Article

General Relativity Fractal for Cosmic Web

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Abstract: A new method for constructing exact solutions of the General Relativity equations for a dusty matter with fractal property is proposed. This method allows to find the solution of the GR-equations in terms of matter velocities u_m: the connection coefficients and the Ricci tensor of space-time are expressed in terms of matter velocities; the metric tensor and the matter density are found as functions of velocity from the GR-equations. The connection coefficients and the Ricci tensor are invariant with respect to the discrete scaling transformation of velocity $u_m \to q \cdot u_m$, where q is constant. Therefore, the found solution can be used to simulate the fractal properties of the cosmic web in terms of matter velocities. This solution includes isotropic and anisotropic distributions of matter density. In an isotropic case, there is a class of exact solutions including both the well-known Friedmann's solution and a solution with a periodic distribution of the matter density in space. This last solution may be used to simulate the quasi-periodic distribution of matter in the cosmic web. It is possible that, the cosmic web and its fractal properties are the space-time primary properties. These properties are described with a deformation tensor of the space-time.

Keywords: fractal; General Relativity; exact solutions; geodetic vector; cosmic web; quasi-periodic distribution of matter; deformation tensor of space-time

1. Introduction

December 2022 marks 100 years since the publication of Friedmann's article "On the Possibility of a World with Constant Negative Curvature of Space" [1]. Friedman was attracted by Einstein's beautiful hypothesis about the connection between gravitational effects and the geometry of space-time, in particular, the space-time geometry of the Universe understood as a single whole consisting of all gravitating matter. Friedman understood that astronomical data did not yet provide a reliable basis for solving the problem of what kind of world our Universe is. At that time scientists did not yet know about the existence of galaxies and the large-scale structure of the Universe (cosmic web). Therefore, he acted strictly mathematically and found the exact solution of Einstein's equations (here and after, GR-equations) for an isotropic three-dimensional space, in which matter is distributed homogeneously and the matter pressure is much less than the energy density of matter (dust). This solution describes a space that changes (expands or contracts) over time. Friedman's cosmological model contributed to the emergence of the idea of a dynamical changing Universe. This idea replaced the philosophical postulate about the eternal and unchanging Universe. Einstein held this view of the eternal Universe.

In the present-day the Friedmann – Lemaitre – Robertson - Walker (FLRW) cosmology model is the basis of the standard cosmology ΛCDM paradigm. This ΛCDM model is very efficient in describing the evolution and structure of the Universe: it is simple, only six cosmological parameters are needed to agree the solutions of GR-equations and observed angular power spectra for the anisotropy of the Cosmic Microwave Background temperature (CMB) and for the cosmic web up to angles of the order

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of 18° (multipole moment $l > 10$), and also explain the average cosmic abundance of chemical elements.

However, there exist "small" contradictions between the theory of the ΛCDM model and observations. For example, amplitudes of the observed power spectrum is larger than the values predicted in the standard cosmology ΛCDM model if multipole moments $l < 10$. This fact indicates the existence of large-scale CMB anisotropy, which is incompatible with the assumption of statistical homogeneity and an isotropy of the Universe space-time. The huge filaments with scales from several hundred megaparsecs to gigaparsecs in the distribution of galaxies and clusters, quasars, gamma-bursters are already discovered (Great Wall, Great GRB Wall, Hyperion, LQG [2 – 6]). The sizes of these structures are several times greater than the maximum scale length for correlation function of galaxies in the ΛCDM model (usually not more than 300 Mpc).

Still, we don't understand the origin of the cosmological constant Λ and its magnitude, and we also have no direct evidence for dark matter. Generally speaking, to a philosophical question "Why is there a cosmic web in a homogeneous and isotropic Universe", we have a "dark" answer: "due to the fact that there is dark matter and dark energy."

Thus, we have only an algorithm for describing the distribution of visible matter, but don't understand of the physics of the cosmic web. In essence, the ΛCDM paradigm is a graceful manipulation of six cosmological parameters. Therefore, these are the motivations to go beyond the ΛCDM paradigm.

Also there is "Hubble tension" – a discrepancy between measurements the Hubble constant H_0 from the CMB observations and from the local distance ladder observations like SHOES [7 - 9].

It should also be noted that the cosmic web has fractal properties. This is indicated by (i) the power law of distribution of red-shifts z of the absorption lines in the spectra of quasars: $\frac{dN}{dz} \sim (1+z)^{\alpha}$, $1.67 < \alpha < 2.09$; and (ii) the geometrical progression consisting of the spatial variations of the Hubble constant [10]. Power laws are characteristic of fractals. Fractals are often found in astrophysical objects that differ both in the processes occurring in them and in spatial scales (examples are given in article [10]). So far, the fractal properties of the cosmic web are not taken into account in the ΛCDM model.

Here we show the possibility that the observable cosmic web is a consequence of the fundamental structure of space-time, and not a consequence of the development of initial gravitational perturbations of the metric in space-time of ΛCDM model. First, we describe a method for constructing an exact solution of the GR-equations. To demonstrate the method, the simple case of a dusty matter without pressure is considered (in the ΛCDM model a dusty matter is used as cold dark matter). The method is based on the use of geodesic vector (or matter velocity) u_i. We determine the connection coefficients Γ^m_{ik} as functions of the velocity u_i and coordinates x_m from geodesic equation (below (1)), and calculate the Ricci tensor of space-time $R_{ik}(u_m, x_m)$. The metric tensor $g_{ik}(u_m, x_m)$ and the matter density $\varepsilon(u_m, x_m)$ are calculated from the GR-equations. At that, the connection coefficients and the Ricci tensor are invariant with respect to the discrete scaling transformation $u_m \to q \cdot u_m$, where q is constant. Therefore, the found solution can be used to simulate the fractal properties of the cosmic web.

The use of vectors u_i is motivated by the assumption that the trajectory of galaxy is a geodesic line, which is an integral line of the GR-equations (these equations are partial

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differential equations for the functions $g_{ik}(u_m, x_m)$). The velocities of galaxies u_i indicate the direction of these lines. The velocities u_m are determined directly from observations. For example, the galaxies velocities tend to align along certain directions. These directions are filaments of cosmic web [11, 12]. Therefore, to compare the results of cosmic web simulations with the results of observations, it is better to use velocities u_i.

Second, the proposed method allows to find solutions for a homogeneous and isotropic distribution of matter with a monotonic expansion of space, as well as for anisotropic distributions of matter with a non-monotonic expansion of space. In isotropic case, there is a class of exact solutions including both the well-known Friedmann's solution. In anisotropic case, there is a solution with a periodic distribution of matter in space. The exact periodic solution may be used to simulate the quasi-periodic distribution of matter in the cosmic web.

The rest of the article is structured as follows. Section 2 describe a method for exact GR-solutions. Section 3 shows how the GR-solution can be interpreted in terms of the deformation theory of continuous medium.

2. Method for Exact Solution of GR-Equation

Consider the dusty matter with the energy-momentum tensor $T_{ik} = \varepsilon u_i u_k$, where an energy density of fluid is ε, 4-velocity is u_i, and $u_i u^i = 1$. The metric tensor is g_{ik}, the lateen indices run through the values 0, 1, 2, 3.

In GR, the axiom is accepted that the movement of particles of matter occurs along geodesic lines of space-time. The equation of geodesic line is

$$u_{i,k} = \frac{\partial u_i}{\partial x^k} - \Gamma^n_{ik} u_n = 0. \tag{1}$$

We determine the connection coefficients Γ^m_{ik} so that the equation (1) will be the identity:

$$\Gamma^n_{ik} = \gamma \frac{\partial u_i}{\partial x^k}\left(u^n + b^n\right), \tag{2}$$

where b_n is some additional velocity, and $b_n b^n = 1$; $u_n b^n = \frac{1-\gamma}{\gamma}$ and γ is a constant. Equation (2) implies that coefficients Γ^m_{ik} do not change under the following discrete scaling transformation:

$$u_m \to q \cdot u_m; \; u^m \to \frac{1}{q} \cdot u^m; \; b^m \to \frac{1}{q} \cdot b^m. \tag{3}$$

Here q is constant.

For the homogeneous and isotropic cosmological model with the dusty matter, the metric tensor is $g_{ik} = a^2 \eta_{ik}$, where η_{ik} is Minkowski's metric. In this case the equation (1) is satisfied if $\gamma = 1$ and b_n is a zero 4-vector, the scale factor $a \sim \left(x^0\right)^2$ and x^0 is the

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conformal time. Then the 4-velocity is $u_i = (u_0, 0, 0, 0)$, and $u_0 = V_0 a\left(x^0\right)$, $u^0 = V^0 \frac{1}{a\left(x^0\right)}$, $V_0 V^0 = 1$, $\Gamma^0_{00} = \frac{1}{a}\frac{da}{dx^0}$.

The vector b_n characterizes anisotropy of space (see Section 3 below) and the deviation from the isotropic expansion of cosmological model.

According to the equation (2), the connection coefficients Γ^m_{ik} are proportional to the acceleration of matter particles $\frac{du_i}{ds} = \frac{\partial u_i}{\partial x^k} u^k$. For covariant conservation of the definition (2), the following covariant derivatives must be equal to zero:

$$u_{i,k} = 0;\ b_{i,k} = 0;\ \left(\frac{\partial u_i}{\partial x^k}\right)_{,m} = 0. \tag{4}$$

We consider the connection coefficients symmetric in the subscripts $\Gamma^n_{ik} = \Gamma^n_{ki}$. In this case $\frac{\partial u_i}{\partial x^k} = \frac{\partial u_k}{\partial x^i}$, and a vorticity is equal to zero, $r_{ik} = u_{i,k} - u_{k,i} = 0$.

Note that when the homogeneous and isotropic cosmological models are considered, the form of the metric tensor $g_{ik} = a^2 \eta_{ik}$ is specified at the first step. Then, the connection coefficients, the Ricci tensor, the velocity and the energy density are calculated as a function of the scale factor a. The dependence of the scale factor on conformal time is determined from the GR-equations. Thus, the scale factor plays the role of the geometric phase coordinate of the system. However, the scale factor not measurable value, and the FLRW model can only be detected by indirect consequences.

In the proposed method, the connection coefficients (2) are functions of the velocity u_i, the Ricci tensor $R_{ik}\left(u_m, x_m\right)$ and the metric tensor $g_{ik}\left(u_m, x_m\right)$ are also calculated as functions of the velocity u_i. The velocity u_i is the physical phase coordinate, which is a measurable value.

Next, we find from the equations (4):

$$\frac{\partial^2 u_i}{\partial x^m \partial x^k} = \gamma\left(\frac{\partial u_m}{\partial x^k}\frac{\partial u_i}{\partial x^n} + \frac{\partial u_m}{\partial x^i}\frac{\partial u_k}{\partial x^n}\right)\left(u^n + b^n\right); \tag{5}$$

$$\frac{\partial u^i}{\partial x^k} = -\gamma \frac{\partial u_n}{\partial x^k} u^n \left(u^i + b^i\right); \tag{6}$$

$$\frac{\partial b^i}{\partial x^k} = -\gamma \frac{\partial u_n}{\partial x^k} b^n \left(u^i + b^i\right). \tag{7}$$

The Ricci tensor is equal to

$$R_{ik} = \gamma^2\left(\frac{\partial u_i}{\partial x^m}\frac{\partial u_k}{\partial x^n}\left(u^m + b^m\right)\left(u^n + b^n\right) - \frac{\partial u_i}{\partial x^k}\frac{\partial u_m}{\partial x^n}\left(u^m b^n + u^n b^m + b^m b^n\right)\right). \tag{8}$$

Here we take into account the identity for an acceleration $w_n = \frac{\partial u_n}{\partial x^m} u^m$:

$$w_n u^n = 0. \tag{9}$$

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The metric tensor is determined from the GR equations:

$$g_{ik} = 2u_i u_k - \frac{2}{\kappa\varepsilon} R_{ik}, \tag{10}$$

where $\kappa = \frac{8\pi G}{c^4}$ is Einstein's constant.

As follows from the equation (10),

$$\kappa\varepsilon = 2R_{mn}u^m u^n = 2\gamma^2 \left(\frac{\partial u_m}{\partial x^n} u^m b^n \right)^2. \tag{11}$$

Thus, we construct the exact solution to the GR-equations: the connection coefficients (2), the Ricci tensor (8), the metric tensor (10) and the energy density (11) in terms of the velocities u_i and b_i. We see from formulae (2) and (8) - (11), that these solutions change under the discrete scaling transformation (3) as follows:

$$R_{ik} \to R_{ik};\; g_{ik} \to q^2 \cdot g_{ik};\; \varepsilon \to \frac{1}{q^2} \cdot \varepsilon. \tag{12}$$

Let consider the exact solution with a function $f\left(v_m x^m\right)$, and

$$\frac{\partial u_n}{\partial x^k} u^n = v_k f\left(v_m x^m\right), \tag{13}$$

where 4-vector v_k does not depend on coordinates x^m. Then we find $v_m u^m = 0$, from the equation (9).

It can be shown by direct calculation that we have the following equations:

$$R_{ik} = \gamma^2 \left(\left(v_n f + \frac{\partial u_i}{\partial x^m} b^m \right)\left(v_k f + \frac{\partial u_k}{\partial x^n} b^n \right) - \frac{\partial u_i}{\partial x^k} \left(2v_n b^n \cdot f + \frac{\partial u_m}{\partial x^n} b^m b^n \right) \right), \tag{14}$$

$$\kappa\varepsilon = 2\gamma^2 \left(v_n b^n \cdot f \right)^2. \tag{15}$$

The solution (13) - (15) depends on the choice of a continuous function $f\left(v_m x^m\right)$. In the general case, this solution corresponds to an inhomogeneous and anisotropic space-time.

We note, first, there is a class of power-law solutions $u_k = V_k \cdot \left(v_m x^m\right)^n$, $V_k V^k = 1$ and $f(v_m x^m) = \frac{n}{v_m x^m}$. This class contains the well known Friedman's solutions for cosmological model with the dusty matter. We can reconstruct Friedman's metric for the exact isotropic solutions $V_1 = V_2 = V_3$ using $n = 2$, $v_m x^m = v_0 x^0$, $a \sim \left(v_0 x^0\right)^2$, $u_k = \left(V_0 a, 0, 0, 0\right)$ and the formulae (13) – (15). Then we have $R_{ik} \sim v_i v_k \left(x^0\right)^{-2}$, $\varepsilon \sim \left(x^0\right)^{-6}$ and $g_{ik} \sim v_i v_k \left(x^0\right)^4$.

Second, there is the exact solution for the quasi-periodic distribution of the matter density. It corresponds to the choice an oscillating function $f(v_m x^m)$, for example

$f(v_m x^m) = \frac{1}{c + v_m x^m} - d \cdot \sin(v_m x^m)$, with constant coefficients c and d. In this case we find the next formulae:

$$u_i = V_i \left(c + v_m x^m\right) e^{d\cos(v_m x^m)}; \ u^i = V^i \frac{1}{c + v_m x^m} e^{-d\cos(v_m x^m)}; \quad (16)$$

$$V_i V^i = 1; \ v_i V^i = 0; \ V_i b^i = \frac{1-\gamma}{\gamma} \frac{e^{-d\cos(v_m x^m)}}{c + v_m x^m}; \quad (17)$$

$$R_{ik} = \gamma^2 \left(v_m b^m\right)\left[1 - d\left(c + v_m x^m\right)\sin(v_m x^m)\right]^2 \left[V_i V_k \left(v_m b^m\right) - \frac{1}{\gamma} V_i v_k \frac{e^{-d\cos(v_m x^m)}}{c + v_m x^m}\right]; \quad (18)$$

$$g_{ik} = \left[V_i V_k \left(c + v_m x^m\right) + \frac{V_i v_k}{\gamma\left(v_m b^m\right)}\right]\left(c + v_m x^m\right) e^{-d\cos(v_m x^m)}. \quad (19)$$

One can see that $g_{ik} \to 2^{-d-2} d \cdot V_i V_k \left(v_m x^m\right)^4$ if $v_m x^m > c$ and $\frac{4}{d} < \left(v_m x^m\right)^2 < 1$.

Thus, we come to the conclusion that the ΛCDM model is not the only possible cosmological model with dust. The found exact solutions with fractal properties may be used to simulate the quasi-periodic and fractal distribution of matter in the cosmic web formed by galactic filaments. Thus, we can consider two hypotheses: (1) the cosmic web is a primary structure formed by primary filaments (scalar and vector gravitational perturbations in the FLRW mode [13]); (2) the cosmic web is a consequence of a fundamental property of space-time, and then the cosmic web properties (e.g. fractality) contain information about the space-time geometry and about a deformation of space-time.

3. Deformation Tensor of Space-Time

Here we show how found here exact solutions can be interpreted in terms of the deformation theory of continuous medium. Suppose, due to deformation of space-time in the vicinity of the world point x^i the coordinate differential dx^i increases by a deviation dy^i : $dx^i \to dx^i + dy^i = dx^i + \frac{\partial y^i}{\partial x^k} dx^k$. Then the space-time interval is $ds^2 = \tilde{g}_{ik}\left(dx^i + dy^i\right)\left(dx^k + dy^k\right) = \left(\tilde{g}_{ik} + 2Y_{ik}\right) dx^i dx^k$. Here the deformation tensor is introduced:

$$Y_{ik} = \frac{1}{2}\left(\tilde{g}_{mk} \frac{\partial y^m}{\partial x^i} + \tilde{g}_{mi} \frac{\partial y^m}{\partial x^k} + \tilde{g}_{mn} \frac{\partial y^m}{\partial x^i} \frac{\partial y^n}{\partial x^k}\right). \quad (20)$$

The local deviation of world lines is characterized by the vector $Y_{ik} dx^i$, and the velocity of this deviation is $Y_{ik} \frac{dx^i}{ds} = Y_{ik} u^i$.

If we consider an isotropic deformation of space-time then, $\frac{\partial y^i}{\partial x^k} = y\delta^i_k$, and $y = const$. In this case, the tensor (20) is the volumetric deformation tensor, $I_{ik} = \frac{1}{4} I g_{ik}$,

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here $I_i^i = I = 2y(2+y)$. The velocity of isotropic deformation is $V_k = I_{ik}u^i = \frac{1}{4}Iu_k$. This velocity V_k is parallel to the velocity u_k, and therefore, the direction of motion of particles does not change due to the volumetric deformation of the reference frame, only an expansion (or compression) of the space volume occurs, that is an expansion (or contraction) of the geodesic grid.

The anisotropic deformation tensor is determined by the following formula $D_{ik} = Y_{ik} - \frac{1}{4}Y\tilde{g}_{ik}$.The velocity of the deviation of the geodesic lines is $W_k = D_{ik}u^i$. In the general case, the direction of velocity W_k does not coincide with the direction of velocity u_k, therefore, the geodesic lines of space-time are deviated relative to world lines in the space-time without anisotropic deformation. This leads, in particular, to the convergence of the world lines of particles in the direction of speed W_k and to the formation of local matter flows and elongated structures. Thus, knowing the distribution of velocities, we can detect the deformation of space-time in a certain region of space-time.

Now, we substitute in formula (10) the metric tensor $g_{ik} = \tilde{g}_{ik} + 2Y_{ik}$, where $\tilde{g}_{ik}$ is the metric tensor for the background space-time (the frame of reference for a distant observer). Then we find an equation for the deformation tensor:

$$Y_{ik} = -\frac{1}{2}\tilde{g}_{ik} + u_i u_k - \frac{R_{ik}}{\kappa\varepsilon}. \qquad (21)$$

The deformation tensor (21) is a local characteristic of space-time. The velocity of the deviation of the geodesic lines is $W_k = \frac{2-Y}{4}u_k - \frac{R_{ik}u^i}{\kappa\varepsilon}$.

Formula (21) allows finding the deformation that corresponds to the general solution (13) - (15). For an oscillating function $f\left(\nu_m x^m\right)$ the solution (13) – (15) corresponds to the oscillating deformation tensor of space-time (21). This tensor changes under the discrete scaling transformation (3) as follows: $Y_{ik} \to q^2 \cdot Y_{ik}$. Therefore the deformation of space-time has fractals properties.

4. Results

Here the main purpose has been to propose a new method for constructing exact solutions of the GR-equations with fractal property. The propose method is based on the use of geodetic vector u_k, and it allows to find solutions for (1) a homogeneous and isotropic distribution of matter with a monotonic expansion of space, (2) an anisotropic distribution of matter with a non-monotonic expansion of space, (3) a quasi-periodic distribution of the density of matter in space.

The exact oscillating solution allows to simulate the quasi-periodic distribution of matter in the cosmic web with fractal property.

It is possible, the cosmic web is a consequence of a fundamental property of space-time, and its properties contain information about the geometry and deformation of space-time.

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