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Traveling Wave Solutions of the Loaded Non-linear Klein-Gordon Equation via Functional Variable Method

Bazar Babajanov [1], **Fakhriddin Abdikarimov**[2*]

[1] Doctor of Science in Mathematical Sciences, Department of Applied Mathematics and Mathematical Physics, Urgench State University, Urgench, Uzbekistan
[2] Phd student, Khorezm Mamun Academy, Khiva, Uzbekistan
* Correspondence: goodluck_0714@mail.ru

Abstract: In this paper, the functional variable method is used to establish solitary wave solutions and periodic wave solutions of the loaded quadratic non-linear Klein-Gordon equation, the loaded cubic non-linear Klein-Gordon equation and the loaded coupled non-linear Klein-Gordon equation. All solutions of these equations have been examined and three dimensional graphics of the obtained solutions have been drawn by using the Matlab software. The main advantage of the proposed functional variable method over other methods is that it provides more new exact traveling wave solutions along with additional free parameters. The graphical representations of the soliton solutions and the periodic wave solutions by using distinct values of random parameter are demonstrated to better understand their physical features. The exact solutions have its great importance to reveal the internal mechanism of the physical phenomena.

Keywords: loaded Klein-Gordon equation; periodic solutions; functional variable method; trigonometric function; soliton solutions; hyperbolic function

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0. INTRODUCTION

The Klein-Gordon(KG) equation is an important group of partial differential equations and is present in relativistic quantum mechanics and field theory, which is immensely important for the high energy physicist [1–3], and is employed for the modeling of various phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles. This equation is expressed in the following basic form

$$w_{tt} - w_{xx} - w_{yy} = b(w).$$

The KG equation most probably first arose in a mathematical context with $b(w) = e^w$ in the theory of constant surfaces in Liouville's work. It appears in very many fields of physics. For a cubic nonlinear $b(w) = w^3 - w$ it has been used as a model field theory[4].

Recently some new exact solutions of general forms of nonlinear KG equation

$$w_{tt} - w_{xx} + \alpha w + \beta w^n = 0,$$

have found by several authors [5–8]. It should be noted that many methods have been developed to find special solutions of the nonlinear KG equation, such as the modified decomposition method [9], the symplectic finite difference approximations method [10], the variational iteration method [11,12], the finite element method [13], the cubic B-spline collocation method [14], the finite difference method [15], the decomposition method [16], exp-function method [17,18], the homotopy perturbation method [19], the tanh method [20] and the Jacobi elliptic function method [21], the Lie symmetry approach method [22], the numerically methods [23–25], the method of functional separation of variables [26] etc. Biswas and et al. studied the adiabatic dynamics of topological and non-topological solitons alike in presence of perturbation terms [27–30].

 doi:10.20944/preprints202206.0344.v1

In recent years, in connection with intensive research of problems optimal management of the agroecosystem, for example, the problem of long-term forecasting and regulation of the level of groundwater and soil moisture, there has been a significant increase in interest in loaded equations. Among the works devoted to loaded equations, one should especially note the works of A. Kneser [31], L. Lichtenstein [32], A. M. Nakhushev [33], and others. A complete explanation of solutions of the nonlinear loaded PDEs and their uses can be found in papers [34–39].

In this paper, we consider the following the loaded KG equation and the loaded coupled KG equation with quadratic and cubic nonlinearity

$$w_{tt} - w_{xx} - w_{yy} + \alpha w + \beta w^n + \gamma(t) w(0,0,t) w = 0, \tag{1}$$

$$\begin{cases} w_{xx} + w_{yy} - w_{tt} - w + 2w^3 + 2wv + \varphi(t) w(0,0,t) w = 0 \\ v_x + v_y - v_t - 4ww_t + \varphi(t) v(0,0,t) v_x = 0, \end{cases} \tag{2}$$

where $w(x,y,t)$ and $v(x,y,t)$ are unknown functions, $x \in R$, $y \in R$, $t \geq 0$, α and β are any constants, $\gamma(t)$ and $\varphi(t)$ are the given real continuous functions.

We construct exact travelling wave solutions of (1) and (2), that is the exact solutions of these equations including solitary wave solutions and periodic wave solutions are obtained by the functional variable method. All solutions of the loaded KG equation and the loaded coupled KG equation have been examined and three dimensional graphics of the obtained solutions have been drawn by using the Matlab software. The main advantage of the proposed functional variable method over other methods is that it provides more new exact traveling wave solutions along with additional free parameters. The graphical representations of the soliton solutions and the periodic wave solutions by using distinct values of random parameter are demonstrated to better understand their physical features. The exact solutions have its great importance to reveal the internal mechanism of the physical phenomena. Apart from the physical relevance, the close-form solutions of nonlinear evolution equations facilitate the numerical solvers to compare the accuracy of their results and help them in the stability analysis.

1. DESCRIPTION OF THE FUNCTIONAL VARIABLE METHOD

We consider NPDE of the form

$$P(w, w_w, w_y, w_t, w_{xx}, w_{tt}, w_{yy}, w_{xy}, w_{xt}, w_{yt} ...) = 0, \tag{3}$$

where P is a polynomial in $w = w(x,y,t)$ and its partial derivatives.

Step 1. It is used the following transformation for a travelling wave solution of (3):

$$w(x,y,t) = w(\xi), \tag{4}$$

with

$$\frac{\partial w}{\partial x} = p\frac{dw}{d\xi}, \frac{\partial w}{\partial y} = q\frac{dw}{d\xi}, \frac{\partial w}{\partial t} = -k\frac{dw}{d\xi}, ..., \tag{5}$$

where

$$\xi = px + qy - kt, \qquad p = const, \;\; q = const \tag{6}$$

and k is the speed of the traveling wave.

Substituting eq. (4) and (5) into NPDE (3) we get the following ODE of the form

$$F(w,\, w', w'', w''', ...) = 0, \qquad w' = \frac{dw}{d\xi} \tag{7}$$

here F is a polynomial in $w(\xi), w(\xi)', w(\xi)'', w(\xi)''', ...$

Step 2. Let

$$w' = F(w). \tag{8}$$

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It follows that

$$\int \frac{dw}{F(w)} = \xi + \xi_0, \tag{9}$$

we suppose $\xi_0 = 0$ for convenience. Now we can calculate higher order derivatives of w:

$$\begin{aligned} w'' &= \tfrac{1}{2}\frac{d\left(F^2(w)\right)}{dw} \\ w''' &= \tfrac{1}{2}\frac{d^2\left(F^2(w)\right)}{dw^2}\sqrt{F^2(w)} \\ w'''' &= \tfrac{1}{2}\left[\frac{d^3\left(F^2(w)\right)}{dw^3}F^2(w) + \frac{d^2\left(F^2(w)\right)}{dw^2}\frac{d\left(F^2(w)\right)}{dw}\right] \end{aligned} \tag{10}$$

Step 3. Putting eq. (10) into eq. (7), we obtain

$$G(w, \frac{dF(w)}{dw}, \frac{d^2F(w)}{dw^2}, \frac{d^3F(w)}{dw^3}, ...) = 0. \tag{11}$$

The key idea of this particular form eq. (11) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, eq. (11) provides the expression of F and this, together with eq. (8), give appropriate solutions to the original problem.

2. SOLUTIONS OF THE LOADED QUADRATIC NON-LINEAR KLEIN-GORDON EQUATION

We will find the exact solution of the loaded quadratic non-linear KG equation by the functional variable method. For doing this, in (1), let use the following transformation.

$$w(x, y, t) = u(\xi), \; \xi = px + qy - kt. \tag{12}$$

It is easy to show that after transformation (12), the nonlinear partial differential eq. (1) can be transformed into an ordinary differential equation of the form

$$w'' = \frac{\beta w^2 + \mu(t)w}{p^2 + q^2 - k^2}, \tag{13}$$

where $\mu(t) = \alpha + \gamma(t)w(0, 0, t)$.

According to (10) eq. (13) can be written as follows

$$\frac{1}{2}\frac{d\left(F^2(w)\right)}{dw} = \frac{\beta w^2 + \mu(t)w}{p^2 + q^2 - k^2}. \tag{14}$$

Integrating eq. (14) and after simple simplification, we get

$$F(w) = w\sqrt{\frac{2\beta w + 3\mu(t)}{3(p^2 + q^2 - k^2)}}. \tag{15}$$

From eq. (8) and eq. (15) we deduce that

$$\frac{dw}{w\sqrt{2\beta w + 3\mu(t)}} = \frac{d\xi}{\sqrt{3(p^2 + q^2 - k^2)}}. \tag{16}$$

After integrating eq. (16), we have

$$w(x, y, t) = \frac{3(\alpha + \gamma(t)w(0, 0, t))}{2\beta}\left(cth^2\left(\frac{1}{2}\sqrt{\frac{\alpha + \gamma(t)w(0, 0, t)}{p^2 + q^2 - k^2}}(px + qy - kt)\right) - 1\right). \tag{17}$$

The function $w(0, 0, t)$ can be easily obtained based on expression (17).

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We get two types of solutions of the loaded quadratic non-linear KG equation (1) as follows:

1) When $\alpha + \gamma(t)w(0,0,t) > 0, p^2 + q^2 - k^2 > 0$, we get the solitary solution

$$w(x,y,t) = \frac{3(\alpha + \gamma(t)w(0,0,t))}{2\beta}\left(cth^2\left(\frac{1}{2}\sqrt{\frac{\alpha + \gamma(t)w(0,0,t)}{p^2+q^2-k^2}}(px+qy-kt)\right) - 1\right). \tag{18}$$

2) When $\alpha + \gamma(t)w(0,0,t) > 0, p^2 + q^2 - k^2 < 0$, we get the periodic solution

$$w(x,y,t) = -\frac{3(\alpha + \gamma(t)w(0,0,t))}{2\beta}\left(ctg^2\left(\frac{1}{2}\sqrt{\frac{\alpha + \gamma(t)w(0,0,t)}{p^2+q^2-k^2}}(px+qy-kt)\right) + 1\right). \tag{19}$$

The graphs of solutions of the loaded quadratic non-linear KG equation by using distinct values of random parameter will be demonstrated.

If $\alpha = 0$, $\beta = 1.5$, $p = 2$, $q = 1$, $k = -2$ and $\gamma(t) = t^2$, then we have

$$w(x,y,t) = t^2 w(0,0,t)\left(cth^2\left(\frac{1}{2}t\sqrt{w(0,0,t)}(2x+y+2t)\right) - 1\right). \tag{20}$$

If $\alpha = 0$, $\beta = -1.5$, $p = 1$, $q = \sqrt{2}$, $k = -2$ and $\gamma(t) = t^2$, then we have

$$w(x,y,t) = t^2 w(0,0,t)\left(ctg^2\left(\frac{1}{2}t\sqrt{w(0,0,t)}(x+\sqrt{2}y+2t)\right) + 1\right). \tag{21}$$

3. PHYSICAL INTERPRETATIONS OF THE LOADED QUADRATIC NON-LINEAR KLEIN-GORDON EQUATION

Graphical representation is an effective tool for communication and it exemplifies evidently the solutions of the problems. The graphical illustrations of the solutions are depicted in the Figure 1 and the Figure 2. After visualizing the graphs of the soliton solutions and the periodic wave solutions by using distinct values of random parameter are demonstrated to better understand their physical features. The amplitude and velocities are controlled by parameters of various kind. These characteristics of the solutions are favorable for investigating certain nonlinear phenomena arising in physics, applied mathematics, and engineering. In particular, the soliton is a self-reinforcing wave packet maintaining its shape while propagating at a constant velocity. Solitons are unscathed in shape and speed by a collision with other solitons and are often studied in quantum mechanics, nuclear physics, and waves along a weakly anharmonic mass-spring chain. Moreover, periodic traveling waves play a fundamental role in several mathematical physics including self-oscillatory systems, reaction-diffusion-advection systems, and excitable chemical reactions.

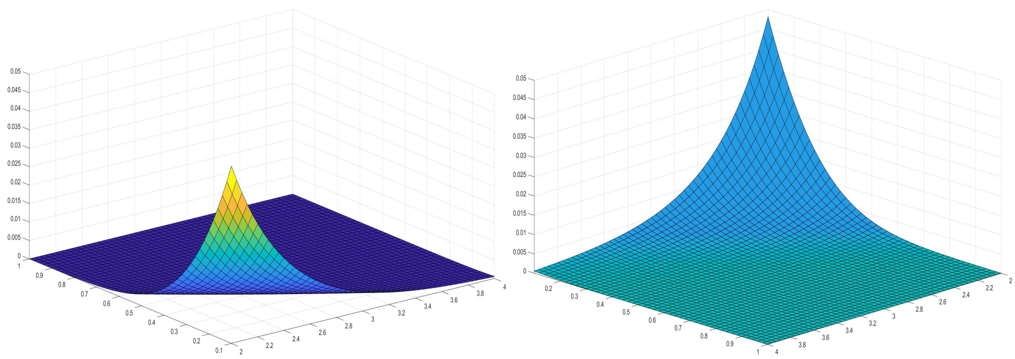

Figure 1. Solitary solution of the loaded quadratic non-linear KG equation for $y = 0$, $\alpha = 0$, $\beta = 1.5$, $p = 2$, $q = 1$, $k = -2$ and $\gamma(t) = t^2$.

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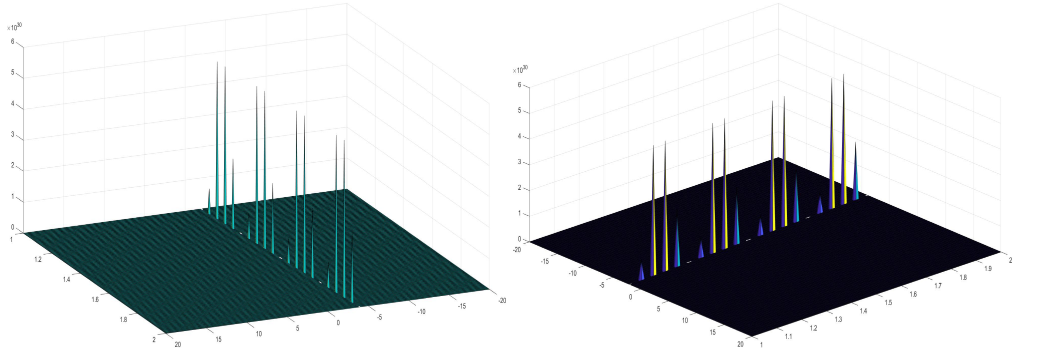

Figure 2. Periodic solution of the loaded quadratic non-linear KG equation for $y = 0$, $\alpha = 0$, $\beta = -1.5$, $p = 1$, $q = \sqrt{2}$, $k = -2$ and $\gamma(t) = t^2$.

4. SOLUTIONS OF THE LOADED CUBIC NON-LINEAR KLEIN-GORDON EQUATION

We will show how to find the exact solution of the loaded cubic non-linear KG equation by the functional variable method. Using the wave variable

$$w(x,y,t) = w(\xi),\ \xi = px + qy - kt, \tag{22}$$

that will convert eq. (1) to an ordinary differential equation

$$w'' = \frac{\beta w^3 + \mu(t)w}{p^2 + q^2 - k^2}, \tag{23}$$

where $\mu(t) = \alpha + \gamma(t)w(0,0,t)$.

Following eq. (23), it is easy to deduce from eq. (10) an expression for the function $F(w)$

$$\frac{1}{2}\frac{d\left(F^2(w)\right)}{dw} = \frac{\beta w^3 + \mu(t)w}{p^2 + q^2 - k^2}. \tag{24}$$

Integrating eq. (24), we have

$$F(w) = w\sqrt{\frac{\beta w^2 + 2\mu(t)}{2(p^2 + q^2 - k^2)}}. \tag{25}$$

From eq. (8) and eq. (25) we deduce that

$$\frac{dw}{w\sqrt{\beta w^2 + 2\mu(t)}} = \frac{d\xi}{\sqrt{2(p^2 + q^2 - k^2)}}. \tag{26}$$

After integrating eq. (26), with zero constant of integration, we obtain following exact solution

$$w(x,y,t) = \sqrt{\frac{2(\alpha + \gamma(t)w(0,0,t))}{\beta}}\sqrt{cth^2\left(\frac{\sqrt{\alpha + \gamma(t)w(0,0,t)}}{\sqrt{p^2 + q^2 - k^2}}(px + qy - kt)\right) - 1}. \tag{27}$$

The function $w(0,0,t)$ can be easily obtained based on expression (27).

We get two types of travelling solutions of the loaded cubic non-linear KG equation (1) as follows:

1) When $\alpha + \gamma(t)w(0,0,t) > 0, \beta > 0, p^2 + q^2 - k^2 > 0$, we have the solitary wave solution

$$w(x,y,t) = \sqrt{\frac{2(\alpha + \gamma(t)w(0,0,t))}{\beta}}\sqrt{cth^2\left(\frac{\sqrt{\alpha + \gamma(t)w(0,0,t)}}{\sqrt{p^2 + q^2 - k^2}}(px + qy - kt)\right) - 1}. \tag{28}$$

 doi:10.20944/preprints202206.0344.v1

2) When $\alpha + \gamma(t)w(0,0,t) > 0, \beta < 0, p^2 + q^2 - k^2 < 0$, we have the periodic wave solution

$$w(x,y,t) = \sqrt{\frac{2(\alpha + \gamma(t)w(0,0,t))}{\beta}} \sqrt{ctg^2\left(\frac{\sqrt{\alpha + \gamma(t)w(0,0,t)}}{\sqrt{p^2 + q^2 - k^2}}(px + qy - kt)\right) + 1}. \quad (29)$$

The graphs of solutions of the loaded quadratic non-linear KG equation via using distinct values of random parameter will be demonstrated.

If $\alpha = 0$, $\beta = 2$, $p = 2$, $q = 1$, $k = -1$ and $\gamma(t) = t^2$, then we have

$$w(x,y,t) = t\sqrt{w(0,0,t)}\sqrt{cth^2\left(\frac{1}{2}t\sqrt{w(0,0,t)}(2x + y + 2t)\right) - 1}. \quad (30)$$

If $\alpha = 0$, $\beta = -2$, $p = 1$, $q = \sqrt{2}$, $k = -1$ and $\gamma(t) = t^2$, then we have

$$w(x,y,t) = t\sqrt{w(0,0,t)}\sqrt{ctg^2\left(\frac{1}{2}t\sqrt{w(0,0,t)}(x + \sqrt{2}y + 2t)\right) + 1}. \quad (31)$$

5. GRAPHICAL REPRESENTATION OF THE LOADED CUBIC NON-LINEARITY KLEIN-GORDON EQUATION

We make graphs of obtained the solitary wave solutions and the periodic wave solutions of the loaded cubic non-linear KG equation, so that they can represent the importance of each obtained solution and physically interpret the consequence of parameters as well. Some of our obtained the soliton solutions and the periodic wave solutions are represented in Figure 3-4 by Matlab software. In the concept of mathematical physics, a soliton wave is defined as a set of self-reinforcing waves that retain their shape. It propagates at a constant amplitude and speed. The existence of periodic travelling waves usually depends on the parameter values in a mathematical equation. If there is a periodic travelling wave solution, then there is typically a family of such solutions, with different wave speeds.

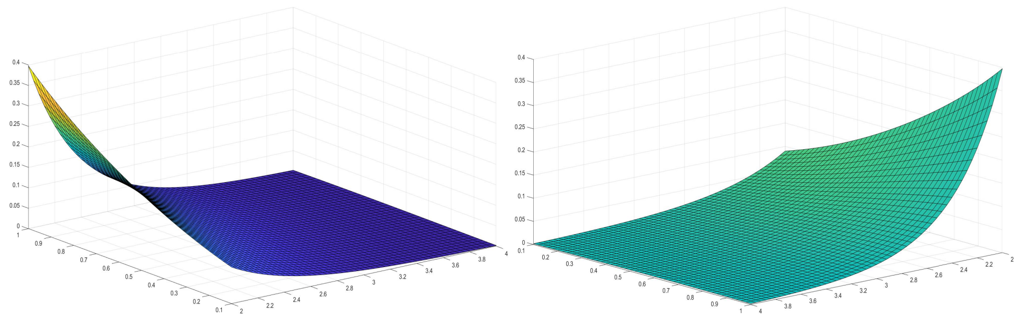

Figure 3. Solitary wave solution of the loaded cubic non-linear KG equation for $y = 0$, $\alpha = 0$, $\beta = 2$, $p = 2$, $q = 1$, $k = -1$ and $\gamma(t) = t^2$.

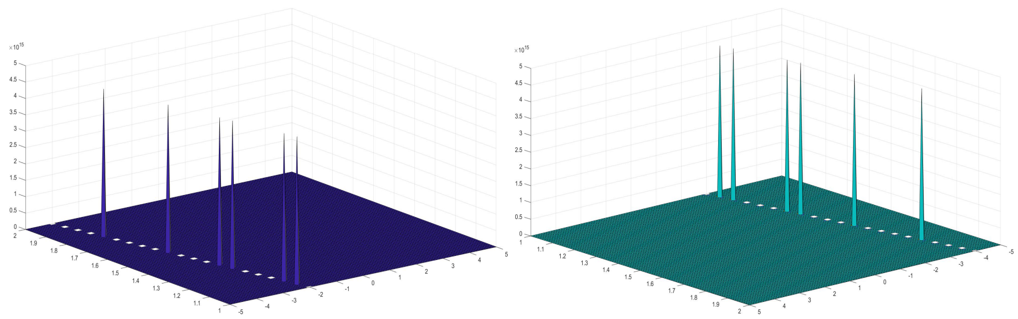

Figure 4. Periodic wave solution of the loaded cubic non-linear KG equation for $y = 0$, $\alpha = 0$, $\beta = -2$, $p = 1$, $q = \sqrt{2}$, $k = -1$ and $\gamma(t) = t^2$.

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6. SOLUTIONS OF THE LOADED COUPLED NON-LINEARITY KLEIN-GORDON EQUATION

Assume that eq. (2) has an exact solution in the form of a travelling wave

$$w(x,y,t) = w(\xi), v(x,y,t) = v(\xi), \xi = px + qy - kt, \tag{32}$$

that will convert eq. (2) to an ordinary differential equation system

$$\begin{cases} (p^2+q^2-k^2)w'' + 2w^3 + 2wv + (\varphi(t)w(0,0,t) - 1)w = 0 \\ (p+q+k)v' + 4kww' + \varphi(t)pv(0,0,t)v' = 0, \end{cases} \tag{33}$$

Integrating once the second equation of (33) and put the constant of integration zero, we get

$$v = -\frac{2kw^2}{p+q+k+\varphi(t)pv(0,0,t)}. \tag{34}$$

We put eq. (34) to the first equation of (33) and write it as follows.

$$w'' = \frac{1}{p^2+q^2-k^2}\left(\left(\frac{4k}{p+q+k+\varphi(t)pv(0,0,t)} - 2\right)w^3 + (1-\varphi(t)w(0,0,t))w\right). \tag{35}$$

Following eq. (10), it is easy to deduce from eq. (35)

$$\frac{1}{2}\frac{d(F^2(w))}{dw} = \frac{1}{p^2+q^2-k^2}\left(\left(\frac{4k}{p+q+k+\varphi(t)pv(0,0,t)} - 2\right)w^3 + (1-\varphi(t)w(0,0,t))w\right). \tag{36}$$

Integrating eq. (36), we have

$$F(w) = \frac{w}{\sqrt{p^2+q^2-k^2}}\sqrt{\mu(t)w^2+\eta(t)}, \tag{37}$$

where $\mu(t) = \frac{2k}{p+q+k+\varphi(t)pv(0,0,t)} - 1, \eta(t) = 1 - \varphi(t)w(0,0,t)$.

From eq. (8) and eq. (37) we deduce that

$$\frac{dw}{w\sqrt{\mu(t)w^2+\eta(t)}} = \frac{1}{\sqrt{p^2+q^2-k^2}}d\xi. \tag{38}$$

After integrating eq. (38), we get exact solution of the loaded coupled non-linear KG equation

$$w(x,y,t) = \sqrt{\frac{\eta(t)}{\mu(t)}}\sqrt{cth^2\left(\sqrt{\frac{\mu(t)}{p^2+q^2-k^2}}(px+qy-kt)\right) - 1}. \tag{39}$$

Following eq. (39), it is easy to find from eq. (34) an expression for the function v

$$v(x,y,t) = -\frac{2k\eta(t)}{\mu(t)(p+q+k+\gamma(t)pv(0,0,t))}\left(cth^2\left(\sqrt{\frac{\mu(t)}{p^2+q^2-k^2}}(px+qy-kt)\right) - 1\right), \tag{40}$$

where $\mu(t) = \frac{2k}{p+q+k+\varphi(t)pv(0,0,t)} - 1, \eta(t) = 1 - \varphi(t)w(0,0,t)$.

The functions $w(0,0,t)$ and $v(0,0,t)$ can be easily obtained based on expression (39) and (40).

1) When $\frac{\mu(t)}{p^2+q^2-k^2} > 0$, we get the solitary solution

$$\begin{cases} w(x,y,t) = \sqrt{\frac{\eta(t)}{\mu(t)}}\sqrt{cth^2\left(\sqrt{\frac{\mu(t)}{p^2+q^2-k^2}}(px+qy-kt)\right)-1} \\ v(x,y,t) = -\frac{2k\eta(t)}{\mu(t)(p+q+k+\gamma(t)pv(0,0,t))}\left(cth^2\left(\sqrt{\frac{\mu(t)}{p^2+q^2-k^2}}(px+qy-kt)\right)-1\right), \end{cases} \tag{41}$$

2) When $\frac{\mu(t)}{p^2+q^2-k^2} < 0$, we get the periodic solution

$$\begin{cases} w(x,y,t) = -\sqrt{\frac{\eta(t)}{\mu(t)}}\sqrt{ctg^2\left(\sqrt{\frac{\mu(t)}{p^2+q^2-k^2}}(px+qy-kt)\right)+1} \\ v(x,y,t) = \frac{2k\eta(t)}{\mu(t)(p+q+k+\gamma(t)pv(0,0,t))}\left(ctg^2\left(\sqrt{\frac{\mu(t)}{p^2+q^2-k^2}}(px+qy-kt)\right)+1\right), \end{cases} \tag{42}$$

7. CONCLUSIONS

The functional variable method has been successfully used to obtain the soliton solutions and the periodic solutions of the loaded quadratic non-linear KG equation, the loaded cubic non-linear KG equation and the loaded coupled non-linear KG equation. We have shown that, this method can provide a useful way to efficiently find the exact structures of solutions to a variety of nonlinear wave equations. After visualizing the graphs of the soliton solutions and the periodic solutions by using distinct values of random parameter are demonstrated to better understand their physical features. These characteristics of the solutions are favorable for investigating certain nonlinear phenomena arising in physics, applied mathematics, and engineering. In particular, the soliton is a self-reinforcing wave packet maintaining its shape while propagating at a constant velocity. We conclude that the exact solutions have its great importance to reveal the internal mechanism of the physical phenomena.

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