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On Fuzzy Locally Convex Spaces

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Abstract

In this paper, the concept of fuzzy locally convex spaces generated by a family of fuzzy seminorms is introduced. We prove that a Minkowski functional of zero neighborhoods generates a seminorm and finally deduce that the topology generated by the family of seminorms coincides with that of the Minkowski functional.

Key words: t-norms, fuzzy seminorm, Minkowski functional, fuzzy topology

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1 Introduction

The concept of fuzzy set has come a long way and its uses cannot be overemphasized. Since its introduction by Zadeh in 1965 [12], its ubiquity has been laid bare in several areas of applications including but not limited to computer science, biomedical engineering, telecommunication, decision making, differential equations, rings, semirings, group, automation and robotics, networking, discrete mathematics and abstract structures of Mathematics, locally convex spaces being a chief. The major ideas of fuzzy vectors, fuzzy topological spaces were introduced by Kastaras in his famous works [5],[6] and [7] with other invariants in the references therein.

For the purpose of this work, basic operations on fuzzy sets will be taken for granted as this can be found in [9]. Classical analysis on fuzzy normed spaces are well elucidated in [1],[2]. For fuzzy locally convex spaces, a gentle introduction can be seen in [3] and [4].

A modern approach to fuzzy analysis is can be seen in [11]. In this work, we introduce the concept of Fuzzy Locally Convex Space generated by family of Fuzzy seminorms. We shall elucidate the properties of Fuzzy set, t-norm, Fuzzy seminormed linear topological spaces, we prove that a Minkowski functional of zero neighbourhood generates a fuzzy seminorm and finally we prove that the topology generated by any fuzzy seminorm is same as the topology generated by the Minkowski functional.

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2 Preliminaries

Definition 2.1. Let X is a nonempty set and $x \in X$. A fuzzy set μ in a universe X is a mapping $\mu : X \to [0,1] \subset \mathbb{R}$. A fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x) \Big| x \in X)\}$$

where

$$\mu_{\tilde{A}} : X \longrightarrow [0,1] \subset \mathbb{R}$$

and $\mu_{\tilde{A}}(x)$ is called the membership function which maps X to the membership space $[0,1]$. The membership space expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 denote that an element completely belong to its corresponding fuzzy set, and the membership degree 0 denotes that an element does not belong to the fuzzy set. The membership degrees in the interval $(0,1)$ denote the partial membership to the fuzzy set.

Definition 2.2. The fuzzy empty set and the fuzzy whole set in a set X are denoted by $\bigcirc_X$ and 1_X and defined as $\bigcirc_X(x) = 0$ and $1_X(x) = 1$ for all $x \in X$ respectively.

Definition 2.3. Let λ be a fuzzy subset of X. A collection τ of fuzzy subsets of λ satisfying

(i) $k \cap \lambda \in \tau \;\; \forall \, k \in I$

(ii) $U_i \in \tau \; \forall \; i \in N \Rightarrow \bigcup\{U_i : i \in N\} \in \tau$.

(iii) $U, V \in \tau \Rightarrow U \cap V \in \tau$

is called a fuzzy topology on λ. The pair (λ, τ) is called a fuzzy topological space. Members of τ will be called fuzzy open sets and their complements with respect to λ are called fuzzy closed sets of (λ, τ).

If B be a given collection of fuzzy subset of λ, then the family of all possible unions and finite intersections of the member of B and the family $\{\lambda \cap k;\; k \in I\}$ is a fuzzy topology on λ and it will be denoted by $\tau(B)$.

Definition 2.4. $B \in \tau$ is called an open base of τ if every member of τ can be expressed as the union of some members of B.

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Definition 2.5. A fuzzy topological space (λ, τ) is said to be fuzzy Hausdorff space if $\forall\ X_p,\ Y_p \in \lambda(X \times Y),\ \exists U, V \in \tau$ such that $X_p \in U, Y_p \in V$ and $U \cap V = \emptyset$

Definition 2.6. A fuzzy topological space (λ, τ) is said to be fuzzy compact if $\forall\ \beta \in \tau$ satisfying $\bigcup\{U_i : U \in \beta\} = \lambda$ and $\forall \varepsilon > 0,\ \exists$ a finite subcollection β_0 of β such that $\bigcup\{U, U \in \beta_0\} \geq \lambda_\varepsilon$ where λ_ε is defined by $\lambda_\varepsilon(x) = \lambda(x) - \varepsilon$ or 0 according as $\lambda(x) > \varepsilon$ or $\lambda(x) \leq \varepsilon$

Definition 2.7. A fuzzy subset U of λ is said to be fuzzy separated. If $\exists\ v, \delta \in \tau$ such that $U = v \cup \delta$ and $v \cap \delta = \emptyset$.

Definition 2.8. A fuzzy topological space (λ, τ) is said to be fuzzy connected if β_0 fuzzy closed subset of (λ, τ) can be fuzzy separated.

Definition 2.9. A fuzzy topological vector space (ftvs) is a vector space Ξ over the field $\mathbb{K}$, Ξ equipped with a fuzzy topology σ and $\mathbb{K}$ equipped with the usual topology κ, such that the two mappings

(i) $(s, t) \longrightarrow s + t$ of $(\Xi, \sigma) \times (\Xi, \sigma)$ into (Ξ, σ)

(ii) $(\alpha, s) \longrightarrow \alpha s$ of $(\mathbb{K}, \kappa) \times (\Xi, \sigma)$ into (Ξ, σ)

are fuzzy continuous.

3 Main Results

Definition 3.1. *[8]* A triangular norm is a binary operation on the unit interval $[0, 1]$. That is, $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$
such that for all $x, y, z \in [0, 1]$ the following axioms are satisfied:

(i) $*(x, y) = *(y, x)$

(ii) $*(x, *(y, z)) = *(*(x, y), z)$

(iii) $*(x, y) \leq *(x, z)$ where $y \leq z$

(iv) $*(x, 1) = x * 1 = x, \quad *(x, 0) = x * 0 = 0.$

Definition 3.2. *[10]* A fuzzy seminorm on a vector space X is a function

$$\rho : X \times \mathbb{R} \longrightarrow [0,1]$$

such that

(P1) $\forall\, t \in \mathbb{R}$ with $t \leq 0,\ \ \rho(x,t) = 0$

(P2) $\forall\, t \in \mathbb{R}$ with $t > 0,\ \ \rho(cx,t) = \rho\left(x, \frac{t}{|c|}\right) \quad \text{if } c \neq 0$

(P3) $\forall\, t, s \in \mathbb{R},\ x, u \in X,\ \ \rho(x+u, t+s) \geq \rho(x,t) * \rho(u,s)$

(P4) $\rho(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} \rho(x,t) = 1$

A fuzzy seminorm ρ is a fuzzy norm N if it satisfies

$$\rho(x,t) = 1 \ \ \forall \ \ t \in \mathbb{R} \quad \text{with} \quad t > 0 \ \ \text{then} \ \ x = 0$$

Definition 3.3. A family $\mathcal{P}$ of fuzzy seminorms on Y is said to be separating if to each $y \neq 0$ corresponds at least one $\rho \in \mathcal{P}$ and $t \in \mathbb{R} \ni$

$$\rho(x,t) \neq 1$$

Let φ be a fuzzy seminorm on a vector space Y. We shall denote by $\bar{B}(0,\alpha,t) = \{y \in Y : \varphi(y,t) \geq 1-\alpha\}$ and $B(0,\alpha,t) = \{y \in Y : \varphi(y,t) > 1-\alpha\}$, the closed and open fuzzy balls respectively.

Definition 3.4. The Minkowski functional of the closed ball $\bar{B}(0,\alpha,1)$ and the open ball $B(0,\alpha,1)$ are given as follows: $\bar{q}(y) = \inf\{t > 0 : q(y,t) \geq 1-\alpha\},\ \ \alpha \in (0,1)$ and $q(y) = \inf\{t > 0 : q(y,t) > 1-\alpha\},\ \ \alpha \in (0,1)$ Define $q_\alpha(y) = \inf\{t > 0 : q(y,t) \geq 1-\alpha\},\ \ \alpha \in (0,1)$. Then $\{q_\alpha : \alpha \in (0,1)\}$ is an ascending family of seminorms on Y. We call these seminorms as $\alpha-$seminorms on Y corresponding to the fuzzy seminorm φ on Y.

Theorem 3.5. $\{q_\alpha : \alpha \in (0,1)\}$ generates a topology on Y which coincides with the topology of absorbing and balanced zero neighbourhoods.

Proof. Let q be the Minkowski functional of the open ball $B(0,\alpha,1)$. Then

$$\begin{aligned} q(y) &= \inf\{\lambda > 0 : y \in \lambda B(0,\alpha,1)\} \\ &= \inf\left\{\lambda > 0 : \frac{y}{\lambda} \in B(0,\alpha,1)\right\} \\ &= \inf\left\{\lambda > 0 : \varphi(\frac{y}{\lambda}, 1) > 1-\alpha\right\} \quad = \inf\{t > 0 : \varphi(y,t) > 1-\alpha\} = \varphi_\alpha(y) \end{aligned}$$

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Next is to show that the topology of a Locally convex topological vector space W is from the collection of fuzzy seminorms obtained as Minkowski functionals W_X associated to a local basis at 0 consisting of convex, balanced open set, since every neighbourhood of zero contains a balanced zero neighbourhood. To do this, we claim as follows:

Claim: For every open $X \in \mathcal{B}$ there exist corresponding fuzzy seminorms W_X such that $X = \{w \in W : W_X(w, \varepsilon) > 1 - \alpha\}$

Proof: For $w \in X$ the scalar multiplication continuity implies that there exist $\varepsilon > 0$ and a neighbourhood V of w such that

$$\begin{aligned} w + \lambda w = w(1+\lambda) = y,\ y \in X \text{ for } |1+\lambda| < \varepsilon \\ = (1+\lambda)^{-1} y = w. \end{aligned}$$

Thus, $w \in (1+\lambda)^{-1}X$. Therefore,

$$\begin{aligned} W_X(w) &= W_X((1+\lambda)^{-1}y) \\ &= (1+\lambda)^{-1} W_X(w) \\ &< (1+\lambda)^{-1} \\ &< 1. \end{aligned}$$

That is, $W_X(w, (1+\lambda)^{-1}) > 1 - \alpha, \ \ \alpha \in (0,1)$.

$W_X(w(1+\lambda), 1) > 1-\alpha = \inf\left\{(1+\lambda)^{-1} : W_X(w, (1+\lambda)^1) > 1 - \alpha\right\} = W_\alpha(w)$.

Hence the proof. □

4 Conclusion

The topology of the Fuzzy Locally convex Space W is the topology of family of seminorms which is same as the topology generated by the Minkowski functional.

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