldsymbol{u}(T)\|^2 + 4\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau \geq \|\boldsymbol{u}(t)\|^2 - 2\int_t^T \|\boldsymbol{u}(\tau)\|\,\|\boldsymbol{f}(\tau)\|\,d\tau.$$

Using the hypothesis (0.7a) and letting $T \to \infty$, this gives, recalling that $\|\boldsymbol{u}(T)\| \to 0$,

$$\|\boldsymbol{u}(t)\|^2 \leq \frac{4\,C_1^2}{2\,\alpha_1 - 1}\,t^{1-2\alpha_1} + 2\int_t^\infty \|\boldsymbol{u}(\tau)\|\,\|\boldsymbol{f}(\tau)\|\,d\tau$$

for all $t > T_0$. If $\boldsymbol{f} = \boldsymbol{0}$, we are done. Otherwise, from the assumption on $\boldsymbol{f}$, we have

$$\|\boldsymbol{u}(t)\|^2 \leq \frac{4\,C_1^2}{2\,\alpha_1 - 1}\,t^{1-2\alpha_1} + 2F_0\int_t^\infty \tau^{-\beta}\,\|\boldsymbol{u}(\tau)\|\,d\tau \tag{0.19a}$$

if $t > T_0$. From this estimate, (0.7b) can be obtained by bootstrapping. By (0.9), we have, for some $a_* \gg 1$ (depending on t_*, τ_0, ν and the function g_0), that

$$\|\boldsymbol{u}(t)\| \leq M_0 = \|\boldsymbol{u}(a_*)\| + \int_{a_*}^\infty \|\boldsymbol{f}(\tau)\|\,d\tau \qquad \forall\ t > a_*, \tag{0.19b}$$

and so we redefine T_0 to be: $T_0 = \max\{1, t_*, \tau_0, \sigma_0, T_1, a_*\}$. Taking (0.19$b$) into (0.19$a$), we get

$$\|\boldsymbol{u}(t)\|^2 \leq \frac{4\,C_1^2}{2\,\alpha_1 - 1}\,t^{1-2\alpha_1} + \frac{2\,F_0\,M_0}{\beta - 1}\,t^{1-\beta}$$

for all $t > T_0$. If $(\beta - 1)/2 \geq \alpha_1 - 1/2$, we are done; otherwise, we obtain

$$\|\boldsymbol{u}(t)\| \leq M_1\,t^{q_1(1-\beta)} \qquad \forall\ t > T_0 \tag{0.19c}$$

with $q_1 = 1/2$ and $M_1 = \{4C_1^2/(2\alpha_1 - 1) + 2F_0M_0/(\beta - 1)\}^{1/2}$. Let $\theta = (\alpha_1 - 1/2)/(\beta - 1)$. Assuming that we have

$$\|\boldsymbol{u}(t)\| \leq M_k\,t^{q_k(1-\beta)} \qquad \forall\ t > T_0 \tag{0.19d}$$

for some $q_k \in [0, \theta)$, we obtain, taking (0.19d) into (0.19a),

$$\|\boldsymbol{u}(t)\|^2 \leq \frac{4\,C_1^2}{2\,\alpha_1 - 1}\,t^{1-2\alpha_1} + \frac{2\,F_0\,M_k}{(1 + q_k)(\beta - 1)}\,t^{(1+q_k)(1-\beta)} \tag{0.19e}$$

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for all $t > T_0$. Let $q_{k+1} = (1+q_k)/2$. If $q_{k+1}(\beta-1) \geq \alpha_1 - 1/2$, we are done; if not, we have

$$\| \boldsymbol{u}(t) \| \;\leq\; M_{k+1}\, t^{q_{k+1}(1-\beta)} \qquad \forall\; t > T_0 \tag{0.19f}$$

where $M_{k+1} = \{4C_1^2/(2\alpha_1 - 1) + F_0 M_k/(q_{k+1}(\beta-1))\}^{1/2}$, and we go to the next iteration. The numbers q_k are given recursively by

$$q_{k+1} \;=\; \frac{1}{2}\,(1+q_k), \qquad k \geq 0,$$

with $q_0 = 0$, so that $q_k = 1 - 2^{-k}$. Now, in the case (i), we have $\theta < 1$ (since $\beta > \alpha_1 + 1/2$), and there will exist $k_* \geq 0$ such that $q_{k_*}(\beta-1) < \alpha_1 - 1/2$ and $q_{k_*+1}(\beta-1) \geq \alpha_1 - 1/2$. For such k, (0.19e) gives that

$$\| \boldsymbol{u}(t) \| \;\leq\; M_{k_*+1}\, t^{-\alpha_1 + 1/2} \tag{0.19g}$$

for all $t > T_0$, showing (0.7b), as claimed. Finally, in the case (ii), where $\theta = 1$, because $2\alpha_1 - 1 > (1+q_k)(\beta-1)$ for all k, the second term on the right hand side of (0.19e) will always decay slower than the first term. However, given $\epsilon > 0$, we have $(1+q_k)(\beta-1) > 2\alpha_1 - 1 - 2\epsilon$ for large k, and so the bootstrap iteration can stop there to give (0.7c). □

Remark 0.11: For completeness, let us show (0.9). Recalling (0.2) and the assumptions (H2) and (H3) with $m = 0$, let $a_* = \max\{t_*, \tau_0\}$. Given $t > a > a_*$, we have, from the equation (0.1),

$$\| \boldsymbol{u}(t) \|^2 \,+\, 2\nu \int_a^t \| D\boldsymbol{u}(\tau) \|^2\, d\tau \;=\; \| \boldsymbol{u}(a) \|^2 \,+\, 2\int_a^t \int_{\mathbb{R}^n} \langle\, \boldsymbol{u}(x,\tau),\, \boldsymbol{f}(x,\tau) - \mathbb{G}(u) \,\rangle\, dx\, d\tau$$

so that, increasing a_* (if necessary) so as to have $g_0(\tau) \leq \nu$ for all $\tau > a_*$, we obtain

$$\| \boldsymbol{u}(t) \|^2 \;\leq\; \| \boldsymbol{u}(a) \|^2 \,+\, 2\int_a^t \| \boldsymbol{u}(\tau) \| \, \| \boldsymbol{f}(\tau) \| \, d\tau,$$

or, in terms of $\mathrm{v}(t) = \| \boldsymbol{u}(t) \|^2$:

$$\mathrm{v}(t) \;=\; \mathrm{v}(a) \,+\, 2\int_a^t \mathrm{v}(\tau)^{1/2}\, \| \boldsymbol{f}(\tau) \| \, d\tau.$$

Now, let $\mathrm{w} \in C^1([a,\infty))$ be given by: $\mathrm{w}'(t) = 2\,\mathrm{w}(t)^{1/2}\, \| \boldsymbol{f}(t) \|$ for $t > a$, and $\mathrm{w}(a) = \mathrm{v}(a)$, that is,

$$\mathrm{w}(t)^{1/2} \;=\; \| \boldsymbol{u}(a) \| \,+\, \int_a^t \| \boldsymbol{f}(\tau) \| \, d\tau.$$

Since $\| \boldsymbol{u}(t) \| = \mathrm{v}(t)^{1/2} \leq \mathrm{w}(t)^{1/2}$, we have obtained (0.9), as claimed. □

Proof of Theorem E:

Let $t_0 = \max\{1, t_*, \tau_0, \sigma_0, t_1, T_0\}$, where T_0 is given in (0.7b), and let $T > t > t_0$. From the equation (0.1), we have

$$\|\boldsymbol{u}(T)\|^2 + 2\nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau = \|\boldsymbol{u}(t)\|^2 + 2\int_t^T \langle\,\boldsymbol{u}(x,\tau), \boldsymbol{f}(x,\tau) - \mathbb{G}(\boldsymbol{u})\,\rangle\,dx\,d\tau,$$

so that, by (H2), we get (increasing t_0 if necessary, depending on ν and the function g_0)

$$\|\boldsymbol{u}(T)\|^2 + \nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau \le \|\boldsymbol{u}(t)\|^2 + 2\int_t^T \|\boldsymbol{u}(\tau)\|\,\|\boldsymbol{f}(\tau)\|\,d\tau.$$

Hence, we have

$$\|\boldsymbol{u}(t)\|^2 \ge \nu\int_t^T \|D\boldsymbol{u}(\tau)\|^2\,d\tau - 2\int_t^T \|\boldsymbol{u}(\tau)\|\,\|\boldsymbol{f}(\tau)\|\,d\tau,$$

so that, letting $T \to \infty$,

$$\|\boldsymbol{u}(t)\|^2 \ge \nu\int_t^\infty \|D\boldsymbol{u}(\tau)\|^2\,d\tau - 2\int_t^\infty \|\boldsymbol{u}(\tau)\|\,\|\boldsymbol{f}(\tau)\|\,d\tau$$

for all $t > t_0$. Using (0.7b), (0.8a) and the assumption (H3) with $m = 0$, this gives

$$\begin{aligned}\|\boldsymbol{u}(t)\|^2 &\ge \frac{c(1)^2}{2\alpha_1 - 1}\,t^{1-2\alpha_1} - \frac{2\,C_0 F_0}{\beta + \alpha_1 - 3/2}\,t^{3/2-\beta-\alpha_1}\\ &\ge \frac{1}{2\alpha_1 - 1}\left\{c(1)^2 - 2\,C_0 F_0\,t^{-\delta}\right\}t^{1-2\alpha_1}\end{aligned}$$

for all $t > t_0$, where $\delta = \beta - (\alpha_1 + 1/2)$. Since $\delta > 0$, increasing t_0 (if necessary) we obtain

$$\|\boldsymbol{u}(t)\|^2 \ge \frac{c(1)^2/4}{\alpha_1 - 1/2}\,t^{1-2\alpha_1}$$

for all $t > t_0$. This is (0.8b), and the proof of Theorem E is now complete. □

In the sequel we will illustrate the theory with six typical examples, ranging from the simple, familiar heat equation and some of its natural extensions to more complex problems like the Navier-Stokes equations, the MHD (magnetohydrodynamics) equations and incompressible micropolar flows. The discussion of these systems is brief and is meant for illustration only, providing a quick, unified derivation of properties that are basically widely known already, but not without some exceptions. Future works will concentrate in applying the theory to generate new results (for the most past, not announced here).

Example 1 (*linear heat equation*).

Given $u_0 \in L^2(\mathbb{R}^n)$, let $u(\cdot,t)$ be the (unique) solution in the space $C([0,\infty), L^2(\mathbb{R}^n))$ of the initial value problem

$$u_t = \nu\,\Delta u, \qquad u(\cdot,0) = u_0, \tag{1.1}$$

which is given by $u(\cdot,t) = e^{\nu\Delta t}u_0$. It is well known that $u(\cdot,t)$ satisfies (0.2) with $t_* = 0$, and also that $\|\,u(\cdot,t)\,\| \to 0$ as $t \to \infty$, and, more generally, for every $m \geq 0$:

$$\|\,D^m u(\cdot,t)\,\| = o(t^{-m/2}) \tag{1.2}$$

as $t \to \infty$, and many other properties. In the case of the equation (1.1), we have $\mathbb{G} = 0$ and $f = 0$, so that only (H1) and (H4) remain to be checked for any particular solution. Beginning with (H1), let us consider that we have

$$\|\,u(\cdot,t)\,\| \leq C_0\,t^{-\alpha} \qquad \forall\; t > T_0 \tag{1.3}$$

for some $\alpha, C_0, T_0 > 0$. Having (1.3), using the Fourier transform it is very easy to obtain

$$\|\,D^m u(\cdot,t)\,\| \leq C_m\,\nu^{-m/2}\,t^{-\alpha-m/2} \qquad \forall\; t \gg 1 \tag{1.4}$$

for every $m \geq 1$, where C_m depends only on m, α and C_0, as predicted by THEOREM A$'$. Lower estimates, on the other hand, are a different matter and are not so easily derived, even for the seemingly trivial equation (1.1) above. Still, assuming (1.3) and that

$$\|\,u(\cdot,t)\,\| \geq c(0)\,t^{-\eta} \qquad \forall\; t > t_0 \tag{1.5}$$

for some $c(0), t_0 > 0$, and some $\eta \geq \alpha$, from THEOREMS B$'$ and C$'$ we immediately obtain

$$\|\,D^m u(\cdot,t)\,\| \geq c(m)\,\nu^{-m/2}\,t^{-\eta-mq/2} \qquad \forall\; t \gg 1 \tag{1.6}$$

for every $m \geq 1$, where $q = \eta/\alpha$, and where $c(m) > 0$ depends only on m, α, $c(0)$ and C_0. Moreover, the approach given in this work shows that obtaining (1.6) from (1.3) and (1.5) is actually almost as easy as obtaining (1.4) from (1.3).

For the heat equation with spatially constant advection, that is,

$$u_t + \boldsymbol{b}(t)\cdot\nabla u = \nu\,\Delta u, \tag{1.7}$$

with a given velocity field $\boldsymbol{b}(t)$, the results are the same. This becomes clear if we change the space variable to $\xi = x - \boldsymbol{B}(t)$, where $\boldsymbol{B}'(t) = \boldsymbol{b}(t)$, or we can apply THEOREMS A′-E′ directly, with $\mathbb{G}(u) = \boldsymbol{b}(t) \cdot \nabla u$ in this case. Observing that, for every $t > 0$, we have

$$\int_{\mathbb{R}^n} u(x,t) \cdot \mathbb{G}(u(x,t))\, dx \;=\; 0 \tag{1.8a}$$

and, for each $m > 0$,

$$\int_{\mathbb{R}^n} D_{\ell_1} \cdots D_{\ell_m} u(x,t) \cdot D_{\ell_1} \cdots D_{\ell_m} \mathbb{G}(u(x,t))\, dx \;=\; 0 \tag{1.8b}$$

for any $1 \leq \ell_1, \ell_2, ..., \ell_m \leq n$, we see that (H2) is clearly satisfied for all m, with $\tau_m = 0$. Hence, for the equation (1.7) we will have again (1.4) following from (1.3), for every m, as well as having (1.6) as consequence of (1.3) and (1.5), for any m, and so forth.

In the case of the inhomogeneous problem

$$u_t \,+\, \boldsymbol{b}(t) \cdot \nabla u \;=\; \nu\, \Delta u \,+\, f, \qquad u(\cdot,0) \,=\, u_0 \in L^2(\mathbb{R}^n), \tag{1.9}$$

with f satisfying (H3) for all m concerned, we proceed similarly, using THEOREMS A-E. For example, if $u(\cdot,t)$ satisfies (1.3) and (1.5) with $\eta = \alpha$, and if (H3) holds valid for all m with some $\beta > \alpha + 1$, then we have, for every m:

$$\|\, D^m u(\cdot,t) \,\| \;\geq\; c(m)\, \nu^{-m/2}\, t^{-\alpha - m/2} \qquad \forall\; t \gg 1 \tag{1.10}$$

for some $c(m) > 0$ depending only on m and on $\alpha, c(0), C_0$ given in (1.3) and (1.5) above. Or, if the property (H3) is satisfied for $m = 0$ and $m = 1$, for some $\beta > \alpha + 1$, and if $\eta = q\alpha$ with $1 < q < (\alpha + \beta)/(2\,\alpha + 1)$, we will then get, from (1.3) and (1.5) above, that

$$\|\, Du(\cdot,t) \,\| \;\geq\; c(1)\, \nu^{-1/2}\, t^{-\eta - q/2} \qquad \forall\; t \gg 1 \tag{1.11}$$

for some $c(1) > 0$ depending only on α, $c(0)$ and C_0 (by THEOREM C), etc.

The more interesting (and much more challenging) problem

$$u_t \,+\, \boldsymbol{b}(t,u) \cdot \nabla u \;=\; \nu\, \Delta u \,+\, f, \qquad u(\cdot,0) \,=\, u_0 \in L^2(\mathbb{R}^n) \tag{1.12}$$

will be taken up in EXAMPLE 2 ($n = 1$) and EXAMPLE 3 ($n \geq 2$) below. It is sufficient to consider the basic case where $\boldsymbol{b} = \boldsymbol{b}(u)$ does not depend explicitly on t, since the analysis in the more general setting (1.12), i.e., when $\boldsymbol{b} = \boldsymbol{b}(t,u)$, turns out to be entirely similar.

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Example 2 (*advection-diffusion equations*: $n = 1$).

Given $b \in C^{\infty}(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R}^n)$, let $u(\cdot,t) \in C([0,\infty), L^2(\mathbb{R})) \cap L^{\infty}_{\text{loc}}((0,\infty), L^{\infty}(\mathbb{R}))$ be a solution to the problem

$$u_t + b(u)\, u_x = \nu\, u_{xx} \qquad u(\cdot,0) = u_0. \tag{2.1}$$

Under the above conditions, it is known that $\|\, u(\cdot,t)\,\| \to 0$ and $t^{1/4}\,\|\, u(\cdot,t)\,\|_{L^{\infty}(\mathbb{R})} \to 0$ as $t \to \infty$ (see [2], Theorem 3.3), as well as the general supnorm estimate

$$\|\, u(\cdot,t)\,\|_{L^{\infty}(\mathbb{R})} \leq \|\, u_0\,\|\, \nu^{-1/4}\, t^{-1/4} \qquad \forall\; t > 0 \tag{2.2}$$

(see [2], Theorem 3.2). Moreover, (0.2) holds with $t_* = 0$, so that there only remains to check whether the condition (H2) is satisfied, where $\mathbb{G}(u) = b(u)\, u_x$ for the equation (2.1). If $m = 0$ this is clearly the case, since

$$\int_{\mathbb{R}} u(x,t)\, b(u)\, u_x\, dx \;=\; \int_{\mathbb{R}} \frac{\partial}{\partial x} E(u(x,t))\, dx \;=\; 0 \tag{2.3}$$

for any $t > 0$, where $E(\mathrm{u}) = \int_0^{\mathrm{u}} \mathrm{v}\, b(\mathrm{v})\, d\mathrm{v}$, recalling the fact that $u(\cdot,t) \in L^2(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$. For $m \geq 1$, checking (H2) is more involved. It will be more convenient to work with

$$\tilde{\mathbb{G}}(\mathrm{u}) := \tilde{b}(\mathrm{u})\, \mathrm{u}_x, \qquad \tilde{b}(\mathrm{u}) := b(\mathrm{u}) - b(0). \tag{2.4a}$$

Clearly, for any given m: (H2) is valid for $\mathbb{G}(u)$ and such m if and only if it is valid for $\tilde{\mathbb{G}}(u)$ and the same value of m. We will be considering $\tilde{\mathbb{G}}(u)$ from now on, with $t > \tau$, where $\tau > 0$ is chosen so that, say: $\|\, u(\cdot,t)\,\|_{L^{\infty}(\mathbb{R})} \leq 1$ for all $t > \tau$ (cf. (2.2) above). Note that, setting $D = \partial/\partial x$,

$$\tilde{\mathbb{G}}(\mathrm{u}) = D[\tilde{B}(\mathrm{u})], \qquad \tilde{B}(\mathrm{u}) = \int_0^{\mathrm{u}} \tilde{b}(\mathrm{v})\, d\mathrm{v}, \tag{2.4b}$$

so that (H2) is valid (for some given m) if we show that

$$\|\, D^m \tilde{B}(u(\cdot,t))\,\| \leq g_m(t)\, \|\, D^{m+1} u(\cdot,t)\,\| \qquad \forall\; t > \tau \tag{2.5}$$

with $g_m(t) \to 0$ as $t \to \infty$. A straightforward computation gives that, for any $m \geq 2$:

$$D^m \tilde{B}(\mathrm{u}) = \tilde{b}(\mathrm{u})\, D^m \mathrm{u} + \sum_{j=1}^{m-1} \binom{m-1}{j} D^j [\, b(\mathrm{u})\,]\, D^{m-j} \mathrm{u} \tag{2.6a}$$

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for all $\mathrm{u} \in H^m(\mathbb{R})$. Similarly, we have

$$D^\ell\, b(\mathrm{u}) \;=\; b'(\mathrm{u})\, D^\ell \mathrm{u} \;+\; \sum_{j=1}^{\ell-1} \binom{\ell-1}{j} D^j\, [\, b'(\mathrm{u})\,]\, D^{\ell-j}\mathrm{u} \qquad (2.6b)$$

for all $\mathrm{u} \in H^\ell(\mathbb{R})$, and any $\ell \geq 2$. It also follows from these expressions that, if $F \in C^m(\mathbb{R})$ and $\mathrm{u} \in W^{m,p}(\mathbb{R})$ for some $1 \leq p \leq \infty$, then $F(\mathrm{u}) \in W^{m,p}(\mathbb{R})$ and

$$\|\, D^m F(\mathrm{u})\,\|_{L^p(\mathbb{R})} \;\leq\; K(m, p; F, M)\, \|\, D^m \mathrm{u}\,\|_{L^p(\mathbb{R})} \qquad (2.7)$$

where the constant K depends on m, p and the quantities $F_\ell = \max\{\, |\, F^{(\ell)}(\mathrm{v})\,|\colon |\,\mathrm{v}\,| < M\,\}$, $1 \leq \ell \leq m$, where $M = \|\,\mathrm{u}\,\|_{L^\infty(\mathbb{R})}$. The estimate (2.7), together with extensions to higher dimensions, was originally obtained by Moser [see ([14], p. 273) or ([10], Lemma 5.1, p. 70)].

2.1. The case $b'(0) = 0$

This is the simplest situation, in which the condition (H2) for $\mathbb{G}(u)$ holds true for all m. To show this fact, we recall the following SNG (Sobolev-Nirenberg-Gagliardo) inequalities: given $\mathrm{u} \in H^{m+1}(\mathbb{R})$, we have

$$\|\, D^\ell \mathrm{u}\,\| \;\leq\; \|\,\mathrm{u}\,\|^{(m-\ell+1)/(m+1)}\, \|\, D^{m+1}\mathrm{u}\,\|^{\ell/(m+1)} \qquad (2.8)$$

and

$$\|\, D^\ell \mathrm{u}\,\|_{L^\infty(\mathbb{R})} \;\leq\; \|\,\mathrm{u}\,\|^{(m-\ell+1/2)/(m+1)}\, \|\, D^{m+1}\mathrm{u}\,\|^{(\ell+1/2)/(m+1)} \qquad (2.9)$$

for all $0 \leq \ell \leq m$. (For both (2.8) and (2.9), the multiplicative constants on the right hand side have been ommitted for the sake of simplicity, as they are not greater than 1.)

Remark 2.1. The estimate (2.8), which is also valid in higher dimensions, is easily shown using Fourier transform. The inequality (2.9) follows from (2.8) and the elementary fact that $\|\,\mathrm{v}\,\|_{L^\infty(\mathbb{R})} \leq \|\,\mathrm{v}\,\|\, \|\, D\mathrm{v}\,\|$ (for any $\mathrm{v} \in H^1(\mathbb{R})$).

In particular, given $\mathrm{u} \in H^2(\mathbb{R})$, we obtain, because $b'(0) = 0$,

$$\|\, D\tilde{B}(\mathrm{u})\,\| \;=\; \|\, \tilde{b}(\mathrm{u})\, D\mathrm{u}\,\| \;\leq\; \|\, \tilde{b}(\mathrm{u})\,\|_{L^\infty(\mathbb{R})} \|\, D\mathrm{u}\,\| \;\leq\; K_1 \|\,\mathrm{u}\,\|^2_{L^\infty(\mathbb{R})} \|\, D\mathrm{u}\,\| \;\leq\; K_1 \|\,\mathrm{u}\,\|^2\, \|\, D^2\mathrm{u}\,\|$$

by (2.8) and (2.9), where $K_1 = \max\{\, |\, b''(\mathrm{v})\,|\colon |\,\mathrm{v}\,| \leq \|\,\mathrm{u}\,\|_{L^\infty(\mathbb{R})}\}/2$. Considering $\mathrm{u} = u(\cdot, t)$, this shows (2.5) for $m = 1$ (since we have $\|\, u(\cdot, t)\,\| \to 0$ as $t \to \infty$), which implies (H2) with $m = 1$. Similarly, the derivation of (H2) for $m = 2$ follows from

$$\| D^2 \tilde{B}(\mathrm{u}) \| \;\leq\; \| \tilde{b}(\mathrm{u}) \|_{L^\infty(\mathbb{R})} \| D^2 \mathrm{u} \| \;+\; \| b'(\mathrm{u}) \|_{L^\infty(\mathbb{R})} \| D\mathrm{u} \|_{L^\infty(\mathbb{R})} \| D\mathrm{u} \|$$

$$\leq K_1 \| \mathrm{u} \|^2_{L^\infty(\mathbb{R})} \| D^2 \mathrm{u} \| \;+\; 2\,K_1 \| \mathrm{u} \|_{L^\infty(\mathbb{R})} \| D\mathrm{u} \|_{L^\infty(\mathbb{R})} \| D\mathrm{u} \| \;\leq\; K_2 \| \mathrm{u} \|^2 \| D^3 \mathrm{u} \|$$

by (2.8) and (2.9), where $K_2 = 3\,K_1$. More generally, the validity of (H2) for (any) $m \geq 3$ follows from (2.10) below, which is similarly obtained using (2.6), (2.7), (2.8) and (2.9).

Lemma 2.1. Given $m \geq 3$, let $\mathrm{u} \in H^{m+1}(\mathbb{R})$ and $b \in C^{m-1}(\mathbb{R})$. If $b'(0) = 0$, then

$$\| D^m \tilde{B}(\mathrm{u}) \| \;\leq\; K_m \| \mathrm{u} \|^2 \| D^{m+1} \mathrm{u} \| \tag{2.10}$$

where $K_m > 0$ depends only on m and the values $B_\ell = \max \{ \,|\, b^{(\ell)}(\mathrm{v}) \,|: \,|\, \mathrm{v} \,| \leq \| \mathrm{u} \|_{L^\infty(\mathbb{R})} \}$, $2 \leq \ell < m$.

Hence, THEOREMS A-E and THEOREMS A′-E′ all apply to the problem (2.1) when $b'(0) = 0$.

2.2. The case $b'(0) \neq 0$

In this case, the condition (H2) holds for $m = 0$, but not (in general) for $m \geq 1$, unless additional assumptions be made. We can still adapt the *proofs* of THEOREMS A-E, A′-E′ and get some partial results in this case, but it seems better to recall that in applications of (2.1) the solution $u(\cdot, t)$ is usually the *density* of some physical quantity, whose total value (or *mass*) is conserved in time and given by

$$\mathcal{M} \;=\; \int_{\mathbb{R}} u(x,t)\, dx \;=\; \int_{\mathbb{R}} u_0(x)\, dx. \tag{2.11}$$

Hence, it is natural for (2.1) to assume that we have $u_0 \in L^1(\mathbb{R})$, with the solution sought in the class $u(\cdot,t) \in C([0,\infty), L^1(\mathbb{R})) \cap L^\infty_{\mathtt{loc}}((0,\infty), L^\infty(\mathbb{R}))$. Some well known properties satisfied by solutions are that, for all $t > 0$,

$$\| u(\cdot,t) \|_{L^1(\mathbb{R})} \leq \| u_0 \|_{L^1(\mathbb{R})} \tag{2.12}$$

and the asymptotic property

$$\lim_{t \to \infty} \| u(\cdot,t) \|_{L^1(\mathbb{R})} \;=\; |\, \mathcal{M} \,| \tag{2.13}$$

(see [28], THEOREM 3.4). Moreover, recalling that, if $u_0 \in L^1(\mathbb{R})$, we have

$$\| u(\cdot,t) \|_{L^\infty(\mathbb{R})} \leq \| u_0 \|_{L^1(\mathbb{R})} \, \nu^{-1/2}\, t^{-1/2} \qquad \forall \; t > 0 \tag{2.14}$$

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(see e.g. ([2], THEOREM 3.2) or ([18], THEOREM 2.1)), it follows form (2.12) that

$$\| u(\cdot,t) \|_{L^2(\mathbb{R})} \leq \| u_0 \|_{L^1(\mathbb{R})} \nu^{-1/4} t^{-1/4} \qquad \forall\ t > 0. \tag{2.15}$$

In particular, regarding the condition

$$\| u(\cdot,t) \|_{L^2(\mathbb{R})} \leq C_0 t^{-\alpha} \qquad \forall\ t > T_0 \tag{2.16}$$

(for some $\alpha > 0$, $T_0 \geq 0$), only the case $\alpha \geq 1/4$ needs to be considered if $u_0 \in L^1(\mathbb{R})$. Writing the equation (2.1) in the form

$$u_t + \mathbb{G}_1(u) + \mathbb{G}_2(u) = \nu u_{xx}, \tag{2.17a}$$

$$\mathbb{G}_1(\mathrm{u}) = b'(0)\,\mathrm{u}\,\mathrm{u}_x, \qquad \mathbb{G}_2(\mathrm{u}) = (b(\mathrm{u}) - b'(0)\mathrm{u})\,\mathrm{u}_x, \tag{2.17b}$$

we know from the previous case that $\mathbb{G}_2(u)$ satisfies the condition (H2) for every $m \geq 0$. Hence, whether or not (H2) is satisfied for some (any) given m in the case of the equation (2.1) depends entirely on the term $\mathbb{G}_1(u)$ alone.

Lemma 2.2. Given $u_0 \in L^1(\mathbb{R})$, let $u(\cdot,t) \in C([0,\infty), L^1(\mathbb{R})) \cap L^\infty_{\mathtt{loc}}((0,\infty), L^\infty(\mathbb{R}))$ be a solution to problem (2.1) satisfying (2.16) above. If $\alpha > 1/4$, or if $\alpha = 1/4$ and $\mathcal{M} = 0$, then $\mathbb{G}_1(u)$ satisfies (H2) for all $m \geq 0$.

Remark 2.2. If (2.16) holds for some $\alpha > 1/4$, then it follows from ([29], THEOREM 2) that the solution mass is necessarily zero, i.e., $\mathcal{M} = 0$ in this case.

Proof of Lemma 2.2: Recalling that $\| \mathrm{v} \|_{L^2} \leq \| \mathrm{v} \|_{L^1}^{1/2} \| \mathrm{v} \|_{L^\infty}^{1/2}$, we get, from (2.8) and (2.9) above, for any $m \geq 0$:

$$\| D^\ell \mathrm{u} \|_{L^2(\mathbb{R})} \leq \| \mathrm{u} \|_{L^1(\mathbb{R})}^{(m-\ell+1)/(m+3/2)} \| D^{m+1} \mathrm{u} \|_{L^2(\mathbb{R})}^{(\ell+1/2)/(m+3/2)} \tag{2.18a}$$

and

$$\| D^\ell \mathrm{u} \|_{L^\infty(\mathbb{R})} \leq \| \mathrm{u} \|_{L^1(\mathbb{R})}^{(m-\ell+1/2)/(m+3/2)} \| D^{m+1} \mathrm{u} \|_{L^2(\mathbb{R})}^{(\ell+1)/(m+3/2)} \tag{2.18b}$$

for all $0 \leq \ell \leq m$. This gives that, for any $m \geq 0$,

$$\| D^j \mathrm{u} \cdot D^{m-j} \mathrm{u} \|_{L^2(\mathbb{R})} \leq \| \mathrm{u} \|_{L^1(\mathbb{R})} \| D^{m+1} \mathrm{u} \|_{L^2(\mathbb{R})} \tag{2.19}$$

for all $0 \leq j \leq m$. Observing that

$$\left| \int_{\mathbb{R}} D^m u(x,t) \cdot D^m \mathbb{G}_1(u(x,t))\, dx \right| \;\leq\; \frac{|\, b'(0)\,|}{2} \,\|\, D^{m+1} u(\cdot,t) \,\|\, \|\, D^m [\, u(\cdot,t)^2 \,] \,\|$$

$$\leq\; \frac{|\, b'(0)\,|}{2} \,\|\, D^{m+1} u(\cdot,t) \,\| \sum_{j=0}^{m} \binom{m}{j} \|\, D^j u(\cdot,t) \cdot D^{m-j} u(\cdot,t) \,\|,$$

we then obtain, from (2.19), that

$$\left| \int_{\mathbb{R}} D^m u(x,t) \cdot D^m \mathbb{G}_1(u(x,t))\, dx \right| \;\leq\; \frac{|\, b'(0)\,|}{2}\, 2^m \,\|\, u(\cdot,t) \|_{L^1(\mathbb{R})} \,\|\, D^{m+1} u(\cdot,t) \|_{L^2(\mathbb{R})}^2. \quad (2.20)$$

This shows that (H2) is valid for $\mathbb{G}_1(u)$, for any m, since $\|\, u(\cdot,t) \,\|_{L^1(\mathbb{R})} \to 0$ as $t \to \infty$. □

Hence, Theorems A-E and Theorems A′-E′ all apply to the problem (2.1) when $u_0 \in L^1(\mathbb{R})$ has zero mass. For $m \geq 1$, from (2.13), (2.20) and the *proofs* of Theorems A-E and A′-E′ we see that the 10 theorems will also be valid for nonzero mass solutions of (2.1) for those values of m such that

$$2^m \,|\, \mathcal{M} \,|\, |\, b'(0) \,| \;<\; 2\nu \quad (2.21)$$

where $\mathcal{M}$ is the solution mass, see (2.11). If condition (2.21) is violated, it appears that lower estimates for $\|\, D^m u(\cdot,t) \,\|$, $m \geq 1$, are not in general valid, even though we have

$$\|\, u(\cdot,t) \,\|_{L^2(\mathbb{R})} \;\geq\; c(0)\, t^{-1/4} \qquad \forall\; t > 0 \quad (2.22)$$

for some $c(0) > 0$ if $\mathcal{M} \neq 0$ [this follows from ([28], Theorem 3.3) and ([29], Theorem 2)]. However, we can still obtain *upper* estimates for $\|\, D^m u(\cdot,t) \,\|_{L^2(\mathbb{R})}$ by adapting the *proofs* of Theorems A and A′, as the following results illustrate.

Theorem 2.1. Let $u_0 \in L^1(\mathbb{R})$ have nonzero mass. Then the solutions of (2.1) satisfy

$$\|\, D^m u(\cdot,t) \,\|_{L^2(\mathbb{R})} \;\leq\; K_m \,\|\, u_0 \,\|_{L^1(\mathbb{R})}\, \nu^{-1/4} \,\mu^{-m/2}\, t^{-1/4 - m/2} \qquad \forall\; t > T_m \quad (2.23)$$

for all $m \geq 0$, where $\mu = \max\{\, \nu,\, \nu^{5/2}/b'(0)^2 \,\}$. Here, the constant K_m depends only on m, and T_m depends on m, ν, $\|\, u_0 \,\|_{L^1(\mathbb{R})}$ and the function $b(\cdot)$ given.

Theorem 2.2. Let $u_0 \in L^2(\mathbb{R})$ be *arbitrary*. Then, for every $m \geq 0$, we necessarily have

$$\lim_{t \to \infty} t^{m/4} \,\|\, D^m u(\cdot,t) \,\|_{L^2(\mathbb{R})} \;=\; 0. \quad (2.24)$$

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Example 3 (*advection-diffusion equations*: $n \geq 2$).

Given a smooth function $\boldsymbol{b} = (b_1, b_2, ..., b_n) \in C^\infty(\mathbb{R})$ and an arbitrary state $u_0 \in L^2(\mathbb{R}^n)$, let $u(\cdot,t) \in C([0,\infty), L^2(\mathbb{R}^n)) \cap L^\infty_{\mathrm{loc}}((0,\infty), L^\infty(\mathbb{R}^n))$ be a solution to the problem

$$u_t + \boldsymbol{b}(u) \cdot \nabla u = \nu \Delta u, \qquad u(\cdot,0) = u_0. \tag{3.1}$$

Under the above conditions, it is known that $\| u(\cdot,t) \| \to 0$ and $t^{n/4} \| u(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} \to 0$ as $t \to \infty$ (see e.g. [2], Theorem 3.3), as well as the general supnorm estimate

$$\| u(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} \leq K_n \| u_0 \| \, \nu^{-n/4} \, t^{-n/4} \qquad \forall \ t > 0 \tag{3.2}$$

for some constant K_n depending only on n (cf. [2], Theorem 3.2, or [18], Theorem 2.1). Moreover, (0.2) is valid with $t_* = 0$, so that there only remains to verify whether the condition (H2) is also satisfied, where $\mathbb{G}(u) = \boldsymbol{b}(u) \cdot \nabla u$ here. For $m = 0$ this is clearly the case, since

$$\int_{\mathbb{R}^n} u(x,t) \, \boldsymbol{b}(u) \cdot \nabla u \, dx = \int_{\mathbb{R}^n} \nabla \cdot \boldsymbol{D}(u(x,t)) \, dx = 0 \tag{3.3}$$

for any $t > 0$, where $\boldsymbol{D}(\mathrm{u}) = \int_0^{\mathrm{u}} \mathrm{v} \, \boldsymbol{b}(\mathrm{v}) \, d\mathrm{v}$, because $u(\cdot,t) \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. For $m \geq 1$, checking (H2) is more involved. We begin by recalling a few basic lemmas.

Lemma 3.1 (*J. Moser*, 1966).
Let $m \geq 1$, $F \in C^m(\mathbb{R})$, with $F(0) = 0$, and let $1 \leq p \leq \infty$ and $\mathrm{u} \in W^{m,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then $F(\mathrm{u}) \in W^{m,p}(\mathbb{R}^n)$ and

$$\| D^m F(\mathrm{u}) \|_{L^p(\mathbb{R}^n)} \leq K(m,n,p) \, \| D^m \mathrm{u} \|_{L^p(\mathbb{R}^n)} \tag{3.4}$$

where $K > 0$ depends only on m, n, p and the values $F_\ell = \sup \{ \, | F^{(\ell)}(\mathrm{v}) | \colon | \mathrm{v} | < \| \mathrm{u} \|_{L^\infty(\mathbb{R}^n)} \}$ for $1 \leq \ell \leq m$, where $F^{(\ell)}$ denotes the derivative of order ℓ of the function F.

Proof: See ([14], p. 273), or ([10], Lemma 5.1, p. 70). □

Given $\mathrm{u} \in \mathbb{R}$, it will be convenient in the sequel to define $\tilde{\boldsymbol{b}}(\mathrm{u}), \tilde{\boldsymbol{B}}(\mathrm{u}) \in \mathbb{R}^n$ given by

$$\tilde{\boldsymbol{b}}(\mathrm{u}) := \boldsymbol{b}(\mathrm{u}) - \boldsymbol{b}(0), \qquad \tilde{\boldsymbol{B}}(\mathrm{u}) := \int_0^{\mathrm{u}} (\boldsymbol{b}(\mathrm{v}) - \boldsymbol{b}(0)) \, d\mathrm{v}, \tag{3.5a}$$

and also

$$\tilde{\mathbb{G}}(u) := (\boldsymbol{b}(u) - \boldsymbol{b}(0)) \cdot \nabla u = \mathbb{G}(u) - \boldsymbol{b}(0) \cdot \nabla u = \nabla \cdot \tilde{\boldsymbol{B}}(u), \tag{3.5b}$$

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where $u = u(x,t)$ is the solution of (3.1) under consideration. Observing that, for any m, (H2) is valid for $\mathbb{G}(u)$ [for such m] if and only it is valid for $\tilde{\mathbb{G}}(u)$ [for that given m], we will from now consider $\tilde{\mathbb{G}}(u)$ instead. The next lemma is obtained by direct computation:

Lemma 3.2.
Let v be a smooth scalar function in $\mathbb{R}^n$, and let $D_\ell = \partial/\partial x_\ell$. Then $D_\ell \tilde{\boldsymbol{B}}(v) = \tilde{\boldsymbol{b}}(v) D_\ell v$ and, for $m \geq 2$:

$$D_{\ell_1} D_{\ell_2} \cdots D_{\ell_m} \tilde{\boldsymbol{B}}(v) \;=\; \tilde{\boldsymbol{b}}(v) D_{\ell_1} D_{\ell_2} \cdots D_{\ell_m} v \;+\; \sum_{j=1}^{m-1} \left\{ \sum_{k=1}^{a(m,j)} \mathbb{D}_{(k)}^{j} \tilde{\boldsymbol{b}}(v) \cdot \mathbb{D}_{(k)}^{m-j} v \right\} \tag{3.6}$$

where $a(m,j) = (m-1)!/(j!(m-1-j)!)$ and, for each k, $\mathbb{D}_{(k)}^{j} = D_{k_1} D_{k_2} \cdots D_{k_j}$ and $\mathbb{D}_{(k)}^{m-j} = D_{k_{j+1}} D_{k_{j+2}} \cdots D_{k_m}$ (with $\{k_1, k_2, ..., k_m\} = \{\ell_1, \ell_2, ..., \ell_m\}$).

Lemma 3.3.
Let $n \geq 2$, $m \geq 1$, and let $F, G \in C^m(\mathbb{R})$ with $F(0) = 0$, $G(0) = 0$. Let $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2 \in \mathbb{Z}^n$ be (nonnegative) multi-indices such that $|\boldsymbol{\lambda}_1| + |\boldsymbol{\lambda}_2| = m$. Then, for any $v \in H^{m+1}(\mathbb{R}^n) \cap H^{n-2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we have $D^{\boldsymbol{\lambda}_1} F(v) \cdot D^{\boldsymbol{\lambda}_2} G(v) \in L^2(\mathbb{R}^n)$ and

$$\| D^{\boldsymbol{\lambda}_1} F(v) \cdot D^{\boldsymbol{\lambda}_2} G(v) \|_{L^2(\mathbb{R}^n)} \leq\; C(m,n) \, \| v \|_{L^2(\mathbb{R}^n)}^{1/2} \| D^{n-2} v \|_{L^2(\mathbb{R}^n)}^{1/2} \| D^{m+1} v \|_{L^2(\mathbb{R}^n)} \tag{3.7}$$

where $C(m,n)$ depends only on m, n and the values $F_\ell = \sup\{ |F^{(\ell)}(\mathrm{v})| \colon |\mathrm{v}| < \|v\|_{L^\infty(\mathbb{R}^n)} \}$ and $G_\ell = \sup\{ |G^{(\ell)}(\mathrm{v})| \colon |\mathrm{v}| < \|v\|_{L^\infty(\mathbb{R}^n)} \}$ for $1 \leq \ell \leq m$. Here, $F^{(\ell)}(\cdot)$ and $G^{(\ell)}(\cdot)$ denote the ℓth-order derivatives of the functions F and G, respectively.

Remark 3.1. The assumption that $v \in L^\infty(\mathbb{R}^n)$ in Lemma 3.3 is needed when $n = 4$ and $m = 1$, but is redundant in the other cases, since the inclusion $H^{m+1}(\mathbb{R}^n) \cap H^{n-2}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ holds for $m \geq 1$ except in the single case $m = 1$, $n = 4$.

Proof: This is shown similarly to ([3], Lemma 3.1), using (3.4), standard SNG inequalities (see e.g. [6], Theorem 9.3) and the Moser supnorm estimates

$$\| \mathrm{v} \|_{L^\infty(\mathbb{R}^n)} \leq\; K(n) \, \| D^{k-1} \mathrm{v} \|_{L^2(\mathbb{R}^n)}^{1/2} \| D^{k+1} \mathrm{v} \|_{L^2(\mathbb{R}^n)}^{1/2} \tag{3.8a}$$

if $n = 2k$, and

$$\| \mathrm{v} \|_{L^\infty(\mathbb{R}^n)} \leq\; K(n) \, \| D^{k} \mathrm{v} \|_{L^2(\mathbb{R}^n)}^{1/2} \| D^{k+1} \mathrm{v} \|_{L^2(\mathbb{R}^n)}^{1/2} \tag{3.8b}$$

if $n = 2k+1$, see e.g. ([24], Ch. 13, Proposition 3.8). □

Lemma 3.4.
Let $u(\cdot,t) \in C([0,\infty), L^2(\mathbb{R}^n)) \cap L^\infty_{\text{loc}}((0,\infty), L^\infty(\mathbb{R}^n))$ solve the problem (3.1). Then, for every $m \geq 1$, there exists $K(m,n,\nu)$ constant, depending only on m, n, ν, the function $\boldsymbol{b}$ and the size of $\| u(\cdot,t) \|_{L^\infty(\mathbb{R}^n)}$ at $t = 1$ (say), such that

$$\| D^m u(\cdot,t) \|_{L^2(\mathbb{R}^n)} \leq K(m,n,\nu) \, \| u_0 \|_{L^2(\mathbb{R}^n)} \qquad \forall \; t > m. \tag{3.9}$$

Proof: From the equation (3.1), we have

$$\| u(t) \|^2 + 2\nu \int_0^t \| Du(\tau) \|^2 \, d\tau \;=\; \| u_0 \|^2 \tag{3.10a}$$

for all $t > 0$. In particular, $\displaystyle\int_0^1 \| Du(\tau) \|^2 \, d\tau \leq (2\nu)^{-1} \| u_0 \|^2$, so that there exists $a_1 \in (0,1)$ such that

$$\| Du(a_1) \|^2 \;\leq\; \frac{1}{2} \, \| u_0 \|^2 \, \nu^{-1}. \tag{3.10b}$$

This gives, from the equation (3.1),

$$\| Du(t) \|^2 + 2\nu \int_{a_1}^t \| D^2 u(\tau) \|^2 \, d\tau \;\leq\; \| Du(a_1) \|^2 + 2 \int_{a_1}^t \| D^2 u(\tau) \| \, \| D [\, \tilde{\boldsymbol{B}}(u) \,] \| \, d\tau$$

for all $t > a_1$. Applying (3.4), Lemma 3.1, we then get

$$\| Du(t) \|^2 + \nu \int_{a_1}^t \| D^2 u(\tau) \|^2 \, d\tau \;\leq\; \| Du(a_1) \|^2 + \frac{K_1}{\nu} \int_{a_1}^t \| Du(\tau) \|^2 \, d\tau \tag{3.10c}$$

for $t > a_1$. Recalling (3.10a) and (3.10b), this shows (3.9) with $m = 1$, and also gives that

$$\int_1^t \| D^2 u(\tau) \|^2 \, d\tau \;\leq\; C(n,\nu) \, \| u_0 \|^2 \tag{3.10d}$$

for all $t > 1$, and some constant $C(n,\nu) > 0$ that depends also on $\boldsymbol{b}$ and $\| u(\cdot,1) \|_{L^\infty(\mathbb{R}^n)}$. In particular, we can find $a_2 \in (1,2)$ such that

$$\| D^2 u(a_2) \|^2 \;\leq\; C(n,\nu) \, \| u_0 \|^2 \tag{3.10e}$$

from which we can consider the energy estimate

$$\| D^2 u(t) \|^2 + 2\nu \int_{a_2}^t \| D^3 u(\tau) \|^2 \, d\tau \;\leq\; \| D^2 u(a_2) \|^2 + 2 \int_{a_2}^t \| D^3 u(\tau) \| \, \| D^2 [\, \tilde{\boldsymbol{B}}(u) \,] \| \, d\tau$$

and proceed in a similar way to obtain (3.9) with $m = 2$, and then $m = 3$, and so on. □

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Theorem 3.1.
Let $n \geq 2$. Then, for the solutions of (3.1), the condition (H2) is satisfied for all $m \geq 0$.

Proof: In fact, given $t \geq 1$, $m \geq 1$, and $\ell_1, \ell_2, ..., \ell_m \in \{1, 2, ..., n\}$, we have, from (3.5),

$$\left| \int_{\mathbb{R}^n} D_{\ell_1} \cdots D_{\ell_m} u(x,t) \cdot D_{\ell_1} \cdots D_{\ell_m} \mathbb{G}(u(x,t))\, dx \right| \leq \| D^{m+1} u(\cdot,t) \| \, \| D^m [\, \tilde{\boldsymbol{B}}(u(\cdot,t)) \,] \|$$
$$\leq C(m,n) \, \| u(\cdot,t) \|^{1/2} \, \| D^{n-2} u(\cdot,t) \|^{1/2} \, \| D^{m+1} u(\cdot,t) \|^2$$

using (3.6), Lemma 3.2, and (3.7), Lemma 3.3. (Here, $C(m,n)$ denotes some constant that depends on m, n and the values $b_\ell = \max \{ | \boldsymbol{b}^{(\ell)}(\mathrm{v}) | \colon | \mathrm{v} | \leq \| u(\cdot,1) \|_{L^\infty(\mathbb{R}^n)} \}$, $1 \leq \ell \leq m$.) From Lemma 3.4, this gives the result. □

Therefore, assuming that we have

$$\| u(\cdot,t) \| \leq C_0 \, t^{-\alpha} \qquad \forall \; t > T_0 \tag{3.11}$$

for some constants $\alpha, C_0, T_0 > 0$, the following result can be obtained from Theorem A:

Theorem 3.2 (*upper estimates for derivatives*).
If (3.11) holds, then: for every $m > 0$, there exists $C_m > 0$ (depending only on m, α, C_0) such that

$$\| D^m u(\cdot,t) \| \leq C_m \, \nu^{-m/2} \, t^{-\alpha - m/2} \qquad \forall \; t > T_m \tag{3.12}$$

for some $T_m > T_0$ that can be chosen to depend only on m, ν, C_0, T_0 and α.

Next, consider the reverse condition

$$\| u(\cdot,t) \| \geq c(0) \, t^{-\eta} \qquad \forall \; t > t_0 \tag{3.13}$$

for some constants $c(0), t_0, \eta > 0$. We then obtain (from Theorem B′ and Theorem C′):

Theorem 3.3 (*lower estimates for derivatives*).
Let (3.11) and (3.13) be valid, for some given $0 < \alpha \leq \eta$. Then, for every $m > 0$, there exists $c(m) > 0$ (depending only on $m, \alpha, c(0)$ and C_0) such that, setting $q = \eta/\alpha$:

$$\| D^m u(\cdot,t) \| \geq c(m) \, \nu^{-m/2} \, t^{-\eta - mq/2} \qquad \forall \; t > t_m \tag{3.14}$$

for some $t_m > t_0$ that can be chosen to depend only on $m, \nu, t_0, T_0, c(0), C_0, \alpha$ and η.

Example 4 (*incompressible Navier-Stokes equations*).

Let $2 \leq n \leq 4$. Given $\boldsymbol{u}_0 \in L^2_\sigma(\mathbb{R}^n)$, let $\boldsymbol{u}(\cdot,t) \in C_{\mathtt{w}}([0,\infty), L^2_\sigma(\mathbb{R}^n)) \cap L^2((0,\infty), \dot{H}_1(\mathbb{R}^n))$ be any given Leray solution to the Navier-Stokes equations

$$\boldsymbol{u}_t \,+\, \boldsymbol{u}\cdot\nabla\boldsymbol{u} \,+\, \nabla p \,=\, \nu\,\Delta\boldsymbol{u}, \tag{4.1a}$$

$$\nabla\cdot\boldsymbol{u} \,=\, 0, \tag{4.1b}$$

with $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0$. The existence of such solutions was originally shown by J. Leray in his 1934 seminal paper (see [12], p. 241), together with the property (0.2) as well ([12], p. 246). The validity of the condition (H2) for every m follows from ([3], Lemma 3.1). Therefore, assuming that we have

$$\|\,\boldsymbol{u}(\cdot,t)\,\| \,\leq\, C_0\,t^{-\,\alpha} \qquad \forall\; t > T_0 \tag{4.2}$$

for some constants $\alpha, C_0, T_0 > 0$, the following result can be obtained from Theorem A:

Theorem 4.1 (*upper estimates for derivatives*).
If (4.2) holds, then: for every $m > 0$, there exists $C_m > 0$ (depending only on m, α, C_0) such that

$$\|\,D^m\boldsymbol{u}(\cdot,t)\,\| \,\leq\, C_m\,\nu^{-\,m/2}\,t^{-\,\alpha\,-\,m/2} \qquad \forall\; t > T_m \tag{4.3}$$

for some $T_m > T_0$ sufficiently large. Moreover, T_m can be chosen to depend only on m, ν, C_0, T_0 and α. If $n = 3$ or 4, T_m can also be chosen to depend only on m, ν, T_0 and $\|\,\boldsymbol{u}_0\,\|$.

Now, consider the reverse condition

$$\|\,\boldsymbol{u}(\cdot,t)\,\| \,\geq\, c(0)\,t^{-\,\eta} \qquad \forall\; t > t_0 \tag{4.4}$$

for some constants $c(0), t_0, \eta > 0$. We then obtain (from Theorem B′ and Theorem C′):

Theorem 4.2 (*lower estimates for derivatives*).
Let (4.2) and (4.4) be valid, for some given $0 < \alpha \leq \eta$. Then, for every $m > 0$, there exists $c(m) > 0$ (depending only on $m, \alpha, c(0)$ and C_0) such that, setting $q = \eta/\alpha$:

$$\|\,D^m\boldsymbol{u}(\cdot,t)\,\| \,\geq\, c(m)\,\nu^{-\,m/2}\,t^{-\,\eta\,-\,m\,q/2} \qquad \forall\; t > t_m \tag{4.5}$$

for some $t_m > t_0$ sufficiently large. Moreover, t_m can be chosen to depend only on m, ν, t_0, T_0, $c(0)$, C_0, α and η.

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Other results can be similarly obtained. For example, rewriting the equation (4.1) as

$$\boldsymbol{u}_t \,=\, \nu\,\Delta\boldsymbol{u} \,+\, \boldsymbol{f}, \qquad \boldsymbol{f} \,=\, -\,\boldsymbol{u}\cdot\nabla\boldsymbol{u} \,-\, \nabla p, \tag{4.6}$$

we have, by (4.3), that $\boldsymbol{f}$ satisfies the assumption (H3) for all m with $\beta = 2\alpha + (n+2)/4$. We then have the following result (from Theorem B and Theorem C):

Theorem 4.3 (*lower componentwise estimates*: $\eta = \alpha$).
Let the solution $\boldsymbol{u} = (u_1, u_2, ..., u_n)$ satisfy the condition (4.2) above for some $\alpha > 0$. If

$$\|\,u_i(\cdot,t)\,\| \,\geq\, c(0)\,t^{-\alpha} \qquad \forall\; t > t_0 \tag{4.7a}$$

for some $1 \leq i \leq n$ (and some $c(0), t_0 > 0$), then: for every $m \geq 1$, there exists $c(m) > 0$ (depending only on m, $\alpha, c(0)$ and C_0) such that

$$\|\,D^m u_i(\cdot,t)\,\| \,\geq\, c(m)\,\nu^{-m/2}\,t^{-\alpha-m/2} \qquad \forall\; t > t_m \tag{4.7b}$$

with t_m depending only on $m, \alpha, \nu, t_*, t_0, c(0), C_0$ and T_0.

Likewise, assuming (4.2) and that we had $\|\,(u_i,u_j)(\cdot,t)\,\| \geq c(0)\,t^{-\alpha}$ instead of (4.7a), it would have been obtained that $\|\,D^m(u_i,u_j)(\cdot,t)\,\| \,\geq\, c(m)\,\nu^{-m/2}\;t^{-\alpha-m/2}$ for $t > t_m$, for all m, applying Theorem B again.

Theorem 4.4 (*lower componentwise estimates*: $\eta > \alpha$).
Let the solution $\boldsymbol{u} = (u_1, u_2, ..., u_n)$ satisfy (4.2) for some $\alpha > 0$, and let $\eta = q\,\alpha$, where $1 < q < (3\alpha + n/4 + 1/2)/(2\alpha + 1)$. If

$$\|\,u_i(\cdot,t)\,\| \,\geq\, c(0)\,t^{-\eta} \qquad \forall\; t > t_0 \tag{4.8a}$$

for some $1 \leq i \leq n$ (and some $c(0), t_0 > 0$), then there is some $c(1) > 0$ (depending only on $\alpha, c(0)$ and C_0) such that

$$\|\,D u_i(\cdot,t)\,\| \,\geq\, c(1)\,\nu^{-1/2}\,t^{-\eta-q/2} \qquad \forall\; t > t_1 \tag{4.8b}$$

with t_1 depending only on $q, \alpha, \nu, t_*, t_0, c(0), C_0$ and T_0.

By Theorem C, we similarly have: if $q < (3\alpha + n/4 + 3/2)/(2\alpha + 2)$, then (4.2) and (4.8$a$) will give (4.8$b$) and also that $\|\,D^2 u_i(\cdot,t)\,\| \,\geq\, c(2)\,\nu^{-1}\;t^{-\eta-q}$ for all $t \gg 1$ large, and so forth.

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As we already mentioned, upper and lower bounds can be pushed both ways, depending on what is available. Illustrating with the present case of the Navier-Stokes, equations, this is illustrated by the following results, which can be derived as with THEOREMS A-E:

Theorem 4.5. Let $2 \leq n \leq 4$, $\alpha \geq 0$, and let $\boldsymbol{u}(\cdot,t)$ be any given Leray solution to (4.1).
(*i*) If, as $t \to \infty$, we have $\| D^{\ell} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha - \ell/2})$ for some $\ell \geq 0$, then we actually have $\| D^{m} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha - m/2})$ for all $m \geq 0$ (and $t \gg 1$).

(*ii*) If $\| D^{\ell} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = o(t^{-\alpha - \ell/2})$ as $t \to \infty$ for some $\ell \geq 0$, then we will have, as $t \to \infty$, that $\| D^{m} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = o(t^{-\alpha - m/2})$ for all $m \geq 0$.

Theorem 4.6. Let $2 \leq n \leq 4$, $\alpha \geq 0$, $c > 0$, and $\boldsymbol{u}(\cdot,t)$ any given Leray solution to (4.1).
(*i*) If we have, for $t \gg 1$, that $\| D^{\ell} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \geq c\, t^{-\alpha - \ell/2}$ for some $\ell \geq 0$, then we will also have $\| D^{m} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \geq c_m\, t^{-\alpha - m/2}$ for every $0 \leq m \leq \ell$ (and $t \gg 1$), for some appropriate constants $c_m > 0$.

(*ii*) If, for $t \gg 1$, $\| D^{k} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha - k/2})$ and $\| D^{\ell} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \geq c\, t^{-\alpha - \ell/2}$ for some particular pair $k, \ell \geq 0$, then we will have, as $t \to \infty$:
$\| D^{m} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \geq c_m\, t^{-\alpha - m/2}$ for all $m \geq 0$ (for some appropriate constants $c_m > 0$).

In a similar way, other classes of decay can be studied using the methods in this report, by easily adapting the argument. In the case of exponential decay, for example, we can derive the following results in much the same way as THEOREMS 4.5 and 4.6 above.

Theorem 4.7. Let $2 \leq n \leq 4$, $\kappa > 0$, and $\boldsymbol{u}(\cdot,t)$ any Leray solution to the system (4.1).
(*i*) If, as $t \to \infty$, we have $\| D^{\ell} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(e^{-\kappa t})$ for some $\ell \geq 0$, then we actually have $\| D^{m} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(e^{-\kappa t})$ for all $m \geq 0$ (and $t \gg 1$).

(*ii*) If $\| D^{\ell} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = o(e^{-\kappa t})$ as $t \to \infty$ for some $\ell \geq 0$, then we will also have, as $t \to \infty$, that $\| D^{m} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = o(e^{-\kappa t})$ for all $m \geq 0$.

Theorem 4.8. Let $2 \leq n \leq 4$, $\kappa > 0$, $c > 0$, and $\boldsymbol{u}(\cdot,t)$ any given Leray solution to (4.1).
(*i*) If, for $t \gg 1$, we have $\| D^{\ell} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \geq c\, e^{-\kappa t}$ for some $\ell \geq 0$, then, for $t \gg 1$: we will have $\| D^{m} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \geq c_m\, e^{-\kappa t}$ for every $0 \leq m \leq \ell$, with constants $c_m > 0$.

(*ii*) If we have, for $t \gg 1$: $\| D^{k} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(e^{-\kappa t})$ and $\| D^{\ell} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \geq c\, e^{-\kappa t}$ for some particular pair $k, \ell \geq 0$, then we will have (for $t \gg 1$):
$\| D^{m} \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} \geq c_m\, e^{-\kappa t}$ for all $m \geq 0$, for some appropriate constants $c_m > 0$.

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Example 5 (*incompressible MHD equations*).

Let $2 \leq n \leq 4$. Given $\boldsymbol{u}_0 \in L^2_\sigma(\mathbb{R}^n)$ and $\boldsymbol{b}_0 \in L^2_\sigma(\mathbb{R}^n)$, let $(\boldsymbol{u}, \boldsymbol{b})(\cdot, t) \in C_{\mathtt{w}}([0, \infty), L^2_\sigma(\mathbb{R}^n))$ $\cap\, L^2((0, \infty), \dot{H}^1(\mathbb{R}^n))$ be any given Leray solution to the MHD equations

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \mu \Delta \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{5.1a}$$

$$\boldsymbol{b}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{b} = \nu \Delta \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{u}, \qquad \nabla \cdot \boldsymbol{b} = 0, \tag{5.1b}$$

with $\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0$, $\boldsymbol{b}(\cdot, 0) = \boldsymbol{b}_0$. In (5.1), we have $\boldsymbol{f} = \boldsymbol{0}$ and

$$\mathbb{G}(\boldsymbol{u}, \boldsymbol{b}) = \begin{pmatrix} \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \boldsymbol{b} \cdot \nabla \boldsymbol{b} \\ \boldsymbol{u} \cdot \nabla \boldsymbol{b} - \boldsymbol{b} \cdot \nabla \boldsymbol{u} \end{pmatrix}, \tag{5.2}$$

so that (H2) is valid for every m, by ([3], Lemma 3.1). Therefore, assuming that we have

$$\| (\boldsymbol{u}, \boldsymbol{b})(\cdot, t) \| \leq C_0\, t^{-\alpha} \qquad \forall\; t > T_0 \tag{5.3}$$

for some $\alpha, C_0, T_0 > 0$, the following result can be obtained from Theorems A and A′:

Theorem 5.1 (*upper estimates for derivatives*).
If (5.3) is valid, then: for every $m > 0$, there exists $C_m > 0$ constant (depending only on m, α and C_0) such that

$$\| D^m \boldsymbol{u}(\cdot, t) \| \leq C_m\, \mu^{-m/2}\, t^{-\alpha - m/2} \qquad \forall\; t > T_m \tag{5.4a}$$

and

$$\| D^m \boldsymbol{b}(\cdot, t) \| \leq C_m\, \nu^{-m/2}\, t^{-\alpha - m/2} \qquad \forall\; t > T_m \tag{5.4b}$$

for some $T_m > 0$ that depends on m, α, μ, ν, C_0 and T_0.

Proof: From the proof of Theorem A′ we obtain, from (5.1) and (5.3) above,

$$\| D^m (\boldsymbol{u}, \boldsymbol{b})(\cdot, t) \| \leq C'_m\, \gamma^{-m/2}\, t^{-\alpha - m/2} \qquad \forall\; t > T'_m$$

for all m, where $\gamma = \min\{\mu, \nu\}$, for some constants C'_m (depending on m, α, C_0) and T'_m (depending on $m, \alpha, \mu, \nu, C_0, T_0$). Therefore, by (5.1$a$), we can write

$$\boldsymbol{u}_t = \mu \Delta \boldsymbol{u} + \boldsymbol{f}, \qquad \boldsymbol{f} = \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla p$$

where $\boldsymbol{f}$ satisfies (H3) for all m with $\beta = 2\,\alpha + (n+2)/4$. Hence, recalling Theorem A, we obtain (5.4a) above. Proceeding similarly with the equation (5.1b), we get (5.4b). □

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Now, consider the reverse conditions

$$\| \boldsymbol{u}(\cdot,t) \| \;\geq\; c(0)\, t^{-\eta} \quad \text{and} \quad \| \boldsymbol{b}(\cdot,t) \| \;\geq\; \hat{c}(0)\, t^{-\eta} \qquad \forall\; t > t_0 \tag{5.5}$$

for some given $c(0), \hat{c}(0), t_0, \eta > 0$. The following result is a consequence of Theorem A′ (or Theorem 5.1) and Theorem B above.

Theorem 5.2 (*lower estimates for derivatives*: *the case* $\eta = \alpha$).
Let (5.3) and (5.5) be valid with $\eta = \alpha > 0$. Then, for every $m > 0$, there exist $c(m) > 0$ (depending only on $m, \alpha, c(0), C_0$) and $\hat{c}(m) > 0$ (depending only on $m, \alpha, \hat{c}(0), C_0$) such that

$$\| D^m \boldsymbol{u}(\cdot,t) \| \;\geq\; c(m)\, \mu^{-m/2}\, t^{-\alpha - m/2} \qquad \forall\; t > t_m \tag{5.6a}$$

and

$$\| D^m \boldsymbol{b}(\cdot,t) \| \;\geq\; \hat{c}(m)\, \nu^{-m/2}\, t^{-\alpha - m/2} \qquad \forall\; t > t_m \tag{5.6b}$$

for some $t_m > 0$ that depends on $m, \alpha, \mu, \nu, c(0), \hat{c}(0), C_0, t_0$ and T_0.

Proof: From (5.1) and Theorem A′ (or Theorem 5.1), we can write

$$\boldsymbol{u}_t \;=\; \mu \Delta \boldsymbol{u} \,+\, \boldsymbol{f}, \qquad \boldsymbol{f} \;=\; \boldsymbol{b}\cdot\nabla \boldsymbol{b} \;-\; \boldsymbol{u}\cdot\nabla\boldsymbol{u} \;-\; \nabla p, \tag{5.7a}$$

$$\boldsymbol{b}_t \;=\; \nu \Delta \boldsymbol{b} \,+\, \boldsymbol{g}, \qquad \boldsymbol{g} \;=\; \boldsymbol{b}\cdot\nabla \boldsymbol{u} \;-\; \boldsymbol{u}\cdot\nabla\boldsymbol{b}, \tag{5.7b}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy (H3) for all m with $\beta = 2\alpha + (n+2)/4$. From Theorem B above, we then get (5.6a) and (5.6b), as claimed. □

Remark 5.1: Writing (5.1) in the form (5.7) brings out more clearly the basic decoupling between the equations (5.1a) and (5.1b) for $t \gg 1$. Thus, for example, assuming (5.3) and that

$$\| \boldsymbol{u}(\cdot,t) \| \;\geq\; c(0)\, t^{-\alpha} \qquad \forall\; t > t_0 \tag{5.8}$$

only (i.e., no lower estimate assumed for $\| \boldsymbol{b}(\cdot,t) \|$), for some given $c(0), t_0 > 0$, from (5.7a) we still get that the lower estimate (5.6a) for $D^m \boldsymbol{u}(\cdot,t)$ will hold for any m, and so forth. A similar decoupling is seen on the individual components of $\boldsymbol{u}(\cdot,t)$ or $\boldsymbol{b}(\cdot,t)$ as well. Thus, for example, if we have (5.3) and, say, only that

$$\| b_i(\cdot,t) \| \;\geq\; \hat{c}(0)\, t^{-\alpha} \qquad \forall\; t > t_0 \tag{5.9}$$

for some particular value of i, then we get from (5.7b) that (5.6b) will be valid for $b_i(\cdot,t)$.

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Applying THEOREM C to (5.7) gives additional results, as illustrated by THEOREM 5.3.

Theorem 5.3 (*lower estimates for derivatives*: *the case* $\eta > \alpha$).
Let (5.3) and (5.5) be valid with $\eta = q\alpha$, where $1 < q < (3\alpha + (n+2)/4)/(2\alpha+1)$. Then, there exist $c(1) > 0$ (depending on α, $c(0), C_0$) and $\hat{c}(1) > 0$ (depending on α, $\hat{c}(0), C_0$) such that

$$\| D\boldsymbol{u}(\cdot,t) \| \;\geq\; c(1)\, \mu^{-1/2}\, t^{-\eta - q/2} \qquad \forall\; t > t_1 \tag{5.10a}$$

and

$$\| D\boldsymbol{b}(\cdot,t) \| \;\geq\; \hat{c}(1)\, \nu^{-1/2}\, t^{-\eta - q/2} \qquad \forall\; t > t_1 \tag{5.10b}$$

for some $t_1 > 0$ that depends on $\alpha, q, \mu, \nu, c(0), \hat{c}(0), C_0, t_0$ and T_0.

Finally, we look at the MHD equations in the form (5.1), that is, for the coupled solution pair $(\boldsymbol{u},\boldsymbol{b})(\cdot,t)$, assuming (5.3) above and the jointly condition

$$\| (\boldsymbol{u},\boldsymbol{b})(\cdot,t) \| \;\geq\; c(0)\, t^{-\eta} \qquad \forall\; t > t_0 \tag{5.11}$$

for some constants $c(0)$, t_0, $\eta > 0$. The following result is a consequence of THEOREM C$'$ (or, more precisely, of its proof):

Theorem 5.4 (*lower estimates for derivatives*: $\eta \geq \alpha$).
Let (5.3) and (5.11) hold with $0 < \alpha \leq \eta$. Then, for every $m > 0$, there exists $c(m) > 0$ (depending only on m, α, $c(0)$ and C_0) such that, setting $q = \eta/\alpha$:

$$\| D^m (\boldsymbol{u},\boldsymbol{b})(\cdot,t) \| \;\geq\; c(m)\, \gamma^{-m/2}\, t^{-\eta - mq/2} \qquad \forall\; t > t_m \tag{5.12}$$

for some $t_m > t_0$, where $\gamma = \max\{\mu,\nu\}$, with t_m depending on $m, \alpha, q, \mu, \nu, c(0), C_0, t_0, T_0$.

Moreover, results similar to THEOREMS 4.5 - 4.8 above can be obtained for the MHD equations as well, as illustrated by the next result.

Theorem 5.5. Let $2 \leq n \leq 4$, $\alpha \geq 0$, and $(\boldsymbol{u},\boldsymbol{b})(\cdot,t)$ any given Leray solution to (5.1).

(*i*) If, as $t \to \infty$, we have $\| D^\ell (\boldsymbol{u},\boldsymbol{b})(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha - \ell/2})$ for some $\ell \geq 0$, then we actually have $\| D^m (\boldsymbol{u},\boldsymbol{b})(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha - m/2})$ for all $m \geq 0$ (and $t \gg 1$).

(*ii*) If, as $t \to \infty$, we have $\| D^k (\boldsymbol{u},\boldsymbol{b})(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha - k/2})$ for some $k \geq 0$, and also $\| D^\ell (\boldsymbol{u},\boldsymbol{b})(\cdot,t) \|_{L^2(\mathbb{R}^n)} \geq c_\ell\, t^{-\alpha - \ell/2}$ for some $\ell \geq 0$, and some constant $c_\ell > 0$, then we will have: $\| D^m (\boldsymbol{u},\boldsymbol{b})(\cdot,t) \|_{L^2(\mathbb{R}^n)} \geq c_m\, t^{-\alpha - m/2}$ for all $m \geq 0$ (and some constants $c_m > 0$).

Example 6 (*incompressible micropolar flows in* $\mathbb{R}^n$, $2 \le n \le 3$).

Consider the micropolar equations for $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\mathbf{w} = (\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3)$ in $\mathbb{R}^3$:

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = (\mu + \chi)\Delta \boldsymbol{u} + 2\chi \nabla \wedge \mathbf{w}, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{6.1a}$$

$$\mathbf{w}_t + \boldsymbol{u} \cdot \nabla \mathbf{w} = \nu \Delta \mathbf{w} + \nabla(\nabla \cdot \mathbf{w}) - 4\chi \mathbf{w} + 2\chi \nabla \wedge \boldsymbol{u}, \tag{6.1b}$$

where μ, χ, ν are positive constants [5, 13]. Given $\boldsymbol{u}_0 \in L^2_\sigma(\mathbb{R}^3)$, $\mathbf{w}_0 \in L^2(\mathbb{R}^3)$, let $(\boldsymbol{u}, \mathbf{w})$ be (any) Leray solution of (6.1a)-(6.1b) having $\boldsymbol{u}_0, \mathbf{w}_0$ as initial data, and let

$$\boldsymbol{\varepsilon}(x,t) = \mathbf{w}(x,t) - \frac{1}{2} \nabla \wedge \boldsymbol{u}(x,t), \tag{6.2}$$

so that the equation (6.1a) can be written as

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \mu \Delta \boldsymbol{u} + \boldsymbol{f}, \qquad \boldsymbol{f} = 2\chi \nabla \wedge \boldsymbol{\varepsilon}. \tag{6.3}$$

For the system (6.1)-(6.2), as well as for the equation (6.3), the validity of condition (H2) for every m follows again from ([3], Lemma 6.1). Assuming that we have

$$\| \boldsymbol{u}(\cdot,t) \| \le C_0\, t^{-\alpha} \qquad \forall\ t > T_0 \tag{6.4}$$

for some constants $C_0, T_0, \alpha > 0$ (and no further assumption on $\mathbf{w}(\cdot,t)$), it follows from the proof of Theorem A$'$ that

$$\| \mathbf{w}(\cdot,t) \| \le \hat{C}_0\, t^{-\alpha - 1/2} \qquad \forall\ t > \hat{T}_0 \tag{6.5}$$

for some $\hat{C}_0, \hat{T}_0$ that depend on μ, χ, ν, C_0, with $\hat{T}_0$ also depending on T_0. (The bound (6.5) was originally obtained in [7, 17].) It then follows the estimates (6.6) given below (as in the proof of Theorems A or A$'$), see also [4, 7, 17].

Theorem 6.1 (*upper estimates for derivatives*).
If (6.4) holds, then: for every $m \ge 0$, there exists $C_m > 0$ (depending on $m, \alpha, C_0, \mu, \chi, \nu$) such that, setting $\gamma = \min\{\mu, \nu\}$,

$$\| D^m \boldsymbol{u}(\cdot,t) \| \le C_m\, \gamma^{-m/2}\, t^{-\alpha - m/2} \qquad \forall\ t > T_m, \tag{6.6a}$$

$$\| D^m \mathbf{w}(\cdot,t) \| \le C_m\, \gamma^{-1/2 - m/2}\, t^{-\alpha - 1/2 - m/2} \qquad \forall\ t > T_m, \tag{6.6b}$$

for some $T_m > 0$ that depends on $m, \mu, \chi, \nu, \alpha, C_0, T_0$.

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This result can be improved by considering the equation (6.3). From (6.1b) and (6.6) it can be shown (see [8]) that, if $\boldsymbol{u}(\cdot,t)$ satisfies (6.4) for some $\alpha > 0$, then

$$\| D^m \boldsymbol{f}(\cdot,t) \| \;\leq\; F_m \, t^{-\alpha - (m+5)/2} \qquad \forall \; t > \sigma_m \tag{6.7}$$

for all $m \geq 0$, with F_m depending on m, α and C_0, while σ_m depends on all the parameters. Therefore, by Theorem A, we obtain the following result.

Theorem 6.1′ (*upper estimates for derivatives*).
If (6.4) holds, then: for every $m \geq 0$, there exists $C_m > 0$ (depending on m, α, C_0) such that

$$\| D^m \boldsymbol{u}(\cdot,t) \| \;\leq\; C_m \, \mu^{-m/2} \, t^{-\alpha - m/2} \qquad \forall \; t > T_m, \tag{6.8a}$$

$$\| D^m \mathbf{w}(\cdot,t) \| \;\leq\; C_m \, \mu^{-1/2 - m/2} \, t^{-\alpha - 1/2 - m/2} \qquad \forall \; t > T_m, \tag{6.8b}$$

for some $T_m > 0$ that depends on $m, \mu, \chi, \nu, \alpha, C_0, T_0$.

Now, consider the reverse condition

$$\| \boldsymbol{u}(\cdot,t) \| \;\geq\; c(0) \, t^{-\eta} \qquad \forall \; t > t_0 \tag{6.9}$$

for some constants $c(0), t_0, \eta > 0$. (Again, no conditions need to be imposed on $\| \mathbf{w}(\cdot,t) \|$.) The following result can then be obtained by applying Theorem B to the equation (6.3).

Theorem 6.2 (*lower estimates for derivatives*: $\eta = \alpha$).
Ley (6.4) and (6.9) be valid, where $\eta = \alpha$. Then, for every $m \geq 0$, there exists $c(m) > 0$ (depending on m, α, $c(0)$ and C_0) such that

$$\| D^m \boldsymbol{u}(\cdot,t) \| \;\geq\; c(m) \, \mu^{-m/2} \, t^{-\alpha - m/2} \qquad \forall \; t > t_m, \tag{6.10a}$$

$$\| D^m \mathbf{w}(\cdot,t) \| \;\geq\; c(m) \, \mu^{-1/2 - m/2} \, t^{-\alpha - 1/2 - m/2} \qquad \forall \; t > t_m, \tag{6.10b}$$

for some $t_m > t_0$ that depends on $m, \mu, \chi, \nu, \alpha, c(0), t_0, C_0, T_0$.

In a similar way, we can apply Theorem C to the equation (6.3) and derive lower bound estimates (under the assumptions (6.4) and (6.9)) in the case $\eta > \alpha$, or we can use Theorems D and/or E to generate results similar to Theorems 4.5 and 4.6, and so forth. However, due to the special features of the equations (6.1), these and other properties will be better left to a separated treatment in a future work [8].

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Example 7 (*inverse Wiegner's theorem*: Navier-Stokes equations, $n \geq 2$).

Let $\boldsymbol{u}(\cdot,t) \in C_{\mathtt{w}}([0,\infty), L^2(\mathbb{R}^n)) \cap L^2((0,\infty), \dot{H}^1(\mathbb{R}^n))$ be any given Leray solution to the Navier-Stokes equations

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p \;=\; \Delta\boldsymbol{u}, \tag{7.1a}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) \;=\; 0, \tag{7.1b}$$

$$\boldsymbol{u}(\cdot,0) \;=\; \boldsymbol{u}_0 \in L^2_\sigma(\mathbb{R}^n) \tag{7.1c}$$

(in dimension $n \geq 2$), and let $\boldsymbol{v}(\cdot,t)$ be the solution in the space $C([0,\infty), L^2_\sigma(\mathbb{R}^n))$ of the associated Stokes problem

$$\boldsymbol{v}_t \;=\; \Delta\boldsymbol{v}, \qquad \boldsymbol{v}(\cdot,0) \;=\; \boldsymbol{u}_0 \tag{7.2}$$

where $\boldsymbol{u}_0 \in L^2_\sigma(\mathbb{R}^n)$ is given in (7.1c) above. Our goal is to show the following result:

Theorem 7.1 (Inverse Wiegner). If $\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha})$ with $0 < \alpha \leq (n+2)/4$, then we also have $\|\,\boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha})$.

This result in the case $n \geq 3$ is an almost immediate corollary of the (direct) Wiegner's theorem (given in [25], p. 305), which says: if we have $\|\,\boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} = O(t^{-\beta})$ for some $0 \leq \beta \leq (n+2)/4$, then we will also have $\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} = O(t^{-\beta})$ and, in addition:

$$\|\,\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \;=\; \begin{cases} O(t^{-2\beta-(n-2)/4}) & \text{se } \; 0 \leq \beta < \frac{1}{2} \\ O(t^{-(n+2)/4}\log t) & \text{se } \; \beta = \frac{1}{2} \\ O(t^{-(n+2)/4}) & \text{se } \; \frac{1}{2} < \beta \leq \frac{n+2}{4} \end{cases} \tag{7.3}$$

Remark 7.1: In dimension $n = 2, 3$, an alternative proof of Wiegner's (direct) theorem was found independently in [11, 26]. The derivation in [11, 26] is actually much simpler than Wiegner's original proof. The inverse Wiegner's theorem above was first obtained by Z. Skalák in dimension $n = 3$ (see [20], Theorem 3.1) using a very elaborated argument. The proof of Theorem 7.1 given here is *much* simpler and is based on monotonicity ideas if $n = 2$. For $n \geq 3$, Theorem 7.1 is a trivial consequence of (7.3), cf. discussion below.

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$$n = 2$$

Letting $\boldsymbol{\theta}(\cdot,t) := \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)$, we have that $\boldsymbol{\theta}(\cdot,t)$ satisfies: $\boldsymbol{\theta}(\cdot,0) = \mathbf{0}$ and

$$\boldsymbol{\theta}_t \,=\, \Delta\boldsymbol{\theta} \,-\, \boldsymbol{f}(\cdot,t), \tag{7.4a}$$

$$\boldsymbol{f}(\cdot,t) \,=\, \boldsymbol{u}(\cdot,t)\cdot\nabla\boldsymbol{u}(\cdot,t) \,+\, \nabla p(\cdot,t) \,=\, \mathbb{P}_H[\,\boldsymbol{u}(\cdot,t)\cdot\nabla\boldsymbol{u}(\cdot,t)\,] \tag{7.4b}$$

(where $\mathbb{P}_H$: $L^2(\mathbb{R}^n) \to L^2_\sigma(\mathbb{R}^n)$ denotes the LERAY-HELMHOLTZ PROJECTOR).

Let then $\boldsymbol{u}_0 \in L^2_\sigma(\mathbb{R}^2)$ be given such that the corresponding Leray solution of (7.1) [which, in dimension $n = 2$, is regular for $t > 0$ (and unique)] satisfies

$$\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} \,=\, O(t^{-\alpha}) \tag{7.5}$$

for some $0 < \alpha \leq (n+2)/4 = 1$ (or, equivalently, that we have: $\|\,\boldsymbol{u}(\cdot,t)\,\| = O(1+t)^{-\alpha}$). Recalling THEOREM A, it follows from (7.5) that

$$\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} \,=\, O(t^{-\alpha-\frac{1}{2}}). \tag{7.6}$$

Before proceeding, it will be convenient to recall the estimates (7.7) next:

Lemma 7.1. *For every $s > 0$, we have, letting $\boldsymbol{Q}(\cdot,t) = \mathbb{P}_H[\,\boldsymbol{u}(\cdot,t)\cdot\nabla\boldsymbol{u}(\cdot,t)\,]$:*

$$\|\,e^{\Delta(t-s)}\boldsymbol{Q}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)} \leq K_1\,(t-s)^{-1/2}\,\|\,\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\,\|\,D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)} \tag{7.7a}$$

and

$$\|\,e^{\Delta(t-s)}\boldsymbol{Q}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)} \,\leq\, K_2\,(t-s)^{-1}\,\|\,\boldsymbol{u}(\cdot,s)\,\|^2_{L^2(\mathbb{R}^2)} \tag{7.7b}$$

for every $t > s$, where $K_1 = (8\pi)^{-1/2}$ and $K_2 = (4\pi e)^{-1/2}$.

Proof: This follows directly from HEAT KERNEL properties (see e.g. [19], THEOREM 3, p. 4) or, if preferred, using the FOURIER TRANSFORM (se e.g. [11], p. 236, or [26], p. 1227). □

Remark 7.2: Lemma 7.1 has a version for $n = 3$ or 4 as well, but for $s \in (0,\infty) \setminus E$, where $E \subset \mathbb{R}$ is a bounded set of zero measure. However, this lemma will only be needed (for the proof of WIEGNER'S INVERSE THEOREM given here) in case of dimension $n = 2$.

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Going back to (7.4) above, we then have

$$\begin{aligned} \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t) \;&=\; -\int_0^t e^{\Delta(t\,-\,s)} \Big[\, \mathbb{P}_H [\, \boldsymbol{u} \cdot \nabla \boldsymbol{u}\,](\cdot,s) \Big]\, ds \\ &=\; -\int_0^t \mathbb{P}_H \Big[\, e^{\Delta(t\,-\,s)} [\, \boldsymbol{u} \cdot \nabla \boldsymbol{u}\,](\cdot,s) \Big]\, ds, \end{aligned}$$

so that

$$\begin{aligned} \|\, \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} &\leq \int_0^t \Big\|\, \mathbb{P}_H \Big[\, e^{\Delta(t\,-\,s)} [\, \boldsymbol{u} \cdot \nabla \boldsymbol{u}\,](\cdot,s) \Big]\, \Big\|_{L^2(\mathbb{R}^2)} ds \\ &\leq \int_0^t \Big\|\, e^{\Delta(t\,-\,s)} [\, \boldsymbol{u} \cdot \nabla \boldsymbol{u}\,](\cdot,s)\, \Big\|_{L^2(\mathbb{R}^2)} ds \end{aligned}$$

$$= \int_0^{t/2} \|\, e^{\Delta(t\,-\,s)} [\, \boldsymbol{u} \cdot \nabla \boldsymbol{u}\,](\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\, ds \;+ \int_{t/2}^t \|\, e^{\Delta(t\,-\,s)} [\, \boldsymbol{u} \cdot \nabla \boldsymbol{u}\,](\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\, ds$$

$$\leq \int_0^{t/2} (t-s)^{-1}\, \|\, \boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}^2\, ds \;+ \int_{t/2}^t (t-s)^{-1/2}\, \|\, \boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\, \|\, D\boldsymbol{u}(\cdot,s)\,\|_{L^2(\mathbb{R}^2)}\, ds$$

for every $t > 0$, where in the last step LEMMA 7.1 was used. Hence, by (7.5) and (7.6):

$$\|\, \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} = O(t^{-1}) \int_0^{t/2} (1+s)^{-2\alpha}\, ds \;+\; O(t^{-2\alpha-\frac{1}{2}}) \int_{t/2}^t (t-s)^{-1/2}\, ds \qquad (7.8)$$

In the case $0 < \alpha < 1/2$, it follows from (7.8) that

$$\|\, \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} = O(t^{-1})\, (1+t)^{-2\alpha+1} + O(t^{-2\alpha-\frac{1}{2}})\, t^{1/2} \;=\; O(t^{-2\alpha}), \qquad (7.9a)$$

so that

$$\begin{aligned} \|\, \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} \;&\leq\; \|\, \boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} + \|\, \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} \\ &=\; O(t^{-\alpha}) \,+\, O(t^{-2\alpha}) \;=\; O(t^{-\alpha}) \end{aligned} \qquad (7.9b)$$

if $\alpha \in (0, 1/2)$. In the case $\alpha = 1/2$, we get from (7.8) that

$$\|\, \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} = O(t^{-1}) \log{(1+t)} + O(t^{-2\alpha-\frac{1}{2}})\, t^{1/2} = O(t^{-1}) \log{(1+t)} \quad (7.10a)$$

and so

$$\begin{aligned} \|\, \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} \;&\leq\; \|\, \boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} + \|\, \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^2)} \\ &=\; O(t^{-\frac{1}{2}}) \,+\, O(t^{-1}) \log{(1+t)} \;=\; O(t^{-\frac{1}{2}}) \end{aligned} \qquad (7.10b)$$

if $\alpha = 1/2$. Finally, in the remaining case $1/2 < \alpha \leq (n+2)/4 = 1$, (7.8) gives

$$\| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^2)} = O(t^{-1}) + O(t^{-2\alpha - \frac{1}{2}})\, t^{1/2} = O(t^{-1}), \tag{7.11a}$$

so that

$$\begin{aligned} \| \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^2)} &\leq \| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^2)} + \| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^2)} \\ &= O(t^{-\alpha}) + O(t^{-1}) = O(t^{-\alpha}) \end{aligned} \tag{7.11b}$$

if $1/2 < \alpha \leq 1$. Therefore, for all values $0 < \alpha \leq 1$ it has been true that $\| \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^2)} = O(t^{-\alpha})$, as claimed, and we had (by (7.9*a*), (7.10*a*) and (7.11*a*) above):

$$\| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t) \| = \begin{cases} O(t^{-2\alpha - (n-2)/4}) & \text{if } \; 0 < \alpha < \frac{1}{2} \\ O(t^{-(n+2)/4} \log t) & \text{if } \; \alpha = \frac{1}{2} \\ O(t^{-(n+2)/4}) & \text{if } \; \frac{1}{2} < \alpha \leq \frac{n+2}{4} \end{cases} \tag{7.12}$$

(in accordance with (7.3)).

Remark 7.3: The key ingredient of the argument above was the validity of (7.6), which turns out to be an immediate consequence of the assumption (7.5) in view of Theorem A. A direct derivation of (7.6) from (7.5) in the spirit of Section 0 (Theorems A-E), that is, exploring simple monotonicity properties of the relevant L^2 norms involved, is very easy to provide, due to the the well known monotonicity of $\| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^2)}$ in the interval $(0, \infty)$. The following argument is adapted from ([11], p. 235): from the basic energy inequality (which is actually an equality if $n = 2$) satisfied by the solution $\boldsymbol{u}(\cdot,t)$, we have

$$\begin{aligned} \| \boldsymbol{u}(\cdot,t/2) \|_{L^2(\mathbb{R}^2)}^2 &\overset{(1)}{\geq} \| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^2)}^2 + 2 \int_{t/2}^{t} \| D\boldsymbol{u}(\cdot,\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau \\ &\geq 2 \int_{t/2}^{t} \| D\boldsymbol{u}(\cdot,\tau) \|_{L^2(\mathbb{R}^2)}^2 \, d\tau \\ &\geq 2 \, \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^2)}^2 \int_{t/2}^{t} 1 \, d\tau \qquad [\text{by monotonicity}] \end{aligned}$$

for every $t > 0$, that is,

$$t \, \| D\boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^2)}^2 \leq \| \boldsymbol{u}(\cdot,t/2) \|_{L^2(\mathbb{R}^2)}^2 \leq K\,(1 + t/2)^{-2\alpha} \qquad [\text{by (7.5)}]$$

for some fixed constant $K > 0$ (by (7.5)), which shows (7.6), as claimed. □

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$$\boxed{\; n = 3 \;}$$

We now consider the case $n = 3$: assuming that we have

$$\| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^3)} \;=\; O(t^{-\alpha}) \tag{7.13}$$

(for some $0 < \alpha \leq (n+2)/4 = 5/4$), let us then show that we will also have

$$\| \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^3)} \;=\; O(t^{-\alpha}) \tag{7.14}$$

(where $\boldsymbol{v}(\cdot,t)$ is the solution to problem (7.2)). Since $\| \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^3)} = o(1)$, we can apply (7.3) with $\beta = 0$ to obtain

$$\| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^3)} \;=\; O(t^{-\frac{1}{4}}). \tag{7.15}$$

In particular,

$$\begin{aligned} \| \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^3)} &\leq \| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^3)} + \| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^3)} \\ &= O(t^{-\alpha}) + O(t^{-\frac{1}{4}}) \;=\; O(t^{-\gamma}) \end{aligned} \tag{7.16}$$

where $\gamma = \min\{\alpha, 1/4\}$. If $\alpha \leq 1/4$, then (7.14) already follows; the remaining cases for α will be considered in the sequel below.

Case I: $1/4 < \alpha < 1/2$.

In this case, by (7.16) it follows that $\| \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^3)} = O(t^{-\frac{1}{4}})$, so that (7.3) gives (with $n = 3$, $\beta = 1/4$):

$$\| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^3)} \;=\; O(t^{-\frac{3}{4}}). \tag{3.17a}$$

In particular,

$$\begin{aligned} \| \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^3)} &\leq \| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^3)} + \| \boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^3)} \\ &= O(t^{-\alpha}) + O(t^{-\frac{3}{4}}) \;=\; O(t^{-\alpha}). \end{aligned} \tag{3.17b}$$

Remark 7.4: applying (17.3) means that we are using Wiegner's (direct) theorem to obtain the inverse Wiegner's theorem as a consequence of the former.

Case II: $\alpha = 1/2$.

In this case, by (7.16) it follows that $\|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = O(t^{-\frac{1}{4}})$, so that, by (7.3) again, we obtain

$$\|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = O(t^{-\frac{3}{4}}). \tag{7.18a}$$

In particular,

$$\begin{aligned} \|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\leq \|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} + \|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \\ &= O(t^{-\frac{1}{2}}) + O(t^{-\frac{3}{4}}) = O(t^{-\frac{1}{2}}). \end{aligned} \tag{7.18b}$$

Case III: $1/2 < \alpha < 5/4$.

Since, in particular, we have $\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = O(t^{-\frac{1}{2}})$ in this case, it then follows from Case II above that $\|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = O(t^{-\frac{1}{2}})$. Hence, by (1.3) with $\beta = \frac{1}{2}$, we have:

$$\|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = O(t^{-\frac{5}{4}}\log t). \tag{7.19a}$$

In particular,

$$\begin{aligned} \|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\leq \|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} + \|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \\ &= O(t^{-\alpha}) + O(t^{-\frac{5}{4}}\log t) = O(t^{-\alpha}). \end{aligned} \tag{7.19b}$$

Case IV: $\alpha = 5/4$.

Since, in particular, we have $\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = O(t^{-1})$ in this case, it follows from Case III that $\|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = O(t^{-1})$, so that, by (7.3) with $\beta = 1$, we have:

$$\|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = O(t^{-\frac{5}{4}}). \tag{7.20a}$$

In particular,

$$\begin{aligned} \|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\leq \|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} + \|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \\ &= O(t^{-\frac{5}{4}}) + O(t^{-\frac{5}{4}}) = O(t^{-\frac{5}{4}}), \end{aligned} \tag{7.20b}$$

which completes the proof of (7.14). □

Remark 7.5: as we can see from the argument above, there was no real need to consider the cases $1/4 < \alpha < 1/2$ and $\alpha = 1/2$ separately.

$$n = 4$$

Having $\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^4)} = O(t^{-\alpha})$ for some $0 < \alpha \leq (n+2)/4 = 3/2$, using (7.3) with $\beta = 0$ e $n = 4$ we obtain, observing that $(n-2)/4 = 1/2$,

$$\|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} = O(t^{-1/2}). \tag{7.21}$$

In particular,

$$\begin{aligned} \|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} &\leq \|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^4)} + \|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} \\ &= O(t^{-\alpha}) + O(t^{-\frac{1}{2}}) = O(t^{-\gamma}) \end{aligned} \tag{7.22}$$

where $\gamma = \min\{\alpha, 1/2\}$. Hence, in the case $\alpha \leq 1/2$ we have $\|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} = O(t^{-\alpha})$ and the result is obtained. For $\alpha > 1/2$, then it is at least known that $\|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} = O(t^{-1/2})$, so that we have, by (7.3):

$$\|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} = O(t^{-3/2}\log t), \tag{7.23}$$

and so

$$\begin{aligned} \|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} &\leq \|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^4)} + \|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} \\ &= O(t^{-\alpha}) + O(t^{-\frac{3}{2}}\log t). \end{aligned} \tag{7.24}$$

Case I: $1/2 < \alpha < 3/2$.

Having $\alpha < 3/2$, it follows directly from (7.24) that $\|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} = O(t^{-\alpha})$, as was to be shown.

Case II: $\alpha = 3/2$.

In this case, it follows from (7.24) that $\|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} = O(t^{-1})$. From (7.3) with $\beta = 1$ and $n = 4$, we get $\|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} = O(t^{-3/2})$, so that

$$\begin{aligned} \|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} &\leq \|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^4)} + \|\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^4)} \\ &= O(t^{-\frac{3}{2}}) + O(t^{-\frac{3}{2}}) = O(t^{-\frac{3}{2}}). \end{aligned} \tag{7.25}$$

$n \geq 5$

Having $\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha})$ for some $0 < \alpha \leq (n+2)/4$, using (7.3) with $\beta = 0$ and $n \geq 5$ we obtain

$$\|\,\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} = O(t^{-(n-2)/4}). \tag{7.26}$$

In particular,

$$\begin{aligned} \|\,\boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} &\leq \|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} + \|\,\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \\ &= O(t^{-\alpha}) + O(t^{-\frac{n-2}{4}}) = O(t^{-\gamma}) \end{aligned} \tag{7.27}$$

where $\gamma = \min\{\alpha, (n-2)/4\}$. Thus, in the case $\alpha \leq (n-2)/4$, we have that $\|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha})$, as was to be shown. If $\alpha > (n-2)/4$, it follows from (7.27) that

$$\|\boldsymbol{v}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = O(t^{-\frac{n-2}{4}}). \tag{7.28}$$

Since $(n-2)/4 > 1/2$, applying (7.3) with $\beta = (n-2)/4$ we obtain, in this case,

$$\|\,\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} = O(t^{-(n+2)/4}), \tag{7.29}$$

so that

$$\begin{aligned} \|\,\boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} &\leq \|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} + \|\,\boldsymbol{u}(\cdot,t) - \boldsymbol{v}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \\ &= O(t^{-\alpha}) + O(t^{-\frac{n+2}{4}}) = O(t^{-\alpha}). \end{aligned} \tag{7.30}$$

This completes the proof of Theorem 7.1. In a similar way, we can prove the following generalization to Leray solutions of the Navier-Stokes equations with external forces,

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \Delta\boldsymbol{u} + \boldsymbol{f}(\cdot,t), \qquad \nabla\cdot\boldsymbol{u}(\cdot,t) = 0 \tag{7.31}$$

(in dimension $n \geq 2$), with $\boldsymbol{f}(\cdot,t) \in L^1((0,\infty), L^2_\sigma(\mathbb{R}^n))$ satisfying the condition

$$\|\,\boldsymbol{f}(\cdot,t)\,\|_{L^2(\mathbb{R}^n)} \leq C_f\,(1+t)^{-\alpha-1} \quad \text{e} \quad \|\,\boldsymbol{f}(\cdot,t)\,\|_{L^n(\mathbb{R}^n)} \leq K_n\,t^{-\alpha-\frac{n+2}{4}} \tag{7.32}$$

for all $t > 0$ (and some $0 < \alpha \leq (n+2)/4$). The associated linear problem is now $\boldsymbol{v}_t = \Delta\boldsymbol{v} + \boldsymbol{f}(\cdot,t)$, with $\boldsymbol{v}(\cdot,0) = \boldsymbol{u}(\cdot,0) \in L^2_\sigma(\mathbb{R}^n)$, and Wiegner's inverse theorem reads

Theorem 7.2 (INVERSE WIEGNER). If $\| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha})$ with $0 < \alpha \leq (n+2)/4$, then we will also have $\| \boldsymbol{v}(\cdot,t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha})$, provided that $\boldsymbol{f}(\cdot,t)$ satisfies (7.32).

The proof is similar to the derivation of THEOREM 7.1 given above.

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