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Article

Impulsive control and synchronization for fractional-order hyper-chaotic financial system

Xinggui Li [1] Ruofeng Rao[1]* Shouming Zhong[2] Xinsong Yang[3] Hu Li[1] Yulin Zhang [1]

[1] Department of Mathematics, Chengdu Normal University, Chengdu 611130, China; 061034@cdnu.edu.cn; ruofengrao@cdnu.edu.cn; 063029@cdnu.edu.cn; 061033@cdnu.edu.cn
[2] College of mathematics, University of Electronic Science and technology of China, Chengdu 611731, China; zhongsm@uestc.edu.cn
[3] College of Electronics and Information Engineering, Sichuan University, Chengdu 610065, China; xinsongyang@scu.edu.cn; xinsongyang@163.com

Abstract: This paper reports a new global Mittag-Leffler synchronization criterion with regard to fractional-order hyper-chaotic financial systems by designing the suitable impulsive control and the state feedback controller. The significance of this impulsive synchronization lies in the fact that the backward economic system can synchronize asymptotically with the advanced economic system under the effective impulse macroeconomic management means. Matlab LMI-toolbox is utilized to deduce the feasible solution in numerical example, which shows the effectiveness of the proposed methods. It is worth mentioning that the LMI-based criterion usually requires the activation function of the system to be Lipschitz, but the activation function in this paper is fixed and truly nonlinear, which cannot be assumed to be Lipschitz continuous. This is another mathematical difficulty overcome in this paper.

Keywords: Mittag-Leffler stability; Caputo fractional-order derivative; non-Lipschitz continuity; hyper-chaotic financial system; Mittag-Leffler function

MSC: 34D06; 34A37

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1. Introduction

The hyperchaotic financial mathematical model takes into account the average profit margin, which can better simulate the actual complex and changeable financial market, and has attracted the attention of researchers ([1-5]). As what has been pointed out in [1,6], an extension of fractality concepts in the investigation of the economic systems has been used. In the fields of mathematics and its applications to physics and engineering, numerous researchers have adopted fractional calculus as an effective method to model and simulate various nonlinear system ([1,7,8]). For example, in [1], a fractional-order hyperchaotic financial system was investigated and the adaptive control scheme for synchronization and chaos suppression of the fractional-order economic systems was proposed. However, impulsive control is not considered in [1]. In fact, impulsive control is always one of the means of macroeconomic management ([2,3]). In [2], a synchronization criterion of hyperchaotic financial system was derived by using impulsive control method and differential mean value theorem. The authors in [3] utilized impulse, Lapalcian semigroup and fixed point theorems to stabilize globally hyperchaotic financial system. But the results in [2,3] are only applicable to integer order differential equation models, and the effective methods used in [2,3] are not suitable to fractional-order hyperchaotic financial mathematical model. Indeed, fractional-order financial system always involves Mittag-Leffler stability, which is different from those of [2,3]. Besides, Riemann-Liouville fractional-order derivative was studied in [1], and so in this paper we consider Caputo fractional-order derivative. To overcome the above-mentioned mathematical difficulties, we shall design a suitable controller and utilize impulsive control to achieve drive-response synchronization. The significance of this impulsive synchronization lies in the fact that the backward economic system can synchronize asymptotically with the advanced economic system under the effective impulse macroeconomic management means.

2. Preliminaries

Generally speaking, there are three common fractional derivatives, including the Grunwald-Letnikov fractional derivative, Riemann-Liouville fractional derivative, as well as the Caputo fractional derivative.

In this paper, the Caputo fractional derivative is considered.

Definition 1.([9]) The fractional integral of order χ for a function $u(t) \in C[[0,+\infty),\mathbb{R}]$ is defined by

$$D^{-\chi}u(t) = \frac{1}{\Gamma(\chi)}\int_0^t (t-s)^{\chi-1}u(s)ds,$$

where $\chi > 0$, and $\Gamma(\chi) = \int_0^{+\infty} e^{-t}t^{\chi-1}dt$.

Definition 2.([9]) The Caputo fractional derivative of order χ for a function $u(t) \in C^{n+1}[[0,+\infty),\mathbb{R}]$ (the set of all n-order continuous differentiable functions on $[0,+\infty]$) is defined as

$$D^{\chi}u(t) = \frac{1}{\Gamma(n-\chi)}\int_0^t (t-s)^{n-\chi-1}u^{(n)}(s)ds,$$

where $\chi > 0$, n is the first integer greater than χ, that is, $n-1 < \chi < n$, and the Laplace transform of $D^{\chi}u(t)$ is given as

$$\mathcal{L}\{D^{\chi}u(t)\} = s^{\chi}\mathcal{H}(s) - \sum_{k=0}^{n-1} s^{\chi-k-1}u^{(k)}(0),$$

where $\mathcal{H}(s) = \mathcal{L}\{u(t)\}$ stands for the Laplace transform of $u(t)$. In particular, when $0 < \chi < 1$, one obtains that

$$D^{\chi}u(t) = \frac{1}{\Gamma(1-\chi)}\int_0^t (t-s)^{-\chi}u'(s)ds,$$

$$\mathcal{L}\{D^{\chi}u(t)\} = s^{\chi}\mathcal{H}(s) - s^{\chi-1}u(0).$$

Definition 3.([9]) The one-parameter and the two-parameter Mittag-Leffler function are defined as

$$E_{\chi}(w) = \sum_{k=0}^{\infty}\frac{w^k}{\Gamma(\chi k+1)}, \quad \chi > 0,\ w \in \mathbb{C},$$

$$E_{\chi,\beta}(w) = \sum_{k=0}^{\infty}\frac{w^k}{\Gamma(\chi k+\beta)}, \quad \chi > 0,\ \beta > 0,\ w \in \mathbb{C},$$

respectively, and the Laplace transform of the two-parameter Mittag-Leffler function is

$$\mathcal{L}[t^{\beta-1}E_{\chi,\beta}(-\gamma t^{\chi})] = \frac{s^{\chi-\beta}}{s^{\chi}+\gamma}, \quad t \geqslant 0,\ Re(s) > |\gamma|^{\frac{1}{\chi}},$$

56 where $\gamma \in \mathbb{R}$ and $Re(s)$ denotes the real parts of s.

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Consider the following fractional-order hyper-chaotic financial system as the drive system:

$$\begin{cases} D^\alpha x = -x \cdot a + (y \cdot x + u + z), \\ D^\alpha y = 1 - y \cdot b - x^2, \\ D^\alpha z = -z \cdot c - x, \\ D^\alpha u = -u \cdot k - x \cdot d \cdot y, \end{cases} \tag{1}$$

which can be written in compact form

$$D^\alpha X(t) = AX(t) + f(X(t)), \quad t \geqslant 0, \tag{2}$$

where $X = (x_1, x_2, x_3, x_4)^T = (x, y, z, u)^T$,

$$A = \begin{pmatrix} -a & 0 & 1 & 1 \\ 0 & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ 0 & 0 & 0 & -k \end{pmatrix}, \quad f(X) = (f_1(X), f_2(X), f_3(X), f_4(X))^T = \begin{pmatrix} x_1 x_2 \\ -x_1^2 + 1 \\ 0 \\ -d x_1 x_2, \end{pmatrix}. \tag{3}$$

and the initial value is of

$$X(0) = X_0. \tag{4}$$

Here, the parameters a, b, c, d and variables x, y, z, u can be referenced in detail in the literature ([1-5]),

Consider the corresponding response system as follows,

$$\begin{cases} D^\alpha Y(t) = AY(t) + f(Y(t)) + W(t), & t \geqslant 0, t \neq t_k, \\ Y(t_k^+) = Y(t_k^-) + g(Y(t_k) - X(t_k)), & k = 1, 2, \cdots \\ Y(0) = Y_0, \end{cases} \tag{5}$$

where $W(t) = (w_1(t), w_2(t), w_3(t), w_4(t))^T$, and each $w_i(t)$ is a controller to be designed. $0 < t_1 < t_2 < \cdots$, and each $t_k (k \in \mathbb{Z}_+)$ represents a fixed impulsive instant, and $Y(t_k^-) = \lim_{t \to t_k^-} Y(t) = Y(t_k)$, $Y(t_k^+) = \lim_{t \to t_k^+} Y(t)$, for all $k = 1, 2, \cdots$.

Set $e(t) = Y(t) - X(t)$, then the systems (1) and (3) synchronization error model is

$$\begin{cases} D^\alpha e(t) = Ae(t) + F(e(t)) + W(t), & t \geqslant 0, t \neq t_k, \\ e(t_k^+) = e(t_k^-) + g(e(t_k)), & k = 1, 2, \cdots \\ e(0) = Y_0 - X_0, \end{cases} \tag{6}$$

where

$$F(e) = f(Y) - f(X). \tag{7}$$

Design the following state feedback controller

$$W(t) = M(X(t) - Y(t)), \tag{8}$$

where M is a constant matrix.

In our study, the following assumption conditions are useful:

(H1) $|x_i(t)| \leqslant m_i$, and $|y_i(t)| \leqslant m_i$, for all $i = 1, 2$.

(H2) $g(s) = Ls$, where $s \in \mathbb{R}^4$, and L is a constant matrix.

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3. Main result

In this section, we present the LMI-based synchronization criterion for chaotic financial systems:

Theorem 1. Assume the conditions (H1)-(H2) hold. If there exists a positive definite symmetric matrix P, and positive real number ε_1 such that

$$PA + A^T P - PM - M^T P + \varepsilon_1^{-1} I + \varepsilon_1 H^T H < 0, \tag{9}$$

$$(I + L)^T P (I + L) < P, \tag{10}$$

then the system (3) achieves global Mittag-Leffler synchronization with System (1), where I is the identity matrix, and

$$H = \begin{pmatrix} m_2 & m_1 & 0 & 0 \\ m_1 + m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ dm_2 & dm_1 & 0 & 0, \end{pmatrix}.$$

Proof. Firstly, it follows from (H1) and (3) that

$$|F(e(t))| \leqslant H|e(t)|,$$

Consider the Lyapunov function as follows,

$$V(t, e(t)) = e^T(t) P e(t),$$

then

$$\begin{aligned} D^\alpha V(t, e(t)) \leqslant & e^T(t) \left(PA + A^T P - PM - M^T P \right) e(t) + [e^T(t) F(e(t)) + F^T(e(t)) e(t)] \\ \leqslant & e^T(t) \left(PA + A^T P - PM - M^T P + \varepsilon_1^{-1} I + \varepsilon_1 H^T H \right) e(t) \\ \leqslant & qV(t, e(t)), \quad t \geqslant 0, t \neq t_k, \end{aligned} \tag{11}$$

where

$$q = \lambda_{\max} \left(PA + A^T P - PM - M^T P + \varepsilon_1^{-1} I + \varepsilon_1 H^T H \right) < 0.$$

Obviously, there exists a function $\nu(t, e(t)) = D^\alpha V(t, e(t))$, and so there is a function $\mu(t) \geqslant 0$ such that

$$\mu(t) + \nu(t, e(t)) = qV(t, e(t)), \quad t \geqslant 0, t \neq t_k, \tag{12}$$

Denote by $*$ the convolution operator, then it follows by taking the Laplace transform and inverse Laplace transform of (12) that

$$V(t, e(t)) = V(0, e(0)) E_\alpha(qt^\alpha) - \mu(s) * (t^{\alpha-1} E_{\alpha,\alpha}(qt^\alpha)), \quad t \geqslant 0, t \neq t_k, \tag{13}$$

which together with $\mu \geqslant 0$ means

$$V(t, e(t)) \leqslant V(0, e(0)) E_\alpha(qt^\alpha), \quad \forall t \in (t_{k-1}, t_k]. \tag{14}$$

On the other hand, (14) and (10) yield

$$\begin{aligned} V(t_k^+, e(t_k^+)) = & [(I + L) e(t_k)]^T P [(I + L) e(t_k)] \\ = & e^T(t_k) (I + L)^T P (I + L) e(t_k) \\ \leqslant & e^T(t_k) P e(t_k) \leqslant V(0, e(0)) E_\alpha(qt^\alpha), \quad \forall k = 1, 2, \cdots, \end{aligned} \tag{15}$$

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which together with (14) implies

$$\lambda_{\min}(P)\|e(t)\|^2 \leqslant V(t,e(t)) \leqslant V(0,e(0))E_\alpha(qt^\alpha), \quad \forall t \geqslant 0,$$

or

$$\|e(t)\| \leqslant \sqrt{\frac{1}{\lambda_{\min}(P)}V(0,e(0))E_\alpha(qt^\alpha)}, \quad \forall t \geqslant 0.$$

In consideration of $q < 0$, the above inequality means that the null solution of System (6) is globally Mittag-Leffler stable. This completes the proof.

4. Numerical example

Example 1. Equip System (1) and System (3) with the following data

$$L = \begin{pmatrix} -0.0111 & 0.0001 & 0 & 0 \\ 0.0001 & -0.0112 & 0 & 0 \\ 0 & 0 & -0.0113 & 0 \\ 0 & 0 & 0 & -0.0111 \end{pmatrix}, M = \begin{pmatrix} 2.11 & 0.0001 & 0 & 0 \\ 0.0001 & 2.2 & 0 & 0 \\ 0 & 0 & 2.3 & 0 \\ 0 & 0 & 0 & 2.1 \end{pmatrix}.$$

Let $a = 0.9, b = 0.2, c = 1.5, d = 0.2, k = 0.17, m_1 = 8, m_2 = 9$, then

$$A = \begin{pmatrix} -0.9 & 0 & 1 & 1 \\ 0 & -0.2 & 0 & 0 \\ -1 & 0 & -1.5 & 0 \\ 0 & 0 & 0 & -0.17 \end{pmatrix}, \quad f(X) = (f_1(X), f_2(X), f_3(X), f_4(X))^T = \begin{pmatrix} x_1x_2 \\ -x_1^2+1 \\ 0 \\ -0.2x_1x_2, \end{pmatrix}.$$

$$H = \begin{pmatrix} 9 & 8 & 0 & 0 \\ 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.8 & 1.6 & 0 & 0 \end{pmatrix}.$$

Using Computer Matlab LMI-toolbox to solve LMI conditions (9) and (10) results in the following feasibility data:

$$P = \begin{pmatrix} 301.5313 & 0 & 0 & 0.0011 \\ 0 & 289.7811 & 0 & 0 \\ 0 & 0 & 313.9712 & 0 \\ 0.0011 & 0 & 0 & 276.7876 \end{pmatrix}, \quad \varepsilon_1 = 0.5013$$

According to Theorem 1, System (3) is global Mittag-Leffler synchronization with System (1).

5. Conclusions

In this paper, the authors design the suitable impulsive control and the state feedback controller to make fractional-order hyper-chaotic financial systems global Mittag-Leffler synchronization, and use computer Matlab LMI-toolbox to verify the effectiveness of newly-obtained criterion. Both theoretical and numerical examples show that as long as the impulsive macroeconomic management measures are appropriate, the backward economic system can gradually synchronize with the advanced economic system. Finally, the impulsive control involving time delay and the impulsive control under trigger event mechanism still need to be studied for the mathematical model of macroeconomics([10-12]). It is an interesting problem how to establish a reasonable model in line with the principles of Macroeconomics.

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Conflicts of Interest: The authors declare no conflict of interest.

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