 doi:10.20944/preprints202206.0224.v1

Small divisors effects in some singularly perturbed initial value problem with irregular singularity

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June, 14 2022

Abstract

We examine a nonlinear initial value problem both singularly perturbed in a complex parameter and singular in complex time at the origin. The study undertaken in this paper is the continuation of a joined work with A. Lastra published in 2015. A change of balance between the leading and a critical subdominant term of the problem considered in our previous work is performed. It leads to a phenomenon of coalescing singularities to the origin in the Borel plane w.r.t time for a finite set of holomorphic solutions constructed as Fourier series in space on horizontal complex strips. In comparison to our former study, an enlargement of the Gevrey order of the asymptotic expansion for these solutions relatively to the complex parameter is induced.
Key words: asymptotic expansion, Borel-Laplace transform, Fourier series, initial value problem, formal power series, singular perturbation. 2000 MSC: 35C10, 35C20.

1 Introduction

In this paper, we focus our attention on a singularly perturbed nonlinear partial differential equation outlined as

$$(1)\quad Q(\partial_z)u(t,z,\epsilon) = \epsilon^{k\delta_D}(t^{k+1}\partial_t)^{\delta_D}R_D(\partial_z)u(t,z,\epsilon) + P(t,z,\epsilon,\partial_t,\partial_z)u(t,z,\epsilon) + c_{Q_1,Q_2}Q_1(\partial_z)u(t,z,\epsilon)Q_2(\partial_z)u(t,z,\epsilon) + f(t,z,\epsilon)$$

for vanishing initial data $u(0,z,\epsilon) \equiv 0$ where

- the constants $k,\delta_D \geq 1$ are integers and $c_{Q_1,Q_2} \in \mathbb{C}^*$ is a given complex number,
- the expressions $Q(X)$,$R_D(X)$,$Q_1(X)$,$Q_2(X)$ stand for polynomials with complex coefficients and $P(t,z,\epsilon,V_1,V_2)$ represents a polynomial in the arguments t,V_1,V_2 with holomorphic coefficients w.r.t the perturbation parameter ϵ on a disc D_{ϵ_0} centered at 0 with prescribed radius $\epsilon_0 > 0$ and holomorphic in the space variable z on a horizontal strip in $\mathbb{C}$ framed as $H_\beta = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta\}$ with assigned width $2\beta > 0$,
- the forcing term $f(t,z,\epsilon)$ entails coefficients that rely polynomially on the time variable t, analytically in ϵ on D_{ϵ_0} and holomorphically in z on H_β.

This work is a natural continuation of the study [8] by A. Lastra and the author. Namely, in that paper, we have investigated a similar initial value problem as (1) shaped as

$$(2)\quad \boldsymbol{Q}(\partial_z)\partial_t \boldsymbol{y}(t,z,\epsilon) = \epsilon^{(\boldsymbol{\delta_D}-1)\boldsymbol{k}} t^{(\boldsymbol{\delta_D}-1)(\boldsymbol{k}+1)} \partial_t^{\boldsymbol{\delta_D}} \boldsymbol{R_D}(\partial_z)\boldsymbol{y}(t,z,\epsilon) + \boldsymbol{H}(t,\epsilon,\partial_t,\partial_z)\boldsymbol{y}(t,z,\epsilon) + \boldsymbol{Q}_1(\partial_z)\boldsymbol{y}(t,z,\epsilon)\boldsymbol{Q}_2(\partial_z)\boldsymbol{y}(t,z,\epsilon) + \boldsymbol{f}(t,z,\epsilon)$$

for prescribed null initial data $\boldsymbol{y}(0,z,\epsilon) \equiv 0$, where $\boldsymbol{k} \geq 1$,$\boldsymbol{\delta_D} \geq 2$ are integers, $\boldsymbol{H}$, $\boldsymbol{Q}_1,\boldsymbol{Q}_2$ are polynomials in their corresponding variables and where the forcing term $\boldsymbol{f}$ is subjected to the same features as the forcing term f appearing in (1). Among other constraints imposed on the profile of (2), we took for granted that the next key condition

$$(3)\quad \deg(\boldsymbol{Q}) \geq \deg(\boldsymbol{R_D})$$

hold. We constructed a set of genuine bounded holomorphic solutions $\boldsymbol{y}_p(t,z,\epsilon)$, for $0 \leq p \leq \boldsymbol{\varsigma}-1$, for some integer $\boldsymbol{\varsigma} \geq 2$, to (2), on domains $\boldsymbol{\mathcal{T}} \times \boldsymbol{H_\beta} \times \boldsymbol{\mathcal{E}}_p$, for a well chosen bounded sector $\boldsymbol{\mathcal{T}}$ edging the origin in $\mathbb{C}^*$, where $\boldsymbol{H_\beta}$ is some horizontal strip in $\mathbb{C}$ with width $2\boldsymbol{\beta} > 0$ and where $(\boldsymbol{\mathcal{E}}_p)_{0\leq p\leq \boldsymbol{\varsigma}-1}$ stands for a set of bounded sectors with radius $\boldsymbol{\epsilon}_0 > 0$ whose union contains a full neighborhood of 0 in $\mathbb{C}^*$, called a good covering in $\mathbb{C}^*$, see Definition 4 in this work. Such functions are modeled as Laplace transform of order $\boldsymbol{k}$ and inverse Fourier transforms on $\mathbb{R}$,

$$\boldsymbol{y}_p(t,z,\epsilon) = \frac{\boldsymbol{k}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\gamma_p}} \boldsymbol{\omega}_p(u,m,\epsilon) \exp\big(-(\frac{u}{\epsilon t})^{\boldsymbol{k}}\big) e^{\sqrt{-1}zm} \frac{du}{u} dm$$

along halflines $L_{\gamma_p} = [0,+\infty)e^{\sqrt{-1}\gamma_p} \subset \boldsymbol{U}_{\boldsymbol{d}_p} \cup \{0\}$, where $\boldsymbol{U}_{\boldsymbol{d}_p}$ is an unbounded sector bisected in direction $\boldsymbol{d}_p \in \mathbb{R}$ and where the so-called Borel map $\boldsymbol{\omega}_p$ represents a holomorphic function with exponential growth w.r.t u on a union $\boldsymbol{U}_{\boldsymbol{d}_p} \cup \boldsymbol{D_r}$, for some radius $\boldsymbol{r} > 0$, continuous w.r.t m on $\mathbb{R}$ with exponential decay and holomorphic w.r.t ϵ on the punctured disc $D_{\boldsymbol{\epsilon}_0} \setminus \{0\}$.

Furthermore, informations concerning asymptotic expansions as ϵ tends to 0 could be extracted. We proved that all the functions $\boldsymbol{y}_p$ share a common asymptotic expansion $\hat{\boldsymbol{y}}(t,z,\epsilon) = \sum_{n\geq 0} \boldsymbol{y}_n(t,z)\epsilon^n$ w.r.t ϵ, uniformly relatively to $(t,z) \in \boldsymbol{\mathcal{T}} \times \boldsymbol{H_\beta}$, where $\hat{\boldsymbol{y}}$ represents a formal power series with bounded coefficients $\boldsymbol{y}_n$ on $\boldsymbol{\mathcal{T}} \times \boldsymbol{H_\beta}$. This asymptotic expansion is (at most) of Gevrey order $1/\boldsymbol{k}$, meaning that

$$(4)\quad \sup_{\substack{t\in\boldsymbol{\mathcal{T}}\\ z\in\boldsymbol{H_\beta}}} |\boldsymbol{y}_p(t,z,\epsilon) - \sum_{l=0}^{n-1} \boldsymbol{y}_l(t,z)\epsilon^l| \leq \boldsymbol{C}\boldsymbol{M}^n \Gamma(1+\frac{n}{\boldsymbol{k}})|\epsilon|^n$$

for all $n \geq 1$, all $\epsilon \in \boldsymbol{\mathcal{E}}_p$, for suitable constants $\boldsymbol{C},\boldsymbol{M} > 0$.

The program undertaken in this work remains the same as in [8] and brings up

- the construction of bounded holomorphic solutions to (1),
- asymptotic expansions of these solutions as the parameter ϵ borders the origin.

As we will acknowledge later on, both aspects of the above record will substantially be altered, compared to [8], by the new assumption

$$(5)\quad \deg(Q) < \deg(R_D)$$

we here require for the equation (1).

The first item is completed in Section 5, Theorem 1. A finite set of bounded holomorphic solutions $u_p(t,z,\epsilon)$, for $0 \leq p \leq \varsigma - 1$, for some integer $\varsigma \geq 2$, to (1) are built up on domains $\mathcal{T} \times H_\beta \times \mathcal{E}_p$, that mirror the ones described above for $\boldsymbol{y}_p(t,z,\epsilon)$, for well chosen bounded sector $\mathcal{T}$ edging the origin, for the given strip H_β and for a set $(\mathcal{E}_p)_{0 \leq p \leq \varsigma-1}$ forming a good covering in $\mathbb{C}^*$. However, in the approach we follow, the restriction (5) disallows the setting up of solutions in the form of Fourier transforms on $\mathbb{R}$ in space z. Instead, they are presented as $2\pi-$periodic *Fourier series* with non negative modes. Indeed, each solution u_p is expressed as a Fourier sum

$$u_p(t,z,\epsilon) = \sum_{m \geq 0} u_{p,m}(t,\epsilon) e^{\sqrt{-1}zm}$$

whose coefficients $u_{p,m}(t,\epsilon)$ are molded as Laplace transforms of order k,

$$u_{p,m}(t,\epsilon) = k \int_{L_{\gamma_p}} \omega_{\mathfrak{d}_p}(u,m,\epsilon) \exp\left(-(\frac{u}{\epsilon t})^k\right) \frac{du}{u} \quad (6)$$

along similar halflines L_{γ_p} as in the Laplace integral part of $\boldsymbol{y}_p(t,z,\epsilon)$, for Borel maps $u \mapsto \omega_{\mathfrak{d}_p}(u,m,\epsilon)$ having at most exponential growth on fitting unbounded sectors $S_{\mathfrak{d}_p}$ bisected in direction $\mathfrak{d}_p \in \mathbb{R}$.

The second item is achieved in Section 6, Theorem 2. The existence of a formal power series $\hat{u}(t,z,\epsilon) = \sum_{n \geq 0} G_n(t,z)\epsilon^n$, with holomorphic bounded coefficients G_n on $\mathcal{T} \times H_\beta$, is established that represents the common asymptotic expansion for all the partial maps $\epsilon \mapsto u_p(t,z,\epsilon)$ on the sectors $\mathcal{E}_p$, uniformly on $\mathcal{T} \times H_\beta$ w.r.t the couple (t,z). This expansion remains of Gevrey type as in our former setting (2) but its order is no longer the inverse $1/k$ of the order of the Laplace transforms (6) involved but a *larger* quantity $1/\kappa$ relying both on $1/k$, on the degrees of Q and R_D and on δ_D, see (176).

We now discuss the origin of the discrepancy between [8] and the present contribution. In [8], the condition (3) allows the Borel map $u \mapsto \boldsymbol{\omega}_p(u,m,\epsilon)$ to be analytic on a mutual small disc $\boldsymbol{D_r}$ enclosing the origin, for all $m \in \mathbb{R}$. Devoided of this assumption, under the constraint (5), the Borel maps $u \mapsto \omega_{\mathfrak{d}_p}(u,m,\epsilon)$ are still analytic on discs centered at 0, but their radius $\rho_m > 0$ are shown to rely on m and furthermore the whole sequence $(\rho_m)_{m \geq 0}$ tends to 0 as m becomes large, see (44). It is worth noticing that both

- the construction of these Borel maps $\omega_{\mathfrak{d}_p}(u,m,\epsilon)$, $m \geq 0$ (reached by induction and fixed points arguments discussed in the technical and outstretched sections 3 and 4),
- the special arrangement of the singularities of the partial maps $u \mapsto \omega_{\mathfrak{d}_p}(u,m,\epsilon)$ and their discret nature

result from the shape of the solutions $u_p(t,z,\epsilon)$ we impose to be written as Fourier series with non negative modes. As shown in Theorem 1, the discret set $\{q_l(m), m \geq 0\}$, for $0 \leq l \leq k\delta_D - 1$ given by (42) of potential complex singularities of the Borel maps $u \mapsto \omega_{\mathfrak{d}_p}(u,m,\epsilon)$, $m \geq 0$, which accumulates at 0, has a direct effect on the order of exponential flatness for the difference of consecutive solutions $u_{p+1} - u_p$. At last, such an order is known to be related to the Gevrey order of the formal expansions of the solutions u_p w.r.t ϵ by the classical Ramis-Sibuya theorem.

The first occurence of such a correlation between small and approching 0 singularities in the Borel place and large multipliers for the asymptotic expansions in the physical plane arises in the seminal paper [2] by B. Braaksma and L. Stolovitch. They study normal forms of nonlinear differential systems

$$z^2 \frac{dx}{dz} = (\Lambda + zA)x(z) + zf(z,x(z)) \quad (7)$$

of so-called irregular type at $z = 0$ where Λ, A are diagonal constant matrices, $x = (x_1, \ldots, x_n)$ denotes a vector of $\mathbb{C}^n$, $n \geq 1$ and f is analytic near 0. By means of analytic changes of coordinates

$$x_i(z) = y_i(z) + g_i(z, y_1, \ldots, y_n), \quad i = 1, \ldots, n,$$

they transform the given system (7) into a so-called normal form consisting in its linear part

$$z^2 \frac{dy}{dz} = (\Lambda + zA)y(z)$$

with $y = (y_1, \ldots, y_n)$. They show that under *diophantine conditions* on Λ, the partial maps

$$z \mapsto g_i(z, u_1, \ldots, u_n) = \sum_{Q=(q_1,\ldots,q_n)\in\mathbb{N}^n} g_Q(z) u_1^{q_1} \cdots u_n^{q_n}$$

for $(u_1, \ldots, u_n)$ in a prescribed small polydisc in $\mathbb{C}^n$, have asymptotic expansions of Gevrey order $1 + \gamma$ on appropriate sectors edging 0 in $\mathbb{C}$, for $\gamma > 0$ related to the condition imposed on Λ. The amplification γ arises from the fact the $1 - Borel$ transform of $g_Q(z)$ (see [1] for an explanation of this terminology) are shown to be analytic only on a disc $D_{c/|Q|^{\gamma}}$, for some constant $c > 0$, whose radius tends to 0 as $|Q| = q_1 + \ldots + q_n$ becomes large.

Later on, the author and colleagues have unveiled comparable small divisors and large multipliers phenomena in different settings, see [6], [7], [9], [14], [16], [18]. Among them, two conspicuous contributions can be distinguished, one in the framework of partial differential equations and the second that concerns $q-$difference differential equations.

- In the paper [15], A. Lastra, J. Sanz and the author consider a nonlinear Cauchy problem framed as

(8) $$\epsilon^{r_0}(z\partial_z)^{r_1}(t^2\partial_t)^{r_2}\partial_z^S X(t,z,\epsilon) = H(t,z,\epsilon,\partial_t,\partial_z)X(t,z,\epsilon) + P(t,z,\epsilon,X(t,z,\epsilon))$$

  for given Cauchy data

(9) $$(\partial_z^j X)(t,0,\epsilon) = \varphi_j(t,\epsilon) \quad , \quad 0 \leq j \leq S-1,$$

  that are holomorphic on a product $\mathcal{T} \times \mathcal{E}$, for a fixed bounded sector $\mathcal{T}$ edging 0 and for a bounded sector $\mathcal{E}$ belonging to a good covering in $\mathbb{C}^*$. The equation (8) is singularly perturbed in the complex parameter ϵ and is both of irregular type at $t = 0$ and of Fuchsian type at $z = 0$. Its leading term is on the left handside of (8), for positive integers $r_0, r_1, r_2, S \geq 1$, H denotes a lower order differential operator, polynomial in t and holomorphic relatively to (z, ϵ) near the origin in $\mathbb{C}^2$ and P is some multivariate polynomial. A genuine holomorphic solution to (8), (9) is achieved as a convergent series

$$X(t,z,\epsilon) = \sum_{\beta\geq 0} X_\beta(t,\epsilon) z^\beta$$

  on some small disc D_r, $r > 0$, whose coefficients $X_\beta(t,\epsilon)$ are expressed as Laplace transforms of order 1 of some Borel maps $\tau \mapsto V_\beta(\tau,\epsilon)$ that turn out to be analytic only on discs $D_{c/(\beta+1)^{r_1/r_2}}$ whose radii tends to 0 as $\beta \to +\infty$ and on well chosen unbounded sectors avoiding its set of singularities. The coalescence to the origin of these singularities at polynomial speed induces a magnification of the Gevrey order of the asymptotic expansion

$$\hat{X}(t,z,\epsilon) = \sum_{n\geq 0} H_n(t,z)\epsilon^n$$

  of $X(t,z,\epsilon)$, uniformly in (t,z) on $\mathcal{T} \times D_r$, w.r.t ϵ on $\mathcal{E}$, which shows to be equal to $\frac{r_1+r_2}{r_0}$.

- In the work [6], A. Lastra and the author address the next linear Cauchy problem

$$\epsilon\partial_t\partial_z^S Y(t,z,\epsilon) = \mathcal{P}(z,\epsilon,\sigma_{q;t},\sigma_{q;z},\partial_t,\partial_z)Y(t,z,\epsilon) \tag{10}$$

for prescribed Cauchy data

$$(\partial_z^j Y)(t,0,\epsilon) = \phi_j(t,\epsilon) \quad , \quad 0 \leq j \leq S-1 \tag{11}$$

which are holomorphic on a product $\tilde{\mathcal{T}} \times \mathcal{E}$, for a suitable unbounded sector $\tilde{\mathcal{T}}$ laying apart of the origin at some large distance $R > 0$ and for a bounded sector $\mathcal{E}$ singled out of a good covering in $\mathbb{C}^*$. Equation (10) is singularly perturbed in the complex parameter ϵ and is of irregular type at $t = \infty$. Its principal term is displayed on the left handside of (10) for a positive integer $S \geq 1$. The lower order linear differential operator $\mathcal{P}$ with polynomial coefficients contains *contraction operators* $\sigma_{q;t}$ and $\sigma_{q;z}$ acting on functions through $\sigma_{q;t}f(t,z) = f(qt,z)$ and $\sigma_{q;z}f(t,z) = f(t,qz)$ for some fixed real number $0 < q < 1$. An actual holomorphic solution to (10), (11) is built up as a convergent series

$$Y(t,z,\epsilon) = \sum_{\beta \geq 0} Y_\beta(t,\epsilon)z^\beta$$

on some small disc D_r, $r > 0$, with coefficients $Y_\beta(t,\epsilon)$ shaped as Laplace transforms of order 1 of analytic Borel maps $\tau \mapsto W_\beta(\tau,\epsilon)$. These Borel maps are shown to be convergent around 0 only on a disc with radius $c_1 q^{c_2\beta}$ for some constants $c_1, c_2 > 0$. Again, their singularities merge at the origin but this time with geometric speed entailing a complete change of nature of the asymptotic expansions in ϵ on $\mathcal{E}$ for $Y(t,z,\epsilon)$. Namely, $Y(t,z,\epsilon)$ admits a formal asymptotic expansion

$$\hat{Y}(t,z,\epsilon) = \sum_{n \geq 0} I_n(t,z)\epsilon^n$$

of so-called $q-$Gevrey type of some order $s \geq 1$ relatively to ϵ, a growth which belongs to a larger scale than the Gevrey rate, meaning that

$$\sup_{\substack{t \in \tilde{\mathcal{T}} \\ z \in D_r}} |Y(t,z,\epsilon) - \sum_{k=0}^{n-1} I_k(t,z)\epsilon^k| \leq L_0 L_1^n q^{-\frac{s}{2}n^2}|\epsilon|^n$$

for all integers $n \geq 1$, all $\epsilon \in \mathcal{E}$, for fitting constants $L_0, L_1 > 0$.

In the context of this paper, the singularly perturbed leading term $\epsilon^{k\delta_D}(t^{k+1}\partial_t)^{\delta_D}R_D(\partial_z)$ of (1) is modeled with a regular differential operator $R_D(\partial_z)$ in space at $z = 0$ and with an irregular operator $t^{k+1}\partial_t$ in time at $t = 0$ and remains essentially the same as in our former work [8]. In contrast with our previous paper [15] depicted above and the two quoted references [14], [16], the small divisor phenomenon does not appear only from the peculiar shape of the leading term, but from a *change of balance* between the principal term of (1) and the lower order term $Q(\partial_z)$, compared to the one considered in [8]. Here, the *couple* of operators $\epsilon^{k\delta_D}(t^{k+1}\partial_t)^{\delta_D}R_D(\partial_z)$ and $Q(\partial_z)$ holds a central place in the origin of the coalescing singularities arising in the Borel plane for the set of sectorial holomorphic solutions built up in the study.

2 Outline of the main initial value problem and associated auxiliary problems

2.1 Laplace transforms

In this tiny paragraph, we include some lead-in definition and features of the Laplace transform of integer order $k \geq 1$, stated in the work [8], that will show up in the upcoming sections.

Definition 1 *Let $k \geq 1$ be an integer. We denote $S_{d,\delta} = \{\tau \in \mathbb{C}^* : |d - \arg(\tau)| < \delta\}$ some unbounded sector with bisecting direction $d \in \mathbb{R}$ and opening $2\delta > 0$ and we consider a disc D_ρ centered at 0 with radius $\rho > 0$. Let $w : S_{d,\delta} \cup D_\rho \to \mathbb{C}$ be a holomorphic function that vanishes at 0 and suffers the bounds : there exist $C > 0$ and $K > 0$ such that*

$$|w(\tau)| \leq C|\tau| \exp(K|\tau|^k) \tag{12}$$

for all $\tau \in S_{d,\delta}$. We define the Laplace transform of w of order k in the direction d as the integral transform

$$\mathcal{L}_k^d(w)(T) = k \int_{L_\gamma} w(u) \exp(-(\frac{u}{T})^k) \frac{du}{u}$$

along a half-line $L_\gamma = [0, +\infty)e^{\sqrt{-1}\gamma} \subset S_{d,\delta} \cup \{0\}$, where γ relies on T and is chosen in such a way that $\cos(k(\gamma - \arg(T))) \geq \delta_1$, for some fixed real number $\delta_1 > 0$. The function $\mathcal{L}_k^d(w)(T)$ is well defined, holomorphic and bounded on any sector

$$S_{d,\theta,R^{1/k}} = \{T \in \mathbb{C}^* : |T| < R^{1/k} \quad , \quad |d - \arg(T)| < \theta/2\},$$

where $0 < \theta < \frac{\pi}{k} + 2\delta$ and $0 < R < \delta_1/K$.

We pinpoint some important feature : if $w(\tau) = \sum_{n\geq 1} w_n \tau^n$ represents an entire function w.r.t $\tau \in \mathbb{C}$ with the bounds (12), its Laplace transform $\mathcal{L}_k^d(w)(T)$ does not depend on the direction d in $\mathbb{R}$ and represents a bounded holomorphic function on $D_{R^{1/k}}$ whose Taylor expansion is represented by the convergent series $X(T) = \sum_{n\geq 1} w_n \Gamma(\frac{n}{k}) T^n$ on $D_{R^{1/k}}$, where $\Gamma(x)$ stands for the Gamma function.

2.2 The main problem displayed

Within this subsection, we introduce the principal nonlinear initial value problem under analysis in this paper. Its shape is stated as

$$\begin{aligned} Q(\partial_z)u(t,z,\epsilon) = \epsilon^{\Delta_D}(t^{k+1}\partial_t)^{\delta_D} R_D(\partial_z)u(t,z,\epsilon) + \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} \partial_t^{\delta_l} a_l(z,\epsilon) R_l(\partial_z) u(t,z,\epsilon) \\ + c_{Q_1,Q_2} Q_1(\partial_z)u(t,z,\epsilon) Q_2(\partial_z)u(t,z,\epsilon) + f(t,z,\epsilon) \end{aligned} \tag{13}$$

where $D \geq 2$, $k \geq 1$ stand for some integers, $c_{Q_1,Q_2} \in \mathbb{C}^*$ is some non zero complex number, for vanishing initial data $u(0,z,\epsilon) \equiv 0$.

The constants Δ_D,δ_D,Δ_l,d_l and δ_l for $1 \leq l \leq D-1$ represent positive integers that are subjected to the next lineup of technical conditions (which are listed in the order of appearence through the work):

1. The next two equalities

$$\Delta_D = k\delta_D \quad , \quad d_l = (k+1)\delta_l + d_{k,l} \tag{14}$$

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hold for some integers $d_{k,l} \geq 1$, for all $1 \leq l \leq D-1$.

2. For all $1 \leq l \leq D-1$, we ask that

$$k\delta_D \geq d_{k,l} + k(\delta_l + 2) \tag{15}$$

for the integer $d_{k,l}$ introduced above.

3. We require that

$$\Delta_l - d_l + \delta_l > 2k \tag{16}$$

for any $1 \leq l \leq D-1$.

The maps $Q(X)$, $R_D(X)$, $R_l(X)$, for $1 \leq l \leq D-1$ and $Q_1(X), Q_2(X)$ are polynomials with complex coefficients that obey the next restrictions:

4. We have

$$\deg(R_D) > \deg(Q). \tag{17}$$

and we assume the existence of some open bounded sector S_{Q,R_D} centered at 0, not containing the origin (that will be determined later on in the work), such that

$$Q(im)/R_D(im) \in S_{Q,R_D} \tag{18}$$

for all integers $m \geq 0$. In particular, one can find two constants $\mathfrak{Q}, \mathfrak{R}_D > 0$ such that

$$|Q(im)| \leq \mathfrak{Q}(1+m)^{\deg(Q)} \quad , \quad |R_D(im)| \geq \mathfrak{R}_D(1+m)^{\deg(R_D)} \tag{19}$$

which yields

$$|\frac{Q(im)}{R_D(im)}| \leq \frac{\mathfrak{Q}/\mathfrak{R}_D}{(1+m)^{\deg(R_D)-\deg(Q)}} \tag{20}$$

for all integers $m \geq 0$.

5. The positive sequence

$$u_m = |Q(im)|/|R_D(im)| \quad , \quad m \geq 0 \tag{21}$$

is decreasing.

6. For each $1 \leq l \leq D-1$, the next inequality on degrees

$$\deg(R_l) \leq \deg(Q) + \frac{\deg(R_D) - \deg(Q)}{k\delta_D}(d_l - \delta_l) \tag{22}$$

holds.

7. We have

$$\deg(Q) \geq \max(\deg(Q_1), \deg(Q_2)) \tag{23}$$

We describe some set of Banach spaces of complex sequences which are discrete versions of Banach spaces of continuous functions used for the first time by the author in the work [17] and established in [4].

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Definition 2 *Let β, μ be real numbers. We denote $SE_{(\beta,\mu)}$ the vector space of all sequences $h : \mathbb{N} \mapsto \mathbb{C}$ such that*

$$||h(m)||_{(\beta,\mu)} = \sup_{m \geq 0} (1+m)^{\mu} \exp(\beta m) |h(m)|$$

is a finite quantity. The space $SE_{(\beta,\mu)}$ endowed with the norm $||.||_{(\beta,\mu)}$ turns out to be a Banach space.

The coefficients $a_l(z, \epsilon)$, $1 \leq l \leq D-1$, are built up in the next manner. For all $1 \leq l \leq D-1$, let $(A_l(m, \epsilon))_{m \geq 0}$ be a sequence

- that belong to $SE_{(\beta,\mu)}$, for some given positive real numbers $\beta, \mu > 0$ that are required to fulfill the conditions

$$\mu > 1 + \deg(Q_1) \quad , \quad \mu > 1 + \deg(Q_2) \tag{24}$$

- that rely analytically on ϵ on a disc D_{ϵ_0} with center at 0 in $\mathbb{C}$ and with radius $\epsilon_0 > 0$ for which a constant $\mathbf{A}_{l,\epsilon_0} > 0$ can be picked out with

$$\sup_{\epsilon \in D_{\epsilon_0}} ||A_l(m, \epsilon)||_{(\beta,\mu)} \leq \mathbf{A}_{l,\epsilon_0} \tag{25}$$

  for all $m \geq 0$.

We set the coefficient a_l as the Fourier series

$$a_l(z, \epsilon) = \sum_{m \geq 0} A_l(m, \epsilon) e^{\sqrt{-1} z m}$$

for all $1 \leq l \leq D-1$. According to the bounds

$$\begin{aligned} |\sum_{m \geq 0} A_l(m, \epsilon) e^{\sqrt{-1} z m}| &\leq \mathbf{A}_{l,\epsilon_0} \sum_{m \geq 0} (1+m)^{-\mu} e^{-\beta m} e^{-\mathrm{Im}(z) m} \\ &\leq \mathbf{A}_{l,\epsilon_0} \sum_{m \geq 0} (1+m)^{-\mu} e^{-(\beta - \beta')m} \end{aligned} \tag{26}$$

provided that z belongs to any horizontal strip

$$H_{\beta'} = \{ z \in \mathbb{C} / |\mathrm{Im}(z)| < \beta' \} \tag{27}$$

for given $0 < \beta' < \beta$, we observe that the maps $(z, \epsilon) \mapsto a_l(z, \epsilon)$ are bounded holomorphic on the product $H_{\beta'} \times D_{\epsilon_0}$, for any fixed $0 < \beta' < \beta$.

The forcing term $f(t, z, \epsilon)$ is constructed in the next way. Let J be a given subset of the positive integers $\mathbb{N}^*$. For $j \in J$, let $(\varphi_j(m, \epsilon))_{m \geq 0}$ be a sequence

- that appertains to the space $SE_{(\beta,\mu)}$, for $\beta, \mu > 0$ prescribed above.
- that depends analytically on ϵ in D_{ϵ_0}, with a constant $\boldsymbol{\varphi}_{j,\epsilon_0}$ such that

$$\sup_{\epsilon \in D_{\epsilon_0}} ||\varphi_j(m, \epsilon)||_{(\beta,\mu)} \leq \boldsymbol{\varphi}_{j,\epsilon_0} \tag{28}$$

 doi:10.20944/preprints202206.0224.v1

We introduce the next polynomial

$$\varphi(u,m,\epsilon) = \sum_{j \in J} \varphi_j(m,\epsilon)u^j \tag{29}$$

in the variable u, with coefficients in $SE_{(\beta,\mu)}$, that depends analytically in ϵ on D_{ϵ_0}. We set

$$F(T,m,\epsilon) = k\int_{L_\gamma} \varphi(u,m,\epsilon)\exp(-(u/T)^k)\frac{du}{u}$$

as Laplace transform of order k of φ, for all integers $m \geq 0$, where $L_\gamma = [0,+\infty)e^{\sqrt{-1}\gamma}$ stands for a halfline in direction $\gamma \in \mathbb{R}$, which relies on T in a way that $\cos(k(\gamma - \arg(T)))$ remains strictly positive.

We set the forcing term f as

$$f(t,z,\epsilon) = \sum_{m \geq 0} F(\epsilon t,m,\epsilon)e^{\sqrt{-1}zm} \tag{30}$$

On grounds of Definition 1, the expression $f(t,z,\epsilon)$ can be written as a polynomial

$$f(t,z,\epsilon) = \sum_{j \in J} f_j(z,\epsilon)\Gamma(\frac{j}{k})(\epsilon t)^j$$

in the variable ϵt, where

$$f_j(z,\epsilon) = \sum_{m \geq 0} \varphi_j(m,\epsilon)e^{\sqrt{-1}zm}$$

for $j \in J$ and represents a bounded holomorphic map relatively to $(z,\epsilon) \in H_{\beta'} \times D_{\epsilon_0}$, for any given $0 < \beta' < \beta$.

2.3 Sequences of related initial value problems

In the first part of this section, we reduce the study of our principal problem (13) to a sequence of parameter depending ordinary differential equations in one single complex variable.

We figure out to seek for solutions $u(t,z,\epsilon)$ to (13) with vanishing initial data at $t = 0$ in the form of a $2\pi-$periodic Fourier series in z and rescaled in time,

$$u(t,z,\epsilon) = \sum_{m \geq 0} U(\epsilon t,m,\epsilon)e^{\sqrt{-1}zm} \tag{31}$$

for some sequence of expressions $U(T,m,\epsilon)$ standing for its Fourier coefficients.

By following the usual derivation rule under the summation sign and product of Fourier series, applied at a *formal level* at this point of the work, we arrive at the next sequence of ordinary differential equations fulfilled by the Fourier coefficients $U(T,m,\epsilon)$,

$$\begin{aligned} Q(im)U(T,m,\epsilon) &= \epsilon^{\Delta_D - k\delta_D}(T^{k+1}\partial_T)^{\delta_D}R_D(im)U(T,m,\epsilon) \\ &+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l + \delta_l}T^{d_l}\partial_T^{\delta_l}\Big(\sum_{m_1+m_2=m} A_l(m_1,\epsilon)R_l(im_2)U(T,m_2,\epsilon)\Big) \\ &+ c_{Q_1,Q_2}\sum_{m_1+m_2=m} Q_1(im_1)U(T,m_1,\epsilon)Q_2(im_2)U(T,m_2,\epsilon) + F(T,m,\epsilon) \end{aligned} \tag{32}$$

 doi:10.20944/preprints202206.0224.v1

for all integers $m \geq 0$.

At a second stage, we convert this latter sequence of ODEs (32) into a sequence of convolution equations that will be analyzed in the forthcoming sections. Now, we search for Fourier coefficients $U(T,m,\epsilon)$ in the form of a Laplace transform of order k,

$$U(T,m,\epsilon) = k\int_{L_{\gamma}} \omega(u,m,\epsilon)\exp(-(u/T)^{k})\frac{du}{u} \tag{33}$$

where $L_{\gamma} = [0,+\infty)e^{\sqrt{-1}\gamma}$ stands for a halfline in well chosen directions γ (described later on in the work). In order to let the above integrals be well defined, we assume that the so-called *Borel maps* $u \mapsto \omega(u,m,\epsilon)$ are holomorphic on some common unbounded sector S_{γ} centered at 0 with bisecting direction γ and subjected to exponential bounds of order k,

$$|\omega(u,m,\epsilon)| \leq K_{m}\exp(A|u|^{k}) \tag{34}$$

for all $u \in S_{\gamma}$, for given constants $K_m > 0$ (relying on m) and $A > 0$. Furthermore, the dependence in ϵ is supposed to be holomorphic on the disc D_{ϵ_0}. More precise bounds will be disclosed in the upcoming sections.

We recall a key formula introduced in the work [19].

Lemma 1 *Let $k,\delta \geq 1$ be integers. One can single out real numbers $A_{\delta,p}$, for $1 \leq p \leq \delta-1$ such that*

$$T^{\delta(k+1)}\partial_{T}^{\delta} = (T^{k+1}\partial_{T})^{\delta} + \sum_{1\leq p\leq \delta-1} A_{\delta,p}T^{k(\delta-p)}(T^{k+1}\partial_{T})^{p} \tag{35}$$

where, by convention, we assume that the sum $\sum_{1\leq p\leq\delta-1}[...]$ vanishes when $\delta=1$ in (35).

Under the assumption (14), by means of the above lemma, we rewrite the sequence (32) into the form

$$\begin{aligned} Q(im)U(T,m,\epsilon) &= (T^{k+1}\partial_{T})^{\delta_D}R_{D}(im)U(T,m,\epsilon) + \sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l+\delta_l}T^{d_{k,l}} \\ &\times \big[(T^{k+1}\partial_T)^{\delta_l} + \sum_{1\leq p\leq\delta_l-1}A_{\delta_l,p}T^{k(\delta_l-p)}(T^{k+1}\partial_T)^{p}\big]\big(\sum_{m_1+m_2=m}A_l(m_1,\epsilon)R_l(im_2)U(T,m_2,\epsilon)\big) \\ &+ c_{Q_1,Q_2}\sum_{m_1+m_2=m}Q_1(im_1)U(T,m_1,\epsilon)Q_2(im_2)U(T,m_2,\epsilon) + F(T,m,\epsilon) \end{aligned} \tag{36}$$

where all the differential operators appearing in (32) are expresses through the one single operator of so-called irregular type $T^{k+1}\partial_T$.

Now, we bring to mind some handy formulas for the Laplace transform under the action of product, product with a monomial and action of the differential operator $T^{k+1}\partial_T$. These identities have already been stated in our forgoing work [8] using formal expansions but a thorough proof in the analytic setting can be found in the paper [10], Lemma 2.

Lemma 2 *Assume that the map ω in the expression (33) undergoes the upper bounds (34). Then, the next three identities hold.*

1. The action of the differential operator $T^{k+1}\partial_T$ on the Laplace transform (33) has the shape

$$T^{k+1}\partial_{T}U(T,m,\epsilon) = k\int_{L_{\gamma}}\{ku^{k}\omega(u,m,\epsilon)\}\exp(-(u/T)^{k})\frac{du}{u}. \tag{37}$$

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2. Let $h \geq 1$ be an integer. The product of T^h with (33) has the form

$$T^{h}U(T,m,\epsilon) = k\int_{L_{\gamma}} \left\{ \frac{u^{k}}{\Gamma(h/k)} \int_{0}^{u^{k}} (u^{k}-s)^{\frac{h}{k}-1}\omega(s^{1/k},m,\epsilon)\frac{ds}{s} \right\} \exp(-(u/T)^{k})\frac{du}{u}. \tag{38}$$

3. For any given integers $m_1, m_2 \geq 0$, the product $U(T,m_1,\epsilon)U(T,m_2,\epsilon)$ has the next Laplace transform profile

$$\begin{aligned} &U(T,m_1,\epsilon)U(T,m_2,\epsilon) \\ &= k\int_{L_{\gamma}} \left\{ u^{k}\int_{0}^{u^{k}} \omega((u^{k}-s)^{1/k},m_1,\epsilon)\omega(s^{1/k},m_2,\epsilon)\frac{1}{(u^{k}-s)s}ds \right\} \exp(-(u/T)^{k})\frac{du}{u}. \end{aligned} \tag{39}$$

This last lemma applied to the recast sequence (36) enables the following statement.

The sequence of maps $U(T,m,\epsilon)$, $m \geq 0$ fulfills the relation (36) **if** the sequence of Borel maps $\omega(u,m,\epsilon)$, $m \geq 0$ is subjected to the next convolution relation

$$\begin{aligned} &Q(im)\omega(u,m,\epsilon) = (ku^{k})^{\delta_D}R_{D}(im)\omega(u,m,\epsilon) \\ &+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l + \delta_l} \Big[\sum_{m_1+m_2=m} A_l(m_1,\epsilon)R_l(im_2)\frac{u^k}{\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}(ks)^{\delta_l}\omega(s^{1/k},m_2,\epsilon)\frac{ds}{s} \\ &\quad + \sum_{m_1+m_2=m} A_l(m_1,\epsilon)R_l(im_2) \sum_{1\leq p\leq \delta_l-1} A_{\delta_l,p}\frac{u^k}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \\ &\quad \times \int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}(ks)^{p}\omega(s^{1/k},m_2,\epsilon)\frac{ds}{s}\Big] \\ &+ \sum_{m_1+m_2=m} c_{Q_1,Q_2}u^k\int_0^{u^k} Q_1(im_1)\omega((u^k-s)^{1/k},m_1,\epsilon)Q_2(im_2)\omega(s^{1/k},m_2,\epsilon)\frac{1}{(u^k-s)s}ds \\ &\quad + \varphi(u,m,\epsilon) \end{aligned} \tag{40}$$

In order to solve this latter relation in the ongoing sections, our strategy consists in reorganizing it into fixed point equations (unveiled later in (47) and (48)). In the process, we ask to perform a division by the next sequence of polynomials

$$P_m(u) = Q(im) - (ku^k)^{\delta_D}R_D(im) \tag{41}$$

for all integers $m \geq 0$. By straight computation, we observe that the roots of $u \mapsto P_m(u)$ are given by the explicit expressions

$$q_l(m) = \left(\frac{|Q(im)|}{|R_D(im)|k^{\delta_D}}\right)^{\frac{1}{k\delta_D}} \exp\left(\sqrt{-1}(\arg(\frac{Q(im)}{R_D(im)k^{\delta_D}})\frac{1}{k\delta_D} + \frac{2\pi l}{k\delta_D})\right) \tag{42}$$

for all $0 \leq l \leq k\delta_D - 1$, all integers $m \geq 0$.

Provided that the aperture of the sector S_{Q,R_D} selected in (18) is taken small enough, one can single out an unbounded sector S_d, centered at 0, with bisecting direction $d \in \mathbb{R}$, with small aperture, such that

$$S_d \cap \left(\cup_{m\geq 0} \cup_{0\leq l\leq \delta_D k-1} \{q_l(m)\} \right) = \emptyset. \tag{43}$$

We set

$$\rho_m = \frac{|q_l(m)|}{2} = \frac{1}{2}\Big(\frac{|Q(im)|}{|R_D(im)|k^{\delta_D}}\Big)^{\frac{1}{k\delta_D}} \tag{44}$$

for all integers $m \geq 0$.

In the subsequent sections, we request sharp lower bounds for $P_m(u)$ provided that u belongs to S_d or to the disc D_{ρ_m}, for any given integer $m \geq 0$.

- When u belongs to S_d, according to the condition (43), one can express u in a factorized form
$$u = \rho e^{\sqrt{-1}\theta} q_l(m)$$
for some radius $\rho > 0$, some real angle $\theta \notin 2\pi\mathbb{Z}$, for some fixed root $q_l(m)$ of $P_m(u)$. As a result, we get the next equality
$$|P_m(u)| = |Q(im) - k^{\delta_D}(\rho e^{\sqrt{-1}\theta})^{k\delta_D}(q_l(m))^{k\delta_D}R_D(im)| = |Q(im)||(1-(\rho e^{\sqrt{-1}\theta})^{k\delta_D}| \tag{45}$$
since, by construction, we observe that
$$(q_l(m))^{k\delta_D} = \frac{Q(im)}{R_D(im)k^{\delta_D}}.$$
- When $u \in D_{\rho_m}$, one can split u in the shape
$$u = \rho e^{\sqrt{-1}\theta} q_l(m)$$
for some radius ρ such that $0 \leq \rho \leq 1/2$ and some angle $\theta \in \mathbb{R}$, for some prescribed root $q_l(m)$. Following the same computation as above, one reaches
$$|P_m(u)| = |Q(im)||(1-(\rho e^{\sqrt{-1}\theta})^{k\delta_D}| \geq |Q(im)|(1-(\frac{1}{2})^{k\delta_D}). \tag{46}$$

According to the above constrution, for $m = 0$, the convolution relation (40) can be rewritten in the form of a nonlinear fixed point equation

$$\omega(u,0,\epsilon) = \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l + \delta_l}\Big[\frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \frac{u^k}{\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}k^{\delta_l}s^{\delta_l}\omega(s^{1/k},0,\epsilon)\frac{ds}{s}$$
$$+ \frac{A_l(0,\epsilon)R_l(0)}{P_0(u)}\sum_{1\leq p\leq\delta_l-1} A_{\delta_l,p}\frac{u^k}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}k^p s^p \omega(s^{1/k},0,\epsilon)\frac{ds}{s}\Big]$$
$$+ c_{Q_1,Q_2}\frac{u^k}{P_0(u)}\int_0^{u^k} Q_1(0)\omega((u^k-s)^{1/k},0,\epsilon)Q_2(0)\omega(s^{1/k},0,\epsilon)\frac{1}{(u^k-s)s}ds + \frac{\varphi(u,0,\epsilon)}{P_0(u)} \tag{47}$$

provided that u belongs to $S_d \cup D_{\rho_0}$, for the unbounded sector S_d and disc D_{ρ_0} introduced overhead.

 doi:10.20944/preprints202206.0224.v1

For any integer $m \geq 1$, we recast the equation (40) in the form of a linear fixed point equation with suitably chosen forcing term

$$
\begin{aligned}
(48)\quad \omega(u,m,\epsilon) = \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l + \delta_l} \\
\times \Big[A_l(0,\epsilon) R_l(im) \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} (ks)^{\delta_l} \omega(s^{1/k}, m, \epsilon) \frac{ds}{s} \\
+ A_l(0,\epsilon) R_l(im) \sum_{1 \leq p \leq \delta_l - 1} A_{\delta_l,p} \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l - p)}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} (ks)^p \omega(s^{1/k}, m, \epsilon) \frac{ds}{s} \Big] \\
+ c_{Q_1,Q_2} \frac{u^k}{P_m(u)} \int_0^{u^k} Q_1(0)\omega((u^k-s)^{1/k}, 0, \epsilon) Q_2(im) \omega(s^{1/k}, m, \epsilon) \frac{1}{(u^k-s)s} ds \\
+ c_{Q_1,Q_2} \frac{u^k}{P_m(u)} \int_0^{u^k} Q_1(im)\omega((u^k-s)^{1/k}, m, \epsilon) Q_2(0) \omega(s^{1/k}, 0, \epsilon) \frac{1}{(u^k-s)s} ds + \psi(u,m,\epsilon)
\end{aligned}
$$

where the forcing term $\psi(u,m,\epsilon)$ is given by the expression

$$
\begin{aligned}
(49)\quad \psi(u,m,\epsilon) = \frac{\varphi(u,m,\epsilon)}{P_m(u)} + \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l + \delta_l} \\
\Big[\sum_{\substack{m_1+m_2=m \\ m_2<m}} A_l(m_1,\epsilon) R_l(im_2) \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}}{k}-1} (ks)^{\delta_l} \omega(s^{1/k}, m_2, \epsilon) \frac{ds}{s} \\
+ \sum_{\substack{m_1+m_2=m \\ m_2<m}} A_l(m_1,\epsilon) R_l(im_2) \sum_{1 \leq p \leq \delta_l - 1} A_{\delta_l,p} \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \\
\times \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} (ks)^p \omega(s^{1/k}, m_2, \epsilon) \frac{ds}{s} \Big] \\
+ \sum_{\substack{m_1+m_2=m \\ m_1<m, m_2<m}} c_{Q_1,Q_2} \frac{u^k}{P_m(u)} \int_0^{u^k} Q_1(im_1)\omega((u^k-s)^{1/k}, m_1, \epsilon) Q_2(im_2) \omega(s^{1/k}, m_2, \epsilon) \frac{1}{(u^k-s)s} ds
\end{aligned}
$$

In the sequel, we introduce a modification of the Banach spaces discussed in [8] in which we plan to solve both fixed point equations introduced above.

Definition 3 *Let $\nu, \beta, \mu > 0$ be positive real numbers, $\epsilon_0 > 0$ a fixed radius and $k \geq 1$ be an integer. We set S_d as an unbounded sector centered at 0 with bisecting direction $d \in \mathbb{R}$. For any integer $m \geq 0$ and any given complex number $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, we denote $F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$ the vector space of holomorphic functions $\tau \mapsto h(\tau)$ on $D_{\rho_m} \cup S_d$ such that*

$$
(50)\quad ||h(\tau)||_{(\nu,\beta,\mu,k,\epsilon,m)} = \sup_{\tau \in D_{\rho_m} \cup S_d} (1+m)^{\mu} \frac{1 + |\frac{\tau}{\epsilon}|^{2k}}{|\tau/\epsilon|} \exp\big(\beta m - \nu |\frac{\tau}{\epsilon}|^k\big) |h(\tau)|
$$

where the radius ρ_m is given by (44). The space $F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$ equipped with the norm $||.||_{(\nu,\beta,\mu,k,\epsilon,m)}$ is a Banach space.

 doi:10.20944/preprints202206.0224.v1

3 Solving the nonlinear convolution equation (47).

Within this section, we intend to provide a solution to the nonlinear fixed point equation displayed in (47) within the Banach space $F^{d}_{(\nu,\beta,\mu,k,\epsilon,0)}$ introduced in Definition 3 for $m = 0$.

For all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, let us consider the nonlinear map

$$(51)\quad \mathcal{G}_{\epsilon}(\omega(u,0)) := \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l + \delta_l} \Big[\frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \frac{u^k}{\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} k^{\delta_l} s^{\delta_l} \omega(s^{1/k},0)\frac{ds}{s}$$
$$+ \frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \sum_{1 \le p \le \delta_l - 1} A_{\delta_l,p} \frac{u^k}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} k^p s^p \omega(s^{1/k},0)\frac{ds}{s} \Big]$$
$$+ c_{Q_1,Q_2} \frac{u^k}{P_0(u)} \int_0^{u^k} Q_1(0)\omega((u^k-s)^{1/k},0) Q_2(0)\omega(s^{1/k},0) \frac{1}{(u^k-s)s} ds + \frac{\varphi(u,0,\epsilon)}{P_0(u)}$$

In the forthcoming proposition, we show that $\mathcal{G}_{\epsilon}$ stands for a $1/2-$Lipschitz map on a well chosen ball of the Banach space $F^{d}_{(\nu,\beta,\mu,k,\epsilon,0)}$.

Proposition 1 *We select the aperture of the sector S_{Q,R_D} introduced in (18) in a way that we can pick up an unbounded sector S_d that fulfills the constraint (43). Then, one can choose a radius $\epsilon_0 > 0$ small enough, a constant $c_{Q_1,Q_2} \in \mathbb{C}^*$ close enough to 0 and an appropriate radius $\varpi > 0$ such that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the map $\mathcal{G}_{\epsilon}$ enjoys the next two qualities*

- *The next inclusion*

$$(52)\qquad \mathcal{G}_{\epsilon}(\bar{B}_{\varpi}) \subset \bar{B}_{\varpi}$$

  holds, where $\bar{B}_{\varpi}$ represents the closed ball of radius $\varpi > 0$ centered at 0 in the space $F^{d}_{(\nu,\beta,\mu,k,\epsilon,0)}$.

- *The shrinking property*

$$(53)\qquad ||\mathcal{G}_{\epsilon}(\omega_1) - \mathcal{G}_{\epsilon}(\omega_2)||_{(\nu,\beta,\mu,k,\epsilon,0)} \le \frac{1}{2} ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)}$$

  for all $\omega_1, \omega_2 \in \bar{B}_{\varpi}$.

Proof We check the first item claiming the inclusion (52). We take some real number $\varpi > 0$ and consider an element $\omega(u,0)$ of $F^{d}_{(\nu,\beta,\mu,k,\epsilon,0)}$ subjected to

$$||\omega(u,0)||_{(\nu,\beta,\mu,k,\epsilon,0)} \le \varpi.$$

It means in particular that the next inequality

$$(54)\qquad |\omega(u,0)| \le \varpi \frac{|u/\epsilon|}{1 + |\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

holds for all $u \in D_{\rho_0} \cup S_d$.
1. We first provide bounds on the disc D_{ρ_0} for the first piece

$$(55)\qquad \frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \frac{u^k}{\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} k^{\delta_l} s^{\delta_l} \omega(s^{1/k},0)\frac{ds}{s}$$

appearing in $\mathcal{G}_{\epsilon}$.

We parametrize the segment $[0,u^k]$ by means of $s=u^k r$ with $0 \leq r \leq 1$ which gives rise to

$$\int_{0}^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} s^{\delta_l} \omega(s^{1/k},0) \frac{ds}{s} = u^{k(\frac{d_{k,l}}{k}-1+\delta_l)} \int_{0}^{1} (1-r)^{\frac{d_{k,l}}{k}-1} r^{\delta_l} \omega(ur^{1/k},0) \frac{dr}{r}. \tag{56}$$

By means of the lower bounds (46) along with the upper estimates (25) and (54), we reach

$$\begin{aligned} &\left|\frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \frac{u^k}{\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}}{k}-1} k^{\delta_l} s^{\delta_l} \omega(s^{1/k},0) \frac{ds}{s}\right| \\ &\leq \mathbf{A}_{l,\epsilon_0} \frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})} \frac{|R_l(0)|}{|Q(0)|(1-(\frac{1}{2})^{k\delta_D})} \rho_0^{d_{k,l}+k\delta_l} \varpi |\frac{u}{\epsilon}| \exp\left(\nu|\frac{u}{\epsilon}|^k\right) \int_0^1 (1-r)^{\frac{d_{k,l}}{k}-1} r^{\delta_l + \frac{1}{k}} \frac{dr}{r} \\ &\qquad \leq \mathbf{A}_{l,\epsilon_0} \frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})} \frac{|R_l(0)|}{|Q(0)|(1-(\frac{1}{2})^{k\delta_D})} \rho_0^{d_{k,l}+k\delta_l} \int_0^1 (1-r)^{\frac{d_{k,l}}{k}-1} r^{\delta_l+\frac{1}{k}} \frac{dr}{r} \\ &\qquad\qquad \times \varpi |\frac{u}{\epsilon}| \exp\left(\nu|\frac{u}{\epsilon}|^k\right) \frac{1}{1+|\frac{u}{\epsilon}|^{2k}} \frac{1}{|\epsilon|^{2k}} [\rho_0^{2k} + \epsilon_0^{2k}] \end{aligned} \tag{57}$$

provided that $u \in D_{\rho_0}$ since

$$1 + |\frac{u}{\epsilon}|^{2k} \leq 1 + \frac{\rho_0^{2k}}{|\epsilon|^{2k}} \leq \frac{1}{|\epsilon|^{2k}} [\rho_0^{2k} + \epsilon_0^{2k}] \tag{58}$$

for all $u \in D_{\rho_0}$.

2. We give bounds on the sector S_d for the first piece (55). By dint of the equality (45) coupled with the upper estimates (25) and (54), we observe that

$$\begin{aligned} &\left|\frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \frac{u^k}{\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}}{k}-1} k^{\delta_l} s^{\delta_l} \omega(s^{1/k},0) \frac{ds}{s}\right| \\ &\leq \mathbf{A}_{l,\epsilon_0} \frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})} |R_l(0)| \frac{1}{|Q(0)||1-(\rho e^{\sqrt{-1}\theta})^{k\delta_D}|} \rho^{d_{k,l}+k\delta_l} (2\rho_0)^{d_{k,l}+k\delta_l} (1 + \frac{(\rho 2\rho_0)^{2k}}{|\epsilon|^{2k}}) \\ &\qquad \times \varpi |\frac{u}{\epsilon}| \frac{1}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right) \times \left(\int_0^1 (1-r)^{\frac{d_{k,l}}{k}-1} r^{\delta_l+\frac{1}{k}} \frac{dr}{r}\right) \end{aligned} \tag{59}$$

whenever $u = \rho e^{\sqrt{-1}\theta} q_l(0) \in S_d$, due to

$$1 + |\frac{u}{\epsilon}|^{2k} \leq 1 + \frac{(\rho 2\rho_0)^{2k}}{|\epsilon|^{2k}} \tag{60}$$

for all $u = \rho e^{\sqrt{-1}\theta} q_l(0) \in S_d$. Furthermore, under the condition (15), we get some constant $\bar{C}_{k,\delta_D,d_{k,l},\delta_l} > 0$ such that

$$\begin{aligned} \frac{1}{|1-(\rho e^{\sqrt{-1}\theta})^{k\delta_D}|} \rho^{d_{k,l}+k\delta_l} (1 + \frac{(\rho 2\rho_0)^{2k}}{|\epsilon|^{2k}}) &\leq \frac{\rho^{d_{k,l}+k\delta_l}((2\rho_0)^{2k}\rho^{2k} + \epsilon_0^{2k})}{|1-(\rho e^{\sqrt{-1}\theta})^{k\delta_D}|} \frac{1}{|\epsilon|^{2k}} \\ &\leq \bar{C}_{k,\delta_D,d_{k,l},\delta_l} \frac{1}{|\epsilon|^{2k}} \end{aligned} \tag{61}$$

for all $\rho > 0$. According to (59) and (61), we deduce that

$$(62)\quad \left|\frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \frac{u^k}{\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} k^{\delta_l} s^{\delta_l} \omega(s^{1/k},0) \frac{ds}{s}\right| \leq \mathbf{A}_{l,\epsilon_0} \frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})} |R_l(0)| \frac{\bar{C}_{k,\delta_D,d_{k,l},\delta_l}}{|Q(0)|} \frac{1}{|\epsilon|^{2k}} (2\rho_0)^{d_{k,l}+k\delta_l} \times \varpi |\frac{u}{\epsilon}| \frac{1}{1+|\frac{u}{\epsilon}|^{2k}} \exp(\nu|\frac{u}{\epsilon}|^k) \times (\int_0^1 (1-r)^{\frac{d_{k,l}}{k}-1} r^{\delta_l + \frac{1}{k}} \frac{dr}{r})$$

for all $u = \rho e^{\sqrt{-1}\theta} q_l(0) \in S_d$.

3. Bounds on the disc D_{ρ_0} for the second piece

$$(63)\quad \frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \sum_{1 \leq p \leq \delta_l - 1} A_{\delta_l,p} \frac{u^k}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} k^p s^p \omega(s^{1/k},0) \frac{ds}{s}$$

of $\mathcal{G}_\epsilon$ are presented.

We parametrize the segment $[0,u^k]$ through $s = u^k r$ for $0 \leq r \leq 1$ which yields

$$(64)\quad \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} s^p \omega(s^{1/k},0) \frac{ds}{s} = u^{k(\frac{d_{k,l}}{k}+\delta_l-1)} \int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^p \omega(ur^{1/k},0) \frac{dr}{r}$$

Based on the lower bounds (46) together with the upper estimates (25) and (54), we obtain

$$(65)\quad \left|\frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \sum_{1 \leq p \leq \delta_l - 1} A_{\delta_l,p} \frac{u^k}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} k^p s^p \omega(s^{1/k},0) \frac{ds}{s}\right| \leq \mathbf{A}_{l,\epsilon_0} \sum_{1 \leq p \leq \delta_l - 1} |A_{\delta_l,p}| \frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \frac{1}{1-(\frac{1}{2})^{k\delta_D}} (\int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^{p+\frac{1}{k}} \frac{dr}{r}) \times \frac{|R_l(0)|}{|Q(0)|} \rho_0^{d_{k,l}+k\delta_l} \varpi |\frac{u}{\epsilon}| \exp(\nu|\frac{u}{\epsilon}|^k) \frac{1}{1+|\frac{u}{\epsilon}|^{2k}} \frac{1}{|\epsilon|^{2k}} [\rho_0^{2k} + \epsilon_0^{2k}]$$

for all $u \in D_{\rho_0}$, according to (58).

4. Estimates on the sector S_d for the second piece (63) are displayed. Due to the equality (45) in addition to the upper estimates (25) and (54), we check that

$$(66)\quad \left|\frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \sum_{1 \leq p \leq \delta_l - 1} A_{\delta_l,p} \frac{u^k}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} k^p s^p \omega(s^{1/k},0) \frac{ds}{s}\right| \leq \mathbf{A}_{l,\epsilon_0} \sum_{1 \leq p \leq \delta_l - 1} |A_{\delta_l,p}| \frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} (2\rho_0)^{d_{k,l}+k\delta_l} \frac{|R_l(0)|}{|Q(0)|} \bar{C}_{k,\delta_D,d_{k,l},\delta_l} \frac{1}{|\epsilon|^{2k}} \times (\int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^{p+\frac{1}{k}} \frac{dr}{r}) \varpi \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp(\nu|\frac{u}{\epsilon}|^k)$$

for all $u = \rho e^{\sqrt{-1}\theta} q_l(0) \in S_d$, owing to (60) and (61).

5. Bounds of the third piece of $\mathcal{G}_{\epsilon}$

$$c_{Q_1,Q_2}\frac{u^k}{P_0(u)}\int_0^{u^k} Q_1(0)\omega((u^k-s)^{1/k},0)Q_2(0)\omega(s^{1/k},0)\frac{1}{(u^k-s)s}ds \tag{67}$$

are disclosed on the union $S_d \cup D_{\rho_0}$. On account of (54), we know that

$$|\omega((u^k-s)^{1/k},0)| \leq \varpi\frac{|(u^k-s)^{1/k}|/|\epsilon|}{1+\frac{|u^k-s|^2}{|\epsilon|^{2k}}}\exp\big(\nu\frac{|u^k-s|}{|\epsilon|^k}\big)$$

and

$$|\omega(s^{1/k},0)| \leq \varpi\frac{|s^{1/k}|/|\epsilon|}{1+\frac{|s|^2}{|\epsilon|^{2k}}}\exp\big(\nu\frac{|s|}{|\epsilon|^k}\big)$$

for all $s \in [0,u^k]$, all $u \in S_d \cup D_{\rho_0}$. We deduce that

$$\Big|u^k\int_0^{u^k} Q_1(0)\omega((u^k-s)^{1/k},0)Q_2(0)\omega(s^{1/k},0)\frac{1}{(u^k-s)s}ds\Big| \leq |Q_1(0)||Q_2(0)|\varpi^2 \\ \times |u|^k\int_0^{|u|^k}\frac{(|u|^k-h)^{1/k}/|\epsilon|}{1+\frac{(|u|^k-h)^2}{|\epsilon|^{2k}}}\frac{h^{1/k}/|\epsilon|}{1+\frac{h^2}{|\epsilon|^{2k}}}\frac{1}{(|u|^k-h)h}dh \times \exp\big(\nu|\frac{u}{\epsilon}|^k\big) \tag{68}$$

for all $u \in S_d \cup D_{\rho_0}$. The following lemma is crucial.

Lemma 3 *There exists a constant K_k (relying on k) such that*

$$|u|^k\int_0^{|u|^k}\frac{(|u|^k-h)^{1/k}/|\epsilon|}{1+\frac{(|u|^k-h)^2}{|\epsilon|^{2k}}}\frac{h^{1/k}/|\epsilon|}{1+\frac{h^2}{|\epsilon|^{2k}}}\frac{1}{(|u|^k-h)h}dh \leq K_k\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \tag{69}$$

for all $u \in S_d \cup D_{\rho_0}$, all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$.

Proof We first make the change of variable $h = |\epsilon|^k h'$ in the integral appearing on the left handside of (69) that yields

$$|u|^k\int_0^{|u|^k}\frac{(|u|^k-h)^{1/k}/|\epsilon|}{1+\frac{(|u|^k-h)^2}{|\epsilon|^{2k}}}\frac{h^{1/k}/|\epsilon|}{1+\frac{h^2}{|\epsilon|^{2k}}}\frac{1}{(|u|^k-h)h}dh \\ = (\frac{|u|}{|\epsilon|})^k\int_0^{(\frac{|u|}{|\epsilon|})^k}\frac{((\frac{|u|}{|\epsilon|})^k-h')^{1/k}}{1+((\frac{|u|}{|\epsilon|})^k-h')^2}\frac{(h')^{1/k}}{1+(h')^2}\frac{1}{(\frac{|u|}{|\epsilon|})^k-h'}\frac{1}{h'}dh' = (\frac{|u|}{|\epsilon|})^k F_k((\frac{|u|}{|\epsilon|})^k) \tag{70}$$

where the map F_k is given by the integral expression

$$F_k(x) = \int_0^x \frac{(x-h')^{1/k}}{1+(x-h')^2}\frac{(h')^{1/k}}{1+(h')^2}\frac{1}{(x-h')}\frac{1}{h'}dh'.$$

For $k=1$, this function F_1 can be explicitely computed in the form

$$F_1(x) = 2\frac{\log(1+x^2)+x\arctan(x)}{x(x^2+4)}$$

for all $x > 0$. As a result, we deduce a constant $K_1 > 0$ for which

$$F_1(x) \leq \frac{K_1}{1+x^2} \tag{71}$$

for all $x > 0$. Gathering (70) and (71) gives rise to the expected bounds (69) for $k = 1$.

Let $k \geq 2$. We make the further change of variable $h' = xu$, for $0 \leq u \leq 1$ in the integral defining F_k and get

$$F_k(x) = x^{\frac{2}{k}-1} \int_0^1 \frac{1}{(1+x^2(1-u)^2)(1+x^2u^2)(1-u)^{1-\frac{1}{k}}u^{1-\frac{1}{k}}} du.$$

A partial fraction decomposition gives rise to the splitting

$$F_k(x) = F_{k,1}(x) + F_{k,2}(x)$$

where

$$F_{k,1}(x) = x^{\frac{2}{k}-1}\frac{1}{x^2+4}\hat{F}_{k,1}(x) \quad , \quad F_{k,2}(x) = x^{\frac{2}{k}-1}\frac{1}{x^2+4}\hat{F}_{k,2}(x)$$

with

$$\hat{F}_{k,1}(x) = \int_0^1 \frac{3-2u}{(1+x^2(1-u)^2)(1-u)^{1-\frac{1}{k}}u^{1-\frac{1}{k}}} du,$$

$$\hat{F}_{k,2}(x) = \int_0^1 \frac{2u+1}{(1+x^2u^2)(1-u)^{1-\frac{1}{k}}u^{1-\frac{1}{k}}} du$$

At last, we provide upper bounds for each piece $\hat{F}_{k,j}(x)$ on $(0,+\infty)$ for $j = 1,2$. Indeed, by carrying out the change of variable $u' = xu$, we get that

$$\hat{F}_{k,2}(x) \leq 3\int_0^1 \frac{1}{(1+x^2u^2)(1-u)^{1-\frac{1}{k}}u^{1-\frac{1}{k}}} du = 3x^{-1/k}\int_0^x \frac{1}{(1+(u')^2)(1-\frac{u'}{x})^{1-\frac{1}{k}}(u')^{1-\frac{1}{k}}} du'$$

which yields a constant $\hat{K}_{k,2} > 0$ such that

$$\hat{F}_{k,2}(x) \leq \hat{K}_{k,2}x^{-1/k}$$

for all $x > 0$. In a similar manner, by means of the change of variable $u' = x(1-u)$, we obtain a constant $\hat{K}_{k,1} > 0$ for which

$$\hat{F}_{k,1}(x) \leq \hat{K}_{k,1}x^{-1/k}$$

holds whenever $x > 0$. Therefore, a constant $K_k > 0$ can be found such that

$$F_k(x) \leq K_k \frac{x^{\frac{1}{k}-1}}{x^2+1} \tag{72}$$

for all $x > 0$ and according to (70) we reach the awaited bounds (69) for any integer $k \geq 2$. □

Besides, owing to the equality (45) and the lower bounds (46), we get a constant $C_{P_0} > 0$ with

$$|P_0(u)| \geq C_{P_0}|Q(0)| \tag{73}$$

provided that $u \in S_d \cup D_{\rho_0}$. Finally, collecting the estimates (68), (69) and (73), we arrive at

$$\tag{74} \left| c_{Q_1,Q_2} \frac{u^k}{P_0(u)} \int_0^{u^k} Q_1(0)\omega((u^k-s)^{1/k},0)Q_2(0)\omega(s^{1/k},0)\frac{1}{(u^k-s)s}ds \right| \\ \leq |c_{Q_1,Q_2}| \frac{|Q_1(0)||Q_2(0)|}{C_{P_0}|Q(0)|}\varpi^2 K_k \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

for all $u \in S_d \cup D_{\rho_0}$.

6. Bounds of the tail piece

$$\tag{75} \frac{\varphi(u,0,\epsilon)}{P_0(u)}$$

of $\mathcal{G}_\epsilon$ are displayed. Departing from the very definition (29) and according to (28), we observe that

$$\tag{76} |\varphi(u,0,\epsilon)| \leq \left[\sum_{j\in J} \boldsymbol{\varphi}_{j,\epsilon_0}|\epsilon|^j|\frac{u}{\epsilon}|^{j-1}(1+|\frac{u}{\epsilon}|^{2k})\exp\left(-\nu|\frac{u}{\epsilon}|^k\right)\right]\frac{|\frac{u}{\epsilon}|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^k\right) \\ \leq \left[\sum_{j\in J}\boldsymbol{\varphi}_{j,\epsilon_0}|\epsilon|^j \sup_{x\geq 0} x^{j-1}(1+x^{2k})e^{-\nu x^k}\right]\frac{|\frac{u}{\epsilon}|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^k\right) \leq \boldsymbol{\varphi}_{\epsilon_0}\frac{|\frac{u}{\epsilon}|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

for all $u \in S_d \cup D_{\rho_0}$, all $\epsilon \in D_{\epsilon_0}\setminus\{0\}$, for the constant

$$\tag{77} \boldsymbol{\varphi}_{\epsilon_0} = \sum_{j\in J}\boldsymbol{\varphi}_{j,\epsilon_0}\epsilon_0^j \sup_{x\geq 0} x^{j-1}(1+x^{2k})e^{-\nu x^k}.$$

Bearing in mind (73), we deduce that

$$\tag{78} \left|\frac{\varphi(u,0,\epsilon)}{P_0(u)}\right| \leq \frac{\boldsymbol{\varphi}_{\epsilon_0}}{C_{P_0}|Q(0)|}\frac{|\frac{u}{\epsilon}|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

for all $u \in S_d \cup D_{\rho_0}$, all $\epsilon \in D_{\epsilon_0}\setminus\{0\}$.

Now, we select the radius $\epsilon_0 > 0$ and the constant $|c_{Q_1,Q_2}| > 0$ small enough and take $\varpi > 0$ suitably in a way that

$$\tag{79} \sum_{l=1}^{D-1}|\epsilon|^{\Delta_l-d_l+\delta_l}\Big\{\max\Big(\mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\frac{|R_l(0)|}{|Q(0)|(1-(\frac{1}{2})^{k\delta_D})}\rho_0^{d_{k,l}+k\delta_l}\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r} \\ \times\varpi\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}], \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}|R_l(0)|\frac{\bar{C}_{k,\delta_D,d_{k,l},\delta_l}}{|Q(0)|}\frac{1}{|\epsilon|^{2k}}(2\rho_0)^{d_{k,l}+k\delta_l} \\ \times\varpi\times\Big(\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r}\Big)\Big) \\ +\max\Big(\mathbf{A}_{l,\epsilon_0}\sum_{1\leq p\leq\delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}\frac{1}{1-(\frac{1}{2})^{k\delta_D}}\Big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\Big) \\ \times\frac{|R_l(0)|}{|Q(0)|}\rho_0^{d_{k,l}+k\delta_l}\varpi\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}], \mathbf{A}_{l,\epsilon_0}\sum_{1\leq p\leq\delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}(2\rho_0)^{d_{k,l}+k\delta_l} \\ \times\frac{|R_l(0)|}{|Q(0)|}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}}\times\Big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\Big)\varpi\Big)\Big\} \\ +|c_{Q_1,Q_2}|\frac{|Q_1(0)||Q_2(0)|}{C_{P_0}|Q(0)|}\varpi^2K_k+\frac{\boldsymbol{\varphi}_{\epsilon_0}}{C_{P_0}|Q(0)|}\leq\varpi$$

holds for all $\epsilon \in D_{\epsilon_0}\setminus\{0\}$. Notice that such an inequality can be reached since

- The quantity $|\epsilon|^{\Delta_l - d_l + \delta_l - 2k}$ can be taken appropriately small when $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, for $\epsilon_0 > 0$ small enough, according to the condition (16).
- The constant φ_{ϵ_0} displayed in (77) is suitably close to 0 provided that $\epsilon_0 > 0$ is taken small enough.

At last, the collection of the six above bounds (57), (62), (65), (66), (74), (78) under the restriction (79) implies that

$$|\mathcal{G}_{\epsilon}(\omega(u,0))| \leq \varpi \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^{k}\right)$$

for all $u \in D_{\rho_0} \cup S_d$ and prescribed ϵ in $D_{\epsilon_0} \setminus \{0\}$, which means that the inclusion (52) holds true.

In the second part of the proof, we focus on the Lipschitz property (53).

Let $\epsilon \in D_{\epsilon_0} \setminus \{0\}$ and set two elements ω_1, ω_2 of the closed ball $\bar{B}_{\varpi}$ in $F^{d}_{(\nu,\beta,\mu,k,\epsilon,0)}$ where the radius $\varpi > 0$ has been prescribed in the first step of the proof.

By construction of the norm, we observe that

$$|\omega_1(u,0) - \omega_2(u,0)| \leq ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)} \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^{k}\right) \tag{80}$$

together with

$$|\omega_j(u,0)| \leq \varpi \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^{k}\right) \tag{81}$$

for $j = 1, 2$, as long as $u \in D_{\rho_0} \cup S_d$. The next list of bounds A. B. C. and D. is a direct aftermath of the bounds 1. 2. 3. and 4. reached in the first part of the proof.

A. Upper bounds on the disc D_{ρ_0} for the first piece

$$\frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \frac{u^k}{\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} k^{\delta_l} s^{\delta_l} \left(\omega_1(s^{1/k},0) - \omega_2(s^{1/k},0)\right) \frac{ds}{s} \tag{82}$$

that shows up in $\mathcal{G}_{\epsilon}(\omega_1) - \mathcal{G}_{\epsilon}(\omega_2)$ are supplied. Indeed,

$$\begin{aligned} &\left|\frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \frac{u^k}{\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} k^{\delta_l} s^{\delta_l} \left(\omega_1(s^{1/k},0) - \omega_2(s^{1/k},0)\right) \frac{ds}{s}\right| \\ &\leq \mathbf{A}_{l,\epsilon_0} \frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})} \frac{|R_l(0)|}{|Q(0)|(1-(\frac{1}{2})^{k\delta_D})} \rho_0^{d_{k,l}+k\delta_l} \int_0^1 (1-r)^{\frac{d_{k,l}}{k}-1} r^{\delta_l + \frac{1}{k}} \frac{dr}{r} \\ &\times ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)} |\frac{u}{\epsilon}| \exp\left(\nu|\frac{u}{\epsilon}|^{k}\right) \frac{1}{1+|\frac{u}{\epsilon}|^{2k}} \frac{1}{|\epsilon|^{2k}} [\rho_0^{2k} + \epsilon_0^{2k}] \end{aligned} \tag{83}$$

provided that $u \in D_{\rho_0}$.

B. Bounds on the sector S_d for the first piece (82) are displayed. Namely,

$$\begin{aligned} &\left|\frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \frac{u^k}{\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} k^{\delta_l} s^{\delta_l} \left(\omega_1(s^{1/k},0) - \omega_2(s^{1/k},0)\right) \frac{ds}{s}\right| \\ &\leq \mathbf{A}_{l,\epsilon_0} \frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})} |R_l(0)| \frac{\bar{C}_{k,\delta_D,d_{k,l},\delta_l}}{|Q(0)|} \frac{1}{|\epsilon|^{2k}} (2\rho_0)^{d_{k,l}+k\delta_l} \\ &\times ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)} |\frac{u}{\epsilon}| \frac{1}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^{k}\right) \times \left(\int_0^1 (1-r)^{\frac{d_{k,l}}{k}-1} r^{\delta_l + \frac{1}{k}} \frac{dr}{r}\right) \end{aligned} \tag{84}$$

for all $u \in S_d$.

C. Bounds on the disc D_{ρ_0} for the second block

$$(85)\quad \frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \sum_{1 \leq p \leq \delta_l - 1} A_{\delta_l,p} \frac{u^k}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} \times k^p s^p \big(\omega_1(s^{1/k},0) - \omega_2(s^{1/k},0)\big) \frac{ds}{s}$$

of the difference $\mathcal{G}_\epsilon(\omega_1) - \mathcal{G}_\epsilon(\omega_2)$ are established. Indeed,

$$(86)\quad \Big| \frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \sum_{1 \leq p \leq \delta_l - 1} A_{\delta_l,p} \frac{u^k}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} \times k^p s^p \big(\omega_1(s^{1/k},0) - \omega_2(s^{1/k},0)\big) \frac{ds}{s} \Big|$$
$$\leq \mathbf{A}_{l,\epsilon_0} \sum_{1 \leq p \leq \delta_l - 1} |A_{\delta_l,p}| \frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \frac{1}{1 - (\frac{1}{2})^{k\delta_D}} \Big(\int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^{p+\frac{1}{k}} \frac{dr}{r} \Big)$$
$$\times \frac{|R_l(0)|}{|Q(0)|} \rho_0^{d_{k,l}+k\delta_l} ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)} |\frac{u}{\epsilon}| \exp\big(\nu|\frac{u}{\epsilon}|^k\big) \frac{1}{1+|\frac{u}{\epsilon}|^{2k}} \frac{1}{|\epsilon|^{2k}} [\rho_0^{2k} + \epsilon_0^{2k}]$$

for all $u \in D_{\rho_0}$.

D. Bounds on the sector S_d for the second block (85) are stated. As expected,

$$(87)\quad \Big| \frac{A_l(0,\epsilon)R_l(0)}{P_0(u)} \sum_{1 \leq p \leq \delta_l - 1} A_{\delta_l,p} \frac{u^k}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} k^p s^p \times \big(\omega_1(s^{1/k},0) - \omega_2(s^{1/k},0)\big) \frac{ds}{s} \Big|$$
$$\leq \mathbf{A}_{l,\epsilon_0} \sum_{1 \leq p \leq \delta_l - 1} |A_{\delta_l,p}| \frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} (2\rho_0)^{d_{k,l}+k\delta_l} \frac{|R_l(0)|}{|Q(0)|} \bar{C}_{k,\delta_D,d_{k,l},\delta_l} \frac{1}{|\epsilon|^{2k}}$$
$$\times \Big(\int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^{p+\frac{1}{k}} \frac{dr}{r} \Big) ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)} \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\big(\nu|\frac{u}{\epsilon}|^k\big)$$

whenever $u \in S_d$.

E. We focus on bounds for the third block

$$(88)\quad c_{Q_1,Q_2} \frac{u^k}{P_0(u)} \int_0^{u^k} Q_1(0)\omega_1((u^k-s)^{1/k},0) Q_2(0)\omega_1(s^{1/k},0) \frac{1}{(u^k-s)s} ds - c_{Q_1,Q_2} \frac{u^k}{P_0(u)} \int_0^{u^k} Q_1(0)\omega_2((u^k-s)^{1/k},0) Q_2(0)\omega_2(s^{1/k},0) \frac{1}{(u^k-s)s} ds$$

of the difference $\mathcal{G}_\epsilon(\omega_1) - \mathcal{G}_\epsilon(\omega_2)$. At first, by means of the identity $ab - cd = (a-c)b + c(b-d)$,

 doi:10.20944/preprints202206.0224.v1

we can rewrite the above difference as

$$
\begin{aligned}
(89)\quad & c_{Q_1,Q_2}\frac{u^k}{P_0(u)}\int_0^{u^k} Q_1(0)\omega_1((u^k-s)^{1/k},0)Q_2(0)\omega_1(s^{1/k},0)\frac{1}{(u^k-s)s}ds\\
&- c_{Q_1,Q_2}\frac{u^k}{P_0(u)}\int_0^{u^k} Q_1(0)\omega_2((u^k-s)^{1/k},0)Q_2(0)\omega_2(s^{1/k},0)\frac{1}{(u^k-s)s}ds\\
&= c_{Q_1,Q_2}Q_1(0)Q_2(0)\frac{1}{P_0(u)}\big[u^k\int_0^{u^k}(\omega_1-\omega_2)((u^k-s)^{1/k},0)\omega_1(s^{1/k},0)\frac{1}{(u^k-s)s}ds\\
&+ u^k\int_0^{u^k}\omega_2((u^k-s)^{1/k},0)(\omega_1-\omega_2)(s^{1/k},0)\frac{1}{(u^k-s)s}ds\big].
\end{aligned}
$$

In view of (80) and (81), we know that

$$|(\omega_1-\omega_2)((u^k-s)^{1/k},0)| \leq ||\omega_1-\omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)}\frac{|(u^k-s)^{1/k}|/|\epsilon|}{1+\frac{|u^k-s|^2}{|\epsilon|^{2k}}}\exp\big(\nu\frac{|u^k-s|}{|\epsilon|^k}\big)$$

and

$$|\omega_1(s^{1/k},0)| \leq \varpi\frac{|s^{1/k}|/|\epsilon|}{1+\frac{|s|^2}{|\epsilon|^{2k}}}\exp\big(\nu\frac{|s|}{|\epsilon|^k}\big)$$

along with

$$|\omega_2((u^k-s)^{1/k},0)| \leq \varpi\frac{|(u^k-s)^{1/k}|/|\epsilon|}{1+\frac{|u^k-s|^2}{|\epsilon|^{2k}}}\exp\big(\nu\frac{|u^k-s|}{|\epsilon|^k}\big)$$

and

$$|(\omega_1-\omega_2)(s^{1/k},0)| \leq ||\omega_1-\omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)}\frac{|s^{1/k}|/|\epsilon|}{1+\frac{|s|^2}{|\epsilon|^{2k}}}\exp\big(\nu\frac{|s|}{|\epsilon|^k}\big)$$

for all $s \in [0,u^k]$, whenever $u \in S_d \cup D_{\rho_0}$. We deduce that

$$
\begin{aligned}
(90)\quad & \Big|u^k\int_0^{u^k} Q_1(0)(\omega_1-\omega_2)((u^k-s)^{1/k},0)Q_2(0)\omega_1(s^{1/k},0)\frac{1}{(u^k-s)s}ds\Big|\\
&\leq |Q_1(0)||Q_2(0)|||\omega_1-\omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)}\varpi\\
&\times |u|^k\int_0^{|u|^k}\frac{(|u|^k-h)^{1/k}/|\epsilon|}{1+\frac{(|u|^k-h)^2}{|\epsilon|^{2k}}}\frac{h^{1/k}/|\epsilon|}{1+\frac{h^2}{|\epsilon|^{2k}}}\frac{1}{(|u|^k-h)h}dh\times\exp\big(\nu|\frac{u}{\epsilon}|^k\big)
\end{aligned}
$$

together with

$$
\begin{aligned}
(91)\quad & \Big|u^k\int_0^{u^k} Q_1(0)\omega_2((u^k-s)^{1/k},0)Q_2(0)(\omega_1-\omega_2)(s^{1/k},0)\frac{1}{(u^k-s)s}ds\Big|\\
&\leq |Q_1(0)||Q_2(0)|||\omega_1-\omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)}\varpi\\
&\times |u|^k\int_0^{|u|^k}\frac{(|u|^k-h)^{1/k}/|\epsilon|}{1+\frac{(|u|^k-h)^2}{|\epsilon|^{2k}}}\frac{h^{1/k}/|\epsilon|}{1+\frac{h^2}{|\epsilon|^{2k}}}\frac{1}{(|u|^k-h)h}dh\times\exp\big(\nu|\frac{u}{\epsilon}|^k\big).
\end{aligned}
$$

for all $u \in S_d \cup D_{\rho_0}$.

By means of Lemma 3 and paying regard to the lower bounds (73), we conclude from (90) and (91) that

$$(92)\quad \Big|c_{Q_1,Q_2}\frac{u^k}{P_0(u)}\int_0^{u^k} Q_1(0)\omega_1((u^k-s)^{1/k},0)Q_2(0)\omega_1(s^{1/k},0)\frac{1}{(u^k-s)s}ds - c_{Q_1,Q_2}\frac{u^k}{P_0(u)}\int_0^{u^k} Q_1(0)\omega_2((u^k-s)^{1/k},0)Q_2(0)\omega_2(s^{1/k},0)\frac{1}{(u^k-s)s}ds\Big| \leq |c_{Q_1,Q_2}|\frac{|Q_1(0)||Q_2(0)|}{C_{P_0}|Q(0)|}2\varpi||\omega_1-\omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)}K_k\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp(\nu|\frac{u}{\epsilon}|^k)$$

for all $u \in S_d \cup D_{\rho_0}$.

We prescribe $\epsilon_0 > 0$ and $|c_{Q_1,Q_2}| > 0$ close enough to 0 enabling the next inequality

$$(93)\quad \sum_{l=1}^{D-1}|\epsilon|^{\Delta_l-d_l+\delta_l}\Big\{\max\Big(\mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\frac{|R_l(0)|}{|Q(0)|(1-(\frac{1}{2})^{k\delta_D})}\rho_0^{d_{k,l}+k\delta_l}\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r} \times\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}], \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}|R_l(0)|\frac{\bar{C}_{k,\delta_D,d_{k,l},\delta_l}}{|Q(0)|}\frac{1}{|\epsilon|^{2k}}(2\rho_0)^{d_{k,l}+k\delta_l} \times(\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r})\Big) + \max\Big(\mathbf{A}_{l,\epsilon_0}\sum_{1\leq p\leq\delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}\frac{1}{1-(\frac{1}{2})^{k\delta_D}}(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}) \times\frac{|R_l(0)|}{|Q(0)|}\rho_0^{d_{k,l}+k\delta_l}\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}], \mathbf{A}_{l,\epsilon_0}\sum_{1\leq p\leq\delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}(2\rho_0)^{d_{k,l}+k\delta_l} \times\frac{|R_l(0)|}{|Q(0)|}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}}\times(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r})\Big)\Big\} + |c_{Q_1,Q_2}|\frac{|Q_1(0)||Q_2(0)|}{C_{P_0}|Q(0)|}2\varpi K_k \leq 1/2$$

to hold. We come to the above inequality by observing that the quantity $|\epsilon|^{\Delta_l-d_l+\delta_l-2k}$ can be taken arbitrarily small when $\epsilon \in D_{\epsilon_0}\setminus\{0\}$, for $\epsilon_0 > 0$ small enough, owing to the condition (16).

Eventually, gathering the bounds (83), (84), (86), (87), (92) subjected to the condition (93), we arrive at

$$(94)\quad |\mathcal{G}_\epsilon(\omega_1(u,0)) - \mathcal{G}_\epsilon(\omega_2(u,0))| \leq \frac{1}{2}||\omega_1-\omega_2||_{(\nu,\beta,\mu,k,\epsilon,0)}\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp(\nu|\frac{u}{\epsilon}|^k)$$

for all $u \in S_d \cup D_{\rho_0}$, for any given $\epsilon \in D_{\epsilon_0}\setminus\{0\}$, which points to the Lipschitz feature (53). □

The next proposition provides a solution to the nonlinear fixed point equation displayed in (47) within the Banach spaces presented in Definition 3 for $m = 0$.

Proposition 2 *We adjust the aperture of the sector S_{Q,R_D} introduced in (18) in a manner that we can single out an unbounded sector S_d that fulfills the constraint (43). Then, for the choice of the radius $\epsilon_0 > 0$, of the constant $c_{Q_1,Q_2} \in \mathbb{C}^*$ and of the radius $\varpi > 0$ made in Proposition 1, there exists a unique solution $u \mapsto \omega_d(u,0,\epsilon)$ of the nonlinear convolution equation (47), for all given $\epsilon \in D_{\epsilon_0}\setminus\{0\}$ with the next features:*

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- *The map* $u \mapsto \omega_d(u,0,\epsilon)$ *belongs to* $F^d_{(\nu,\beta,\mu,k,\epsilon,0)}$ *under the constraint*

$$|\omega_d(u,0,\epsilon)| \leq \varpi \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right) \tag{95}$$

for all $u \in S_d \cup D_{\rho_0}$, *granted that* $\epsilon \in D_{\epsilon_0} \setminus \{0\}$.

- *The partial map* $\epsilon \mapsto \omega_d(u,0,\epsilon)$ *is holomorphic from* $D_{\epsilon_0} \setminus \{0\}$ *into* $\mathbb{C}$, *for any prescribed* $u \in S_d \cup D_{\rho_0}$.

Proof We select the constants $\epsilon_0 > 0$, $c_{Q_1,Q_2} \in \mathbb{C}^*$ and $\varpi > 0$ as in Proposition 1. Since $F^d_{(\nu,\beta,\mu,k,\epsilon,0)}$ is a Banach space for the norm $||.||_{(\nu,\beta,\mu,k,\epsilon,0)}$, it follows that the closed ball $\bar{B}_{\varpi} \subset F^d_{(\nu,\beta,\mu,k,\epsilon,0)}$ represents a complete space for the metric $d(X,Y) = ||X-Y||_{(\nu,\beta,\mu,k,\epsilon,0)}$. The proposition 1 asserts that $\mathcal{G}_{\epsilon}$ induces a $1/2-$Lipschitz map from $(\bar{B}_{\varpi}, d)$ into itself. Then, owing to the classical Banach fixed point theorem, $\mathcal{G}_{\epsilon}$ possesses a unique fixed point denoted $\omega_d(u,0,\epsilon)$ in $\bar{B}_{\varpi}$, meaning that

$$\mathcal{G}_{\epsilon}(\omega_d(u,0,\epsilon)) = \omega_d(u,0,\epsilon) \tag{96}$$

provided that $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. Observe that this latter equation (96) exactly confirms that $u \mapsto \omega_d(u,0,\epsilon)$ solves the nonlinear equation (47) on the domain $S_d \cup D_{\rho_0}$, granted that $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. Furthermore, since $\mathcal{G}_{\epsilon}$ relies holomorphically on ϵ on $D_{\epsilon_0} \setminus \{0\}$, it follows that $\omega_d(u,0,\epsilon)$ itself hinges analytically on ϵ on the same punctured disc. Proposition 2 follows. □

4 Solving the linear convolution equation (48) with forcing term (49)

The principal objective of this section is to prove the next proposition.

Proposition 3 *One can select a radius* $\epsilon_0 > 0$ *and a constant* $c_{Q_1,Q_2} \in \mathbb{C}^*$ *close enough to the origin in a way that the next property holds: for each integer* $m \geq 1$, *there exists a unique solution* $u \mapsto \omega_d(u,m,\epsilon)$ *of the linear convolution equation (48), for all prescribed* $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, *with the next two features:*

- *The map* $u \mapsto \omega_d(u,m,\epsilon)$ *appertains to* $F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$ *and is subjected to the upper bounds*

$$|\omega_d(u,m,\epsilon)| \leq \varpi \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right)(1+m)^{-\mu}\exp(-\beta m) \tag{97}$$

for all $u \in S_d \cup D_{\rho_m}$, *granted that* $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, *where the radius* ρ_m *is given by (44) and* $\varpi > 0$ *together with the sector* S_d *are given in Proposition 2.*

- *The partial map* $\epsilon \mapsto \omega_d(u,m,\epsilon)$ *stands for an holomorphic function from* $D_{\epsilon_0} \setminus \{0\}$ *into* $\mathbb{C}$, *for any given* $u \in S_d \cup D_{\rho_m}$.

The proof of the above proposition is grounded on a recursion procedure. Namely, let $m \geq 1$ be a fixed integer. We *assume* that for all $0 \leq h < m$, there exists a map $\omega_d(u,h,\epsilon)$ which solves the convolution relation (40) where m is replaced by h, such that

- the map $u \mapsto \omega_d(u,h,\epsilon)$ belongs to $F^d_{(\nu,\beta,\mu,k,\epsilon,h)}$ with the bounds

$$|\omega_d(u,h,\epsilon)| \leq \varpi \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right)(1+h)^{-\mu}\exp(-\beta h) \quad (98)$$

  for all $u \in S_d \cup D_{\rho_h}$, provided that $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, where $\varpi > 0$ is given in Proposition 2.

- The partial map $\epsilon \mapsto \omega_d(u,h,\epsilon)$ stands for an holomorphic function from $D_{\epsilon_0} \setminus \{0\}$ into $\mathbb{C}$, for any given $u \in S_d \cup D_{\rho_h}$.

Notice that the existence of such a map $\omega_d(u,0,\epsilon)$ for the case $h=0$ results from Proposition 2.

Since the complete chain of reasoning is rather lenghty, we break it up in two separate subsections, one dedicated to bounds related to the forcing term (49) and the other devoted to the very contruction of the map $\omega_d(u,m,\epsilon)$ by means of a fixed point argument.

4.1 Bounds for the forcing term (49)

In this subsection, we discuss the next result. Let $m \geq 1$ be the integer fixed above.

Proposition 4 *One can prescribe the radius $\epsilon_0 > 0$ and the constant $|c_{Q_1,Q_2}| > 0$ independently of m and close enough to 0 such that the forcing term $u \mapsto \psi(u,m,\epsilon)$ given by the expression (49) is submitted to the next two properties*

- *The map $u \mapsto \psi(u,m,\epsilon)$ belongs to $F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$ and is constrained to*

$$|\psi(u,m,\epsilon)| \leq \frac{\varpi}{2} \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right)(1+m)^{-\mu}\exp(-\beta m) \quad (99)$$

  for all $u \in S_d \cup D_{\rho_m}$, provided that $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, where $\varpi > 0$ is given in Proposition 2.

- *The partial map $\epsilon \mapsto \psi(u,m,\epsilon)$ is a holomorphic function from $D_{\epsilon_0} \setminus \{0\}$ into $\mathbb{C}$, for any fixed $u \in S_d \cup D_{\rho_m}$.*

Proof By construction, the forcing term (49) contains only maps $\omega_d(u,h,\epsilon)$ with $0 \leq h < m$ for which the bounds (98) can be applied.

1) We first provide bounds for the piece

$$\sum_{\substack{m_1+m_2=m \\ m_2<m}} A_l(m_1,\epsilon)R_l(im_2)\frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}(ks)^{\delta_l}\omega_d(s^{1/k},m_2,\epsilon)\frac{ds}{s} \quad (100)$$

on the disc D_{ρ_m}. We bear in mind that the sequence (21) is assumed to be decreasing. Hence, the sequence of radius $m \mapsto \rho_m$ is also decreasing and we observe that $D_{\rho_m} \subset D_{\rho_{m_2}}$ as long as $m_2 < m$. From the induction hypothesis, we know that

$$|\omega_d(u,m_2,\epsilon)| \leq \varpi \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right)(1+m_2)^{-\mu}\exp(-\beta m_2) \quad (101)$$

for all $u \in S_d \cup D_{\rho_m} \subset S_d \cup D_{\rho_{m_2}}$, provided that $m_2 < m$. We parametrize the segment $[0, u^k]$ by means of $s = u^k p$ for $0 \leq p \leq 1$, which yields

$$(102) \quad \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} s^{\delta_l} \omega_d(s^{1/k}, m_2, \epsilon) \frac{ds}{s} = u^{d_{k,l}+k(\delta_l - 1)} \int_0^1 (1-p)^{\frac{d_{k,l}}{k}-1} p^{\delta_l} \omega_d(up^{1/k}, m_2, \epsilon) \frac{dp}{p}$$

By means of the lower bounds (46) along with the upper estimates (25) and (101), we reach

$$(103) \quad \left| A_l(m_1,\epsilon) R_l(im_2) \frac{u^k}{P_m(u)} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} s^{\delta_l} \omega_d(s^{1/k}, m_2, \epsilon) \frac{ds}{s} \right| \leq \mathbf{A}_{l,\epsilon_0} (1+m_1)^{-\mu} e^{-\beta m_1} |R_l(im_2)| \frac{1}{|Q(im)|(1 - (\frac{1}{2})^{k\delta_D})} \rho_m^{d_{k,l}+k\delta_l} \varpi (1+m_2)^{-\mu} \exp(-\beta m_2) \times \frac{|u|}{|\epsilon|} \exp\left(\nu |\frac{u}{\epsilon}|^k\right) \left(\int_0^1 (1-p)^{\frac{d_{k,l}}{k}-1} p^{\delta_l + \frac{1}{k}} \frac{dp}{p} \right)$$

for all $u \in D_{\rho_m}$. By definition of the polynomial $R_l(X)$ and by construction of $Q(X)$, two constants $\mathfrak{Q}_1 > 0$ and $\mathfrak{R}_l > 0$ can be singled out with

$$(104) \quad |R_l(im_2)| \leq \mathfrak{R}_l (1+m_2)^{\deg(R_l)} \quad , \quad |Q(im)| \geq \mathfrak{Q}_1 (1+m)^{\deg(Q)}$$

for all given non negative integers $m_2 < m$. Keeping in mind the definition of ρ_m set up in (44) and the bounds (20), (22), we deduce a constant $\hat{C}_{l,Q,R_D,\delta_D} > 0$ (independent of m,m_2) such that

$$(105) \quad \frac{|R_l(im_2)|}{|Q(im)|} \rho_m^{d_{k,l}+k\delta_l} \leq \frac{\mathfrak{R}_l}{\mathfrak{Q}_1} (\frac{1}{2})^{d_{k,l}+k\delta_l} \left(\frac{\mathfrak{Q}/\mathfrak{R}_D}{k^{\delta_D}}\right)^{\frac{d_{k,l}+k\delta_l}{k\delta_D}} \left[\frac{1}{(1+m)^{\frac{\deg(R_D)-\deg(Q)}{k\delta_D}}} \right]^{d_l - \delta_l} \times \frac{(1+m)^{\deg(R_l)}}{(1+m)^{\deg(Q)}} \leq \hat{C}_{l,Q,R_D,\delta_D}$$

The next lemma will be helpful.

Lemma 4 *A constant $\check{C}_\mu > 0$ (depending on μ) can be found with*

$$(106) \quad \sum_{\substack{m_1+m_2=m \\ m_2<m}} (1+m_1)^{-\mu} (1+m_2)^{-\mu} \leq \check{C}_\mu (1+m)^{-\mu}$$

for all integers $m \geq 1$.

Proof The proof follows the same lines of arguments as in Lemma 2.2 from [4] or in Lemma 4 from [17]. Namely, let us break up the sum

$$(107) \quad (1+m)^\mu \sum_{m_1=1}^{m} \frac{1}{(1+m-m_1)^\mu (1+m_1)^\mu} = (1+m)^\mu (\mathcal{A}_m + \mathcal{B}_m)$$

in two pieces

$$\mathcal{A}_m = \sum_{0<m_1<\frac{m}{2}} \frac{1}{(1+m-m_1)^\mu (1+m_1)^\mu} \quad , \quad \mathcal{B}_m = \sum_{\frac{m}{2} \leq m_1 \leq m} \frac{1}{(1+m-m_1)^\mu (1+m_1)^\mu}$$

 doi:10.20944/preprints202206.0224.v1

On the one hand, from the inequality $1+m-m_1 \geq \frac{1}{2}(m+1)$ provided that $0 < m_1 < m/2$, we get a constant $\check{C}_{\mu,1} > 0$ (relying on μ) such that

$$\mathcal{A}_m \leq \frac{2^{\mu}}{(1+m)^{\mu}} \sum_{0<m_1<\frac{m}{2}} \frac{1}{(1+m_1)^{\mu}} \leq \frac{2^{\mu}}{(1+m)^{\mu}} \sum_{m_1 \geq 0} \frac{1}{(1+m_1)^{\mu}} = \frac{\check{C}_{\mu,1}}{(1+m)^{\mu}} \quad (108)$$

for all $m \geq 1$, according to (24). On the other hand, owing to the inequality $1+m_1 \geq \frac{1}{2}(m+1)$ for $m_1 \geq m/2$, we observe that

$$\mathcal{B}_m \leq \frac{2^{\mu}}{(1+m)^{\mu}} \sum_{\frac{m}{2} \leq m_1 \leq m} \frac{1}{(1+m-m_1)^{\mu}} \leq \frac{2^{\mu}}{(1+m)^{\mu}} \sum_{m_1 \geq 0} \frac{1}{(1+m_1)^{\mu}} = \frac{\check{C}_{\mu,1}}{(1+m)^{\mu}} \quad (109)$$

for all $m \geq 1$. Gathering (107), (108) and (109) yields the lemma. □

The collection of bounds (103), (105) and (106) begets the upper estimates for the piece (100)

$$\Big|\sum_{\substack{m_1+m_2=m \\ m_2<m}} A_l(m_1,\epsilon)R_l(im_2)\frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}(ks)^{\delta_l}\omega_d(s^{1/k},m_2,\epsilon)\frac{ds}{s}\Big| \quad (110)$$
$$\leq \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\frac{\hat{C}_{l,Q,R_D,\delta_D}}{(1-(\frac{1}{2})^{k\delta_D})}\Big(\int_0^1(1-p)^{\frac{d_{k,l}}{k}-1}p^{\delta_l+\frac{1}{k}}\frac{dp}{p}\Big) \times \varpi$$
$$\times \check{C}_{\mu}(1+m)^{-\mu}e^{-\beta m}\frac{|u|}{|\epsilon|}\exp\left(\nu|\frac{u}{\epsilon}|^k\right)\frac{1}{1+|\frac{u}{\epsilon}|^{2k}}\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}]$$

for all $u \in D_{\rho_m}$, since

$$1+|\frac{u}{\epsilon}|^{2k} \leq 1+\frac{\rho_m^{2k}}{|\epsilon|^{2k}} \leq 1+\frac{\rho_0^{2k}}{|\epsilon|^{2k}} \leq \frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}] \quad (111)$$

holds for $u \in D_{\rho_m}$, according to the assumption that $m \mapsto \rho_m$ is decreasing.

2) We exhibit bounds for the piece (100) on the sector S_d. Paying regard to (45) and owing to the bounds (101), by means of the expression (102), we notice that

$$\Big|A_l(m_1,\epsilon)R_l(im_2)\frac{u^k}{P_m(u)}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}s^{\delta_l}\omega_d(s^{1/k},m_2,\epsilon)\frac{ds}{s}\Big| \quad (112)$$
$$\leq \mathbf{A}_{l,\epsilon_0}(1+m_1)^{-\mu}e^{-\beta m_1}|R_l(im_2)|\frac{1}{|Q(im)||(\rho e^{\sqrt{-1}\theta})^{k\delta_D}-1|}\rho^{d_{k,l}+k\delta_l}(2\rho_m)^{d_{k,l}+k\delta_l}(1+\frac{(\rho 2\rho_m)^{2k}}{|\epsilon|^{2k}})$$
$$\times \varpi(1+m_2)^{-\mu}e^{-\beta m_2}\frac{|u|}{|\epsilon|}\frac{1}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^k\right)\Big(\int_0^1(1-p)^{\frac{d_{k,l}}{k}-1}p^{\delta_l+\frac{1}{k}}\frac{dp}{p}\Big)$$

for $u \in S_d$, under the factorization $u = \rho e^{\sqrt{-1}\theta}q_l(m)$, for $\rho > 0$ and $\theta \notin 2\pi\mathbb{Z}$.

Under the condition (15) and the assumption that $m \mapsto \rho_m$ is decreasing, according to (61), we check that

$$\frac{1}{|1-(\rho e^{\sqrt{-1}\theta})^{k\delta_D}|}\rho^{d_{k,l}+k\delta_l}(1+\frac{(\rho 2\rho_m)^{2k}}{|\epsilon|^{2k}}) \leq \frac{1}{|1-(\rho e^{\sqrt{-1}\theta})^{k\delta_D}|}\rho^{d_{k,l}+k\delta_l}(1+\frac{(\rho 2\rho_0)^{2k}}{|\epsilon|^{2k}}) \quad (113)$$
$$\leq \bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}}$$

for the constant $\bar{C}_{k,\delta_D,d_{k,l},\delta_l} > 0$ appearing in (61), for all $\rho > 0$.

Piling up the bounds (105), (106) and (113), we come to the estimates for the piece (100)

$$(114)\quad \Big| \sum_{\substack{m_1+m_2=m \\ m_2<m}} A_l(m_1,\epsilon)R_l(im_2)\frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}(ks)^{\delta_l}\omega_d(s^{1/k},m_2,\epsilon)\frac{ds}{s}\Big| \\ \leq \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}2^{d_{k,l}+k\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\hat{C}_{l,Q,R_D,\delta_D}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}}\Big(\int_0^1(1-p)^{\frac{d_{k,l}}{k}-1}p^{\delta_l+\frac{1}{k}}\frac{dp}{p}\Big)\varpi \\ \times \check{C}_\mu(1+m)^{-\mu}e^{-\beta m}\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)$$

for all $u \in S_d$.

3) We display bounds for the piece

$$(115)\quad \sum_{\substack{m_1+m_2=m \\ m_2<m}} A_l(m_1,\epsilon)R_l(im_2)\sum_{1\leq p\leq\delta_l-1}A_{\delta_l,p}\frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \\ \times\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}(ks)^p\omega_d(s^{1/k},m_2,\epsilon)\frac{ds}{s}$$

on the disc D_{ρ_m}. The segment $[0,u^k]$ is parametrized through $s = u^k r$, for $0 \leq r \leq 1$, which brings

$$(116)\quad \int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}s^p\omega_d(s^{1/k},m_2,\epsilon)\frac{ds}{s} \\ = u^{d_{k,l}+k(\delta_l-1)}\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^p\omega_d(ur^{1/k},m_2,\epsilon)\frac{dr}{r}$$

Calling to mind the lower bounds (46) along with the upper estimates (25) and (101), we come up to

$$(117)\quad \Big|A_l(m_1,\epsilon)R_l(im_2)\frac{u^k}{P_m(u)}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}s^p\omega_d(s^{1/k},m_2,\epsilon)\frac{ds}{s}\Big| \\ \leq \mathbf{A}_{l,\epsilon_0}(1+m_1)^{-\mu}e^{-\beta m_1}|R_l(im_2)|\frac{1}{|Q(im)|(1-(\frac{1}{2})^{k\delta_D})}\rho_m^{d_{k,l}+k\delta_l}\varpi(1+m_2)^{-\mu}\exp(-\beta m_2) \\ \times\frac{|u|}{|\epsilon|}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)\times\Big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\Big)$$

for all $u \in D_{\rho_m}$.

Flocking the bounds (105), (106) and (111), we deduce from (117) that

$$(118)\quad \Big| \sum_{\substack{m_1+m_2=m\\ m_2<m}} A_l(m_1,\epsilon)R_l(im_2) \sum_{1\leq p\leq \delta_l-1} A_{\delta_l,p} \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \times \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}(ks)^p \omega_d(s^{1/k},m_2,\epsilon)\frac{ds}{s}\Big| \leq \mathbf{A}_{l,\epsilon_0}\frac{1}{1-(\frac{1}{2})^{k\delta_D}} \times \sum_{1\leq p\leq \delta_l-1} |A_{\delta_l,p}| \frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \hat{C}_{l,Q,R_D,\delta_D}\Big(\int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^{p+\frac{1}{k}}\frac{dr}{r}\Big) \times \varpi \check{C}_\mu (1+m)^{-\mu}e^{-\beta m}\frac{|u|}{|\epsilon|}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)\frac{1}{1+|\frac{u}{\epsilon}|^{2k}}\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}]$$

for all $u \in D_{\rho_m}$.

4) We handle the piece (115) on the sector S_d. Owing to the equality (45), the upper estimates (25), (101) and by means of the parametrization (116), we get

$$(119)\quad \Big| A_l(m_1,\epsilon)R_l(im_2)\frac{u^k}{P_m(u)}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}s^p\omega_d(s^{1/k},m_2,\epsilon)\frac{ds}{s}\Big| \leq \mathbf{A}_{l,\epsilon_0}(1+m_1)^{-\mu}e^{-\beta m_1}|R_l(im_2)|\frac{1}{|Q(im)||(\rho e^{\sqrt{-1}\theta})^{k\delta_D}-1|}\rho^{d_{k,l}+k\delta_l}(2\rho_m)^{d_{k,l}+k\delta_l}\Big(1+\frac{(\rho 2\rho_m)^{2k}}{|\epsilon|^{2k}}\Big) \times \varpi(1+m_2)^{-\mu}e^{-\beta m_2}\frac{|u|}{|\epsilon|}\frac{1}{1+|\frac{u}{\epsilon}|^{2k}}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)\Big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\Big)$$

that is valid for all $u \in S_d$ by means of the factorization $u = \rho e^{\sqrt{-1}\theta}q_l(m)$, for $\rho > 0$ and $\theta \notin 2\pi\mathbb{Z}$. Grouping the bounds (105), (106) and (113), we derive from (119) that

$$(120)\quad \Big| \sum_{\substack{m_1+m_2=m\\ m_2<m}} A_l(m_1,\epsilon)R_l(im_2) \sum_{1\leq p\leq \delta_l-1} A_{\delta_l,p} \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \times \int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}(ks)^p\omega_d(s^{1/k},m_2,\epsilon)\frac{ds}{s}\Big| \leq \mathbf{A}_{l,\epsilon_0}\sum_{1\leq p\leq \delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \times \varpi\check{C}_\mu(1+m)^{-\mu}e^{-\beta m}\hat{C}_{l,Q,R_D,\delta_D}2^{d_{k,l}+k\delta_l}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}}\Big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\Big) \times \frac{|u|}{|\epsilon|}\frac{1}{1+|\frac{u}{\epsilon}|^{2k}}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)$$

as long as $u \in S_d$.

5) We treat the nonlinear piece

$$(121)\quad \sum_{\substack{m_1+m_2=m\\ m_1<m,m_2<m}} c_{Q_1,Q_2}\frac{u^k}{P_m(u)}\int_0^{u^k}Q_1(im_1)\omega_d((u^k-s)^{1/k},m_1,\epsilon) \times Q_2(im_2)\omega_d(s^{1/k},m_2,\epsilon)\frac{1}{(u^k-s)s}ds$$

on the union $S_d \cup D_{\rho_m}$. We check from (101) that

$$|\omega_d((u^k-s)^{1/k},m_1,\epsilon)| \leq \varpi(1+m_1)^{-\mu}e^{-\beta m_1}\frac{|(u^k-s)^{1/k}|/|\epsilon|}{1+\frac{|u^k-s|^2}{|\epsilon|^{2k}}}\exp\left(\nu\frac{|u^k-s|}{|\epsilon|^k}\right)$$

and

$$|\omega_d(s^{1/k},m_2,\epsilon)| \leq \varpi(1+m_2)^{-\mu}e^{-\beta m_2}\frac{|s^{1/k}|/|\epsilon|}{1+\frac{|s|^2}{|\epsilon|^{2k}}}\exp\left(\nu\frac{|s|}{|\epsilon|^k}\right)$$

for all $s \in [0,u^k]$, all $u \in S_d \cup D_{\rho_m}$, according to the fact that $D_{\rho_m} \subset D_{\rho_{m_1}}$ and $D_{\rho_m} \subset D_{\rho_{m_2}}$ for any integers $0 \leq m_1,m_2 < m$, since the sequence of radius ρ_m is assumed to be decreasing.

We deduce that

$$(122)\quad \left|u^k\int_0^{u^k} Q_1(im_1)\omega_d((u^k-s)^{1/k},m_1,\epsilon)Q_2(im_2)\omega_d(s^{1/k},m_2,\epsilon)\frac{1}{(u^k-s)s}ds\right| \\ \leq \varpi^2e^{-\beta m}|Q(im_1)|(1+m_1)^{-\mu}|Q(im_2)|(1+m_2)^{-\mu} \\ \times |u|^k\int_0^{|u|^k}\frac{(|u|^k-h)^{1/k}/|\epsilon|}{1+\frac{(|u|^k-h)^2}{|\epsilon|^{2k}}}\frac{h^{1/k}/|\epsilon|}{1+\frac{h^2}{|\epsilon|^{2k}}}\frac{1}{(|u|^k-h)h}dh \times \exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

provided that $u \in S_d \cup D_{\rho_m}$.

The next technical lemma is useful.

Lemma 5 *Let $D = \max(\deg(Q_1),\deg(Q_2))$. A constant $\check{C}_{\mu,Q_1,Q_2} > 0$ (relying on μ and Q_1,Q_2) can be singled out with*

$$(123)\quad \sum_{\substack{m_1+m_2=m\\ m_1<m,m_2<m}} |Q_1(im_1)|(1+m_1)^{-\mu}|Q_2(im_2)|(1+m_2)^{-\mu} \leq \check{C}_{\mu,Q_1,Q_2}(1+m)^{-\mu+D}$$

for all integers $m \geq 1$.

Proof By construction, we can select two constant $\check{\mathfrak{Q}}_1,\check{\mathfrak{Q}}_2 > 0$ with

$$(124)\quad |Q_1(im_1)| \leq \check{\mathfrak{Q}}_1(1+m_1)^{\deg(Q_1)} \quad , \quad |Q_2(im_2)| \leq \check{\mathfrak{Q}}_2(1+m_2)^{\deg(Q_2)}$$

for all integers $m_1,m_2 \geq 0$. The remaining part of the proof is an adjustment of the one of Lemma 4. Namely, we split up the sum

$$(125)\quad (1+m)^{\mu-D}\sum_{m_1=1}^{m-1}\frac{1}{(1+m_1)^{-\deg(Q_1)+\mu}}\frac{1}{(1+m-m_1)^{-\deg(Q_2)+\mu}} = (1+m)^{\mu-D}(\check{\mathcal{A}}_m+\check{\mathcal{B}}_m)$$

in two parts

$$\check{\mathcal{A}}_m = \sum_{0<m_1<\frac{m}{2}}\frac{1}{(1+m_1)^{-\deg(Q_1)+\mu}(1+m-m_1)^{-\deg(Q_2)+\mu}},$$

$$\check{\mathcal{B}}_m = \sum_{\frac{m}{2}\leq m_1<m}\frac{1}{(1+m_1)^{-\deg(Q_1)+\mu}(1+m-m_1)^{-\deg(Q_2)+\mu}}$$

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From the inequality $1+m-m_1 \geq \frac{1}{2}(m+1)$ for $0 < m_1 < m/2$, we obtain a constant $\check{C}_{1,\mu,Q_1,Q_2} > 0$ (depending on μ,Q_1,Q_2) such that

$$(126) \quad \check{\mathcal{A}}_m \leq \frac{2^{-\deg(Q_2)+\mu}}{(1+m)^{-\deg(Q_2)+\mu}} \sum_{0<m_1<\frac{m}{2}} \frac{1}{(1+m_1)^{-\deg(Q_1)+\mu}} \leq \frac{2^{-\deg(Q_2)+\mu}}{(1+m)^{-\deg(Q_2)+\mu}} \sum_{m_1 \geq 0} \frac{1}{(1+m_1)^{-\deg(Q_1)+\mu}} = \frac{\check{C}_{1,\mu,Q_1,Q_2}}{(1+m)^{-\deg(Q_2)+\mu}}$$

for all $m \geq 1$, according to (24). Futhermore, owing to the inequality $1+m_1 \geq \frac{1}{2}(1+m)$ for $m_1 \geq m/2$, we reach a constant $\check{C}_{2,\mu,Q_1,Q_2} > 0$ (depending on μ,Q_1,Q_2) for which

$$(127) \quad \check{\mathcal{B}}_m = \frac{2^{-\deg(Q_1)+\mu}}{(1+m)^{-\deg(Q_1)+\mu}} \sum_{\frac{m}{2} \leq m_1 < m} \frac{1}{(1+m-m_1)^{-\deg(Q_2)+\mu}} \leq \frac{2^{-\deg(Q_1)+\mu}}{(1+m)^{-\deg(Q_1)+\mu}} \sum_{m_1 \geq 0} \frac{1}{(1+m_1)^{-\deg(Q_2)+\mu}} = \frac{\check{C}_{2,\mu,Q_1,Q_2}}{(1+m)^{-\deg(Q_1)+\mu}}$$

for all $m \geq 1$, based on (24).

Eventually, gathering (124), (125), (126) and (127) gives rise to the awaited bounds (123). □

On the other hand, in line with the equality (45) on the sector S_d and the lower bounds (46) on the disc D_{ρ_m}, we deduce a constant $C_{d,k,\delta_D} > 0$ for which

$$(128) \quad |P_m(u)| \geq C_{d,k,\delta_D}|Q(im)|$$

for all $u \in S_d \cup D_{\rho_m}$, all integers $m \geq 1$.

At last, the collection of (122), (123), (128) and (69) from Lemma 3 begets the bounds

$$(129) \quad \Big| \sum_{\substack{m_1+m_2=m \\ m_1<m, m_2<m}} c_{Q_1,Q_2} \frac{u^k}{P_m(u)} \int_0^{u^k} Q_1(im_1)\omega_d((u^k-s)^{1/k}, m_1, \epsilon) \times Q_2(im_2)\omega_d(s^{1/k}, m_2, \epsilon) \frac{1}{(u^k-s)s} ds \Big| \leq |c_{Q_1,Q_2}| \varpi^2 (1+m)^{-\mu} e^{-\beta m} \frac{(1+m)^D}{C_{d,k,\delta_D}|Q(im)|} \check{C}_{\mu,Q_1,Q_2} \times K_k \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right) \leq |c_{Q_1,Q_2}| \varpi^2 (1+m)^{-\mu} e^{-\beta m} \frac{\hat{C}_{D,Q}\check{C}_{\mu,Q_1,Q_2}}{C_{d,k,\delta_D}} \times K_k \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

for all $u \in S_d \cup D_{\rho_m}$, for some constant $\hat{C}_{D,Q} > 0$ under the restriction (23).

6) We deal with the remaining piece $\varphi(u,m,\epsilon)/P_m(u)$ on the domain $S_d \cup D_{\rho_m}$. Owing to (128) and the assumption (18), we first notice that

$$(130) \quad \frac{1}{|P_m(u)|} \leq \frac{1}{C_{d,k,\delta_D}|Q(im)|} \leq \frac{1}{C_{d,k,\delta_D}} \max_{m \geq 0} \frac{1}{|Q(im)|}$$

for all $u \in S_d \cup D_{\rho_m}$, all integers $m \geq 1$. According to the computations already made in (76), the very definition (29) of $\varphi(u,m,\epsilon)$ and the constraint (28) give rise to the bounds

$$(131) \quad |\varphi(u,m,\epsilon)| \leq \sum_{j \in J} \boldsymbol{\varphi}_{j,\epsilon_0}(1+m)^{-\mu}e^{-\beta m}|u|^j \leq \boldsymbol{\varphi}_{\epsilon_0}(1+m)^{-\mu}e^{-\beta m} \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

for all $u \in S_d \cup D_{\rho_m}$, all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, for the constant $\varphi_{\epsilon_0} > 0$ introduced in (77). By dint of (130), we deduce

$$\left|\frac{\varphi(u,m,\epsilon)}{P_m(u)}\right| \leq \frac{\varphi_{\epsilon_0}}{C_{d,k,\delta_D}}\left(\max_{m\geq 0}\frac{1}{|Q(im)|}\right)(1+m)^{-\mu}e^{-\beta m}\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^k\right) \tag{132}$$

as long as $u \in S_d \cup D_{\rho_m}$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$.

We prescribe $\epsilon_0 > 0$ and the quantity $|c_{Q_1,Q_2}| > 0$ small enough, warranting the next constraint

$$\begin{aligned}
&\frac{\varphi_{\epsilon_0}}{C_{d,k,\delta_D}}\max_{m\geq 0}\frac{1}{|Q(im)|} \\
&+ \sum_{l=1}^{D-1}|\epsilon|^{\Delta_l-d_l+\delta_l}\Big\{\max\Big(\mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\frac{\hat{C}_{l,Q,R_D,\delta_D}}{(1-(\frac{1}{2})^{k\delta_D})}\Big(\int_0^1(1-p)^{\frac{d_{k,l}}{k}-1}p^{\delta_l+\frac{1}{k}}\frac{dp}{p}\Big)\times\varpi \\
&\times \check{C}_\mu\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}], \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\hat{C}_{l,Q,R_D,\delta_D}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}}\Big(\int_0^1(1-p)^{\frac{d_{k,l}}{k}-1}p^{\delta_l+\frac{1}{k}}\frac{dp}{p}\Big)\varpi \\
&\times 2^{d_{k,l}+k\delta_l}\check{C}_\mu\Big) + \max\Big(\mathbf{A}_{l,\epsilon_0}\frac{1}{1-(\frac{1}{2})^{k\delta_D}} \\
&\times \sum_{1\leq p\leq\delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}\hat{C}_{l,Q,R_D,\delta_D}\Big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\Big) \\
&\times \varpi\check{C}_\mu\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}], \mathbf{A}_{l,\epsilon_0}\sum_{1\leq p\leq\delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \\
&\times \varpi\check{C}_\mu\hat{C}_{l,Q,R_D,\delta_D}2^{d_{k,l}+k\delta_l}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}}\Big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\Big)\Big)\Big\} \\
&+ |c_{Q_1,Q_2}|\varpi^2\frac{\hat{C}_{D,Q}\check{C}_{\mu,Q_1,Q_2}}{C_{d,k,\delta_D}}K_k \leq \varpi/2
\end{aligned} \tag{133}$$

Such an inequality is justified from the facts that

- The coefficient $|\epsilon|^{\Delta_l-d_l+\delta_l-2k}$ can be taken appropriately small when $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, for $\epsilon_0 > 0$ small enough, according to the condition (16).
- The constant φ_{ϵ_0} displayed in (77) is suitably close to 0 provided that $\epsilon_0 > 0$ is taken small enough.

Finally, the expression of $\psi(u,m,\epsilon)$ displayed in (49) and the list of bounds (110), (114), (118), (120), (129) and (132) under the restriction (133) spawn the bounds (99), which implies the first item of Proposition 4. The second item is a direct consequence of the fact that the forcing term $\psi(u,m,\epsilon)$ contains only maps $\omega_d(u,h,\epsilon)$ for $0 \leq h < m$ which are, by the recursion hypothesis, holomorphic from $D_{\epsilon_0} \setminus \{0\}$ into $\mathbb{C}$ for any given $u \in S_d \cup D_{\rho_h}$. □

4.2 Construction of $\omega_d(u,m,\epsilon)$ as a fixed point

Let $m \geq 1$ be the integer fixed at the beginning of Section 4, after Proposition 3. In this subsection, we plan to build up a map $\omega_d(u,m,\epsilon)$ which fulfills the two items of Proposition 3 and solves the linear convolution equation (48). Thereupon, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, we introduce the linear map

$$\begin{aligned}(134)\quad \mathcal{H}_{\epsilon}(\omega(u,m)) := \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l + \delta_l} \\ \times \Big[A_l(0,\epsilon) R_l(im) \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}}{k})} \int_0^{u^k} (u^k - s)^{\frac{d_{k,l}}{k}-1} (ks)^{\delta_l} \omega(s^{1/k},m) \frac{ds}{s} \\ + A_l(0,\epsilon) R_l(im) \sum_{1 \le p \le \delta_l - 1} A_{\delta_l,p} \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} (ks)^p \omega(s^{1/k},m) \frac{ds}{s} \Big] \\ + c_{Q_1,Q_2} \frac{u^k}{P_m(u)} \int_0^{u^k} Q_1(0)\omega_d((u^k-s)^{1/k},0,\epsilon) Q_2(im)\omega(s^{1/k},m) \frac{1}{(u^k-s)s} ds \\ + c_{Q_1,Q_2} \frac{u^k}{P_m(u)} \int_0^{u^k} Q_1(im)\omega((u^k-s)^{1/k},m) Q_2(0)\omega_d(s^{1/k},0,\epsilon) \frac{1}{(u^k-s)s} ds + \psi(u,m,\epsilon)\end{aligned}$$

where $\omega_d(u,0,\epsilon)$ is constructed in Proposition 2 and where the forcing term $\psi(u,m,\epsilon)$ is given by the expression (49) that comprises maps $\omega_d(u,h,\epsilon)$, for $0 \le h < m$ that are assumed to be already built up.

In the upcoming proposition, we show that $\mathcal{H}_\epsilon$ represents a $1/2-$Lipschitz map on a properly selected ball of the Banach space $F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$.

Proposition 5 *One can specify the radius $\epsilon_0 > 0$ and the constant $|c_{Q_1,Q_2}| > 0$ independently of m and proximate to the origin in a way that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the map $\mathcal{H}_\epsilon$ upholds the next two attributes*

- *The inclusion*

  $$(135)\qquad \mathcal{H}_\epsilon(\bar{B}_\varpi) \subset \bar{B}_\varpi$$

  holds, where $\bar{B}_\varpi$ stands for the closed ball of radius $\varpi > 0$ centered at 0 in the space $F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$.

- *The contractive feature*

  $$(136)\qquad ||\mathcal{H}_\epsilon(\omega_1) - \mathcal{H}_\epsilon(\omega_2)||_{(\nu,\beta,\mu,k,\epsilon,m)} \le \frac{1}{2} ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,m)}$$

  occurs for all $\omega_1, \omega_2 \in \bar{B}_\varpi$.

where the radius $\varpi > 0$ is given in Proposition 2.

Proof We inspect the first item asserting the inclusion (135). We consider an element $\omega(u,m)$ of $F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$ that obeys

$$||\omega(u,m)||_{(\nu,\beta,\mu,k,\epsilon,m)} \le \varpi$$

where $\varpi > 0$ is given in Proposition 2. In particular, the next bounds

$$(137)\qquad |\omega(u,m)| \le \varpi(1+m)^{-\mu} e^{-\beta m} \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

hold for all $u \in D_{\rho_m} \cup S_d$.

1. At first, we supply bounds on the disc D_{ρ_m} for the starting piece

$$A_l(0,\epsilon)R_l(im)\frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}(ks)^{\delta_l}\omega(s^{1/k},m)\frac{ds}{s} \tag{138}$$

of $\mathcal{H}_\epsilon$.

We parametrize the segment $[0,u^k]$ through $s=u^k r$ with $0 \le r \le 1$ which allows

$$\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}s^{\delta_l}\omega(s^{1/k},m)\frac{ds}{s} = u^{k(\frac{d_{k,l}}{k}-1+\delta_l)}\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l}\omega(ur^{1/k},m)\frac{dr}{r}. \tag{139}$$

By means of the lower bounds (46) along with the upper estimates (25) and (137), we reach

$$\begin{aligned} &\left|\frac{A_l(0,\epsilon)R_l(im)}{P_m(u)}\frac{u^k}{\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}k^{\delta_l}s^{\delta_l}\omega(s^{1/k},m)\frac{ds}{s}\right| \\ &\le \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\frac{|R_l(im)|}{|Q(im)|(1-(\frac{1}{2})^{k\delta_D})}\rho_m^{d_{k,l}+k\delta_l}\varpi(1+m)^{-\mu}e^{-\beta m} \\ &\times|\frac{u}{\epsilon}|\exp\left(\nu|\frac{u}{\epsilon}|^k\right)\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r} \end{aligned} \tag{140}$$

for $u \in D_{\rho_m}$. Besides, similarly to the the upper bounds (105), under the constraints (20) and (22), we deduce that

$$\frac{|R_l(im)|}{|Q(im)|}\rho_m^{d_{k,l}+k\delta_l} \le \hat{C}_{l,Q,R_D,\delta_D} \tag{141}$$

for the constant $\hat{C}_{l,Q,R_D,\delta_D}$ appearing in (105). As a result, we arrive at

$$\begin{aligned} &\left|A_l(0,\epsilon)R_l(im)\frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}(ks)^{\delta_l}\omega(s^{1/k},m)\frac{ds}{s}\right| \\ &\le \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\frac{\hat{C}_{l,Q,R_D,\delta_D}}{1-(\frac{1}{2})^{k\delta_D}}\times\left(\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r}\right)\varpi(1+m)^{-\mu}e^{-\beta m}\times|\frac{u}{\epsilon}|\exp\left(\nu|\frac{u}{\epsilon}|^k\right)\frac{1}{1+|\frac{u}{\epsilon}|^{2k}} \\ &\times\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}] \end{aligned} \tag{142}$$

provided that $u \in D_{\rho_m}$ since

$$1+|\frac{u}{\epsilon}|^{2k} \le 1+\frac{\rho_m^{2k}}{|\epsilon|^{2k}} \le \frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}] \tag{143}$$

for all $u \in D_{\rho_m}$, bearing in mind that the sequence $(\rho_m)_{m\ge0}$ is descreasing.

2. Bounds on the sector S_d for the first piece (138) of $\mathcal{H}_\epsilon$ are disclosed.

Through the parametrization (139), the equality (45), the upper estimates (25) and (137),

 doi:10.20944/preprints202206.0224.v1

we check that

$$(144)\quad \Big|\frac{A_l(0,\epsilon)R_l(im)}{P_m(u)}\frac{u^k}{\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}k^{\delta_l}s^{\delta_l}\omega(s^{1/k},m)\frac{ds}{s}\Big|$$
$$\leq \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\frac{|R_l(im)|}{|Q(im)||(\rho e^{\sqrt{-1}\theta})^{k\delta_D}-1|}(\rho 2\rho_m)^{d_{k,l}+k\delta_l}(1+\frac{(\rho 2\rho_m)^{2k}}{|\epsilon|^{2k}})$$
$$\times\,\varpi(1+m)^{-\mu}e^{-\beta m}|\frac{u}{\epsilon}|\frac{1}{1+|\frac{u}{\epsilon}|^{2k}}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)\Big(\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r}\Big)$$

for all $u\in S_d$, under the factorization $u=\rho e^{\sqrt{-1}\theta}q_l(m)$, for $\rho>0$ with real angle $\theta\notin 2\pi\mathbb{Z}$. Besides, paying regard to (113) and (141), we come to

$$(145)\quad \Big|\frac{A_l(0,\epsilon)R_l(im)}{P_m(u)}\frac{u^k}{\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}k^{\delta_l}s^{\delta_l}\omega(s^{1/k},m)\frac{ds}{s}\Big|$$
$$\leq \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}2^{d_{k,l}+k\delta_l}\Big(\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r}\Big)\hat{C}_{l,Q,R_D,\delta_D}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}}$$
$$\times\,\varpi(1+m)^{-\mu}e^{-\beta m}|\frac{u}{\epsilon}|\frac{1}{1+|\frac{u}{\epsilon}|^{2k}}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)$$

for all $u\in S_d$.

3. We discuss bounds for the second piece of $\mathcal{H}_\epsilon$ given by

$$(146)\quad A_l(0,\epsilon)R_l(im)\sum_{1\leq p\leq\delta_l-1}A_{\delta_l,p}\frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}$$
$$\times\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}(ks)^p\omega(s^{1/k},m)\frac{ds}{s}$$

on the disc D_{ρ_m}. The segment $[0,u^k]$ is parametrized through $s=u^kr$ for $0\leq r\leq 1$ which enable to rewrite

$$(147)\quad \int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}s^p\omega(s^{1/k},m)\frac{ds}{s}$$
$$=u^{d_{k,l}+k(\delta_l-1)}\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^p\omega(ur^{1/k},m)\frac{dr}{r}$$

Thanks to the lower bounds (46) along with the upper estimates (25) and (137), we get

$$(148)\quad \Big|A_l(0,\epsilon)R_l(im)\frac{u^k}{P_m(u)}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}s^p\omega(s^{1/k},m)\frac{ds}{s}\Big|$$
$$\leq \mathbf{A}_{l,\epsilon_0}|R_l(im)|\frac{1}{|Q(im)|(1-(\frac{1}{2})^{k\delta_D})}\rho_m^{d_{k,l}+k\delta_l}\varpi(1+m)^{-\mu}\exp(-\beta m)$$
$$\times\frac{|u|}{|\epsilon|}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)\times\Big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\Big)$$

 doi:10.20944/preprints202206.0224.v1

provided that $u \in D_{\rho_m}$. According to (141), we are pined down to

$$
(149) \quad \Big| A_l(0,\epsilon)R_l(im) \sum_{1 \le p \le \delta_l - 1} A_{\delta_l,p} \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \times \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}(ks)^p \omega(s^{1/k},m)\frac{ds}{s} \Big| \le \mathbf{A}_{l,\epsilon_0} \frac{1}{1-(\frac{1}{2})^{k\delta_D}} \times \sum_{1 \le p \le \delta_l - 1} |A_{\delta_l,p}| \frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \hat{C}_{l,Q,R_D,\delta_D} \Big(\int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^{p+\frac{1}{k}} \frac{dr}{r} \Big) \times \varpi (1+m)^{-\mu} e^{-\beta m} \frac{|u|}{|\epsilon|} \exp\big(\nu |\frac{u}{\epsilon}|^k\big) \frac{1}{1+|\frac{u}{\epsilon}|^{2k}} \frac{1}{|\epsilon|^{2k}} [\rho_0^{2k} + \epsilon_0^{2k}]
$$

for all $u \in D_{\rho_m}$, keeping in mind (143).

4. Bounds for the second piece (146) are given on the sector S_d. Based on the equality (45), the upper estimates (25) and (137), the parametrization (147) yields

$$
(150) \quad \Big| A_l(0,\epsilon)R_l(im)\frac{u^k}{P_m(u)} \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} s^p \omega(s^{1/k},m)\frac{ds}{s} \Big| \le \mathbf{A}_{l,\epsilon_0} |R_l(im)| \frac{1}{|Q(im)||(\rho e^{\sqrt{-1}\theta})^{k\delta_D} - 1|} \rho^{d_{k,l}+k\delta_l} (2\rho_m)^{d_{k,l}+k\delta_l} \big(1 + \frac{(\rho 2\rho_m)^{2k}}{|\epsilon|^{2k}}\big) \times \varpi(1+m)^{-\mu} \exp(-\beta m) \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\big(\nu |\frac{u}{\epsilon}|^k\big) \times \Big(\int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^{p+\frac{1}{k}} \frac{dr}{r} \Big)
$$

for all $u \in S_d$, under the factorization $u = \rho e^{\sqrt{-1}\theta} q_l(m)$, for $\rho > 0$ with real angle $\theta \notin 2\pi\mathbb{Z}$. Besides, having in mind (113) and (141), we arrive at

$$
(151) \quad \Big| A_l(0,\epsilon)R_l(im) \sum_{1 \le p \le \delta_l - 1} A_{\delta_l,p} \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \times \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}(ks)^p \omega(s^{1/k},m)\frac{ds}{s} \Big| \le \mathbf{A}_{l,\epsilon_0} \sum_{1 \le p \le \delta_l - 1} |A_{\delta_l,p}| \frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} 2^{d_{k,l}+k\delta_l} \hat{C}_{l,Q,R_D,\delta_D} \bar{C}_{k,\delta_D,d_{k,l},\delta_l} \frac{1}{|\epsilon|^{2k}} \times \varpi(1+m)^{-\mu} \exp(-\beta m) \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\big(\nu |\frac{u}{\epsilon}|^k\big) \times \Big(\int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^{p+\frac{1}{k}} \frac{dr}{r} \Big)
$$

as long as $u \in S_d$.

5. Estimates for the third piece

$$
(152) \quad c_{Q_1,Q_2} \frac{u^k}{P_m(u)} \int_0^{u^k} Q_1(0)\omega_d((u^k-s)^{1/k},0,\epsilon) Q_2(im)\omega(s^{1/k},m) \frac{1}{(u^k-s)s} ds
$$

of $\mathcal{H}_\epsilon$ are presented on the union $S_d \cup D_{\rho_m}$. Owing to (95) from Proposition 2, we know that

$$
|\omega_d((u^k-s)^{1/k},0,\epsilon)| \le \varpi \frac{|(u^k-s)^{1/k}|/|\epsilon|}{1+\frac{|u^k-s|^2}{|\epsilon|^{2k}}} \exp\big(\nu \frac{|u^k-s|}{|\epsilon|^k}\big)
$$

and from (137) we check that

$$|\omega(s^{1/k},m)| \leq \varpi(1+m)^{-\mu}e^{-\beta m}\frac{|s^{1/k}|/|\epsilon|}{1+\frac{|s|^2}{|\epsilon|^{2k}}}\exp\left(\nu\frac{|s|}{|\epsilon|^k}\right)$$

for all $s \in [0,u^k]$, all $u \in S_d \cup D_{\rho_m}$. We deduce that

$$\left|u^k \int_0^{u^k} Q_1(0)\omega_d((u^k-s)^{1/k},0,\epsilon)Q_2(im)\omega(s^{1/k},m)\frac{1}{(u^k-s)s}ds\right| \tag{153}$$
$$\leq |Q_1(0)||Q_2(im)|\varpi^2(1+m)^{-\mu}e^{-\beta m}$$
$$\times |u|^k \int_0^{|u|^k}\frac{(|u|^k-h)^{1/k}/|\epsilon|}{1+\frac{(|u|^k-h)^2}{|\epsilon|^{2k}}}\frac{h^{1/k}/|\epsilon|}{1+\frac{h^2}{|\epsilon|^{2k}}}\frac{1}{(|u|^k-h)h}dh \times \exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

for all $u \in S_d \cup D_{\rho_m}$. By mindful of the condition (23), we get a constant $\hat{C}_{Q,Q_2} > 0$ (independent of m) such that

$$\left|\frac{Q_2(im)}{Q(im)}\right| \leq \hat{C}_{Q,Q_2}$$

and paying heed to (69), (128), we conclude that

$$\left|c_{Q_1,Q_2}\frac{u^k}{P_m(u)}\int_0^{u^k} Q_1(0)\omega_d((u^k-s)^{1/k},0,\epsilon)Q_2(im)\omega(s^{1/k},m)\frac{1}{(u^k-s)s}ds\right| \tag{154}$$
$$\leq |c_{Q_1,Q_2}|\frac{|Q_1(0)|}{C_{d,k,\delta_D}}\hat{C}_{Q,Q_2}\varpi^2(1+m)^{-\mu}e^{-\beta m}K_k\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

whenever $u \in S_d \cup D_{\rho_m}$.

6. We seek bounds for the fourth piece

$$c_{Q_1,Q_2}\frac{u^k}{P_m(u)}\int_0^{u^k} Q_1(im)\omega((u^k-s)^{1/k},m)Q_2(0)\omega_d(s^{1/k},0,\epsilon)\frac{1}{(u^k-s)s}ds$$

of $\mathcal{H}_\epsilon$ on the domain $S_d \cup D_{\rho_m}$. From (137) we see that

$$|\omega((u^k-s)^{1/k},m)| \leq \varpi(1+m)^{-\mu}e^{-\beta m}\frac{|(u^k-s)^{1/k}|/|\epsilon|}{1+\frac{|u^k-s|^2}{|\epsilon|^{2k}}}\exp\left(\nu\frac{|u^k-s|}{|\epsilon|^k}\right)$$

and due to (95) in Proposition 2, we notice that

$$|\omega_d(s^{1/k},0,\epsilon)| \leq \varpi\frac{|s^{1/k}|/|\epsilon|}{1+\frac{|s|^2}{|\epsilon|^{2k}}}\exp\left(\nu\frac{|s|}{|\epsilon|^k}\right)$$

for all $s \in [0,u^k]$, all $u \in S_d \cup D_{\rho_m}$. It follows that

$$\left|u^k \int_0^{u^k} Q_1(im)\omega((u^k-s)^{1/k},m)Q_2(0)\omega_d(s^{1/k},0,\epsilon)\frac{1}{(u^k-s)s}ds\right| \tag{155}$$
$$\leq |Q_1(im)||Q_2(0)|\varpi^2(1+m)^{-\mu}e^{-\beta m}$$
$$\times |u|^k \int_0^{|u|^k}\frac{(|u|^k-h)^{1/k}/|\epsilon|}{1+\frac{(|u|^k-h)^2}{|\epsilon|^{2k}}}\frac{h^{1/k}/|\epsilon|}{1+\frac{h^2}{|\epsilon|^{2k}}}\frac{1}{(|u|^k-h)h}dh \times \exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

 doi:10.20944/preprints202206.0224.v1

for all $u \in S_d \cup D_{\rho_m}$. Paying regard to the condition (23), a constant $\hat{C}_{Q,Q_1} > 0$ (independent of m) can be singled out with

$$\left|\frac{Q_1(im)}{Q(im)}\right| \leq \hat{C}_{Q,Q_1}$$

and not forgetting (69), (128), it entails that

$$(156)\quad \left|c_{Q_1,Q_2}\frac{u^k}{P_m(u)}\int_0^{u^k} Q_1(im)\omega((u^k-s)^{1/k},m)Q_2(0)\omega_d(s^{1/k},0,\epsilon)\frac{1}{(u^k-s)s}ds\right| \\ \leq |c_{Q_1,Q_2}|\frac{|Q_2(0)|}{C_{d,k,\delta_D}}\hat{C}_{Q,Q_1}\varpi^2(1+m)^{-\mu}e^{-\beta m}K_k\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

whenever $u \in S_d \cup D_{\rho_m}$.

7. At last, we remember the bounds (99) from Proposition 4 which gives rise to upper estimates for the tail piece of $\mathcal{H}_\epsilon$.

The quantities $\epsilon_0 > 0$ and $|c_{Q_1,Q_2}| > 0$ are prescribed nearby 0, granting the next constraint

$$(157)\quad \sum_{l=1}^{D-1}|\epsilon|^{\Delta_l-d_l+\delta_l}\Big\{\max\Big(\mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\frac{\hat{C}_{l,Q,R_D,\delta_D}}{1-(\frac{1}{2})^{k\delta_D}} \\ \times\Big(\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r}\Big)\varpi\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}], \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}2^{d_{k,l}+k\delta_l}\Big(\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r}\Big) \\ \times\hat{C}_{l,Q,R_D,\delta_D}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}}\varpi\Big) + \max\Big(\mathbf{A}_{l,\epsilon_0}\frac{1}{1-(\frac{1}{2})^{k\delta_D}} \\ \times\sum_{1\leq p\leq\delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}\hat{C}_{l,Q,R_D,\delta_D}\Big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\Big) \\ \times\varpi\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}], \mathbf{A}_{l,\epsilon_0}\sum_{1\leq p\leq\delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}2^{d_{k,l}+k\delta_l}\hat{C}_{l,Q,R_D,\delta_D}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}} \\ \times\varpi\times\Big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\Big)\Big)\Big\} + |c_{Q_1,Q_2}|\frac{|Q_1(0)|}{C_{d,k,\delta_D}}\hat{C}_{Q,Q_2}\varpi^2K_k \\ + |c_{Q_1,Q_2}|\frac{|Q_2(0)|}{C_{d,k,\delta_D}}\hat{C}_{Q,Q_1}\varpi^2K_k + \frac{\varpi}{2} \leq \varpi$$

This latter inequality is confirmed from the fact that the coefficient $|\epsilon|^{\Delta_l-d_l+\delta_l-2k}$ can be taken appropriately small when $\epsilon \in D_{\epsilon_0}\setminus\{0\}$, for $\epsilon_0 > 0$ small enough, according to the condition (16).

At last, the collection of the seven above bounds (142), (145), (149), (151), (154), (156) and (99) subjected to the restriction (157) entails

$$|\mathcal{H}_\epsilon(\omega(u,m))| \leq \varpi(1+m)^{-\mu}e^{-\beta m}\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^k\right)$$

for all $u \in S_d \cup D_{\rho_m}$ and given $\epsilon \in D_{\epsilon_0}\setminus\{0\}$. This means that the inclusion (135) holds.

In the second part of the proof, the Lipschitz property (136) is discussed. Let $\epsilon \in D_{\epsilon_0}\setminus\{0\}$ and set two elements ω_1,ω_2 of the closed ball $\bar{B}_\varpi$ in $F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$ where the radius $\varpi > 0$ has been prescribed in Proposition 2 and considered in the first part of the proof.

 doi:10.20944/preprints202206.0224.v1

By construction of the norm, we observe that

$$|\omega_1(u,m) - \omega_2(u,m)| \leq ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,m)}(1+m)^{-\mu}e^{-\beta m}\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^{k}\right) \tag{158}$$

as long as $u \in D_{\rho_m} \cup S_d$. The next list of bounds A. B. C. D. E. and F. is a direct upshot of the bounds 1. 2. 3. 4. 5. and 6. reached in the first part of the proof.

A. We provide upper bounds on the disc D_{ρ_m} for the first piece

$$A_l(0,\epsilon)R_l(im)\frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}(ks)^{\delta_l}\left(\omega_1(s^{1/k},m)-\omega_2(s^{1/k},m)\right)\frac{ds}{s} \tag{159}$$

that crops up in $\mathcal{H}_\epsilon(\omega_1) - \mathcal{H}_\epsilon(\omega_2)$. Namely,

$$\begin{aligned} &\left|A_l(0,\epsilon)R_l(im)\frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}(ks)^{\delta_l}\left(\omega_1(s^{1/k},m)-\omega_2(s^{1/k},m)\right)\frac{ds}{s}\right| \\ &\leq \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\frac{\hat{C}_{l,Q,R_D,\delta_D}}{1-(\frac{1}{2})^{k\delta_D}}\times\left(\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r}\right)||\omega_1-\omega_2||_{(\nu,\beta,\mu,k,\epsilon,m)} \\ &\times(1+m)^{-\mu}e^{-\beta m}\times|\frac{u}{\epsilon}|\exp\left(\nu|\frac{u}{\epsilon}|^k\right)\frac{1}{1+|\frac{u}{\epsilon}|^{2k}}\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}] \end{aligned} \tag{160}$$

provided that $u \in D_{\rho_m}$.

B. Estimates for the first piece (159) are given on the sector S_d. Indeed,

$$\begin{aligned} &\left|\frac{A_l(0,\epsilon)R_l(im)}{P_m(u)}\frac{u^k}{\Gamma(\frac{d_{k,l}}{k})}\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}}{k}-1}k^{\delta_l}s^{\delta_l}\left(\omega_1(s^{1/k},m)-\omega_2(s^{1/k},m)\right)\frac{ds}{s}\right| \\ &\leq \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}2^{d_{k,l}+k\delta_l}\left(\int_0^1(1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r}\right)\hat{C}_{l,Q,R_D,\delta_D}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}} \\ &\times||\omega_1-\omega_2||_{(\nu,\beta,\mu,k,\epsilon,m)}(1+m)^{-\mu}e^{-\beta m}|\frac{u}{\epsilon}|\frac{1}{1+|\frac{u}{\epsilon}|^{2k}}\exp\left(\nu|\frac{u}{\epsilon}|^k\right) \end{aligned} \tag{161}$$

for all $u \in S_d$.

C. The second piece

$$\begin{aligned} &A_l(0,\epsilon)R_l(im)\sum_{1\leq p\leq\delta_l-1}A_{\delta_l,p}\frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \\ &\times\int_0^{u^k}(u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}(ks)^p\left(\omega_1(s^{1/k},m)-\omega_2(s^{1/k},m)\right)\frac{ds}{s} \end{aligned} \tag{162}$$

of the difference $\mathcal{H}_{\epsilon}(\omega_1) - \mathcal{H}_{\epsilon}(\omega_2)$ is considered on the disc D_{ρ_m}. We arrive at

$$
\begin{aligned}
(163)\quad & \Big| A_l(0,\epsilon)R_l(im) \sum_{1 \le p \le \delta_l - 1} A_{\delta_l,p} \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \\
& \times \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}(ks)^p \big(\omega_1(s^{1/k},m) - \omega_2(s^{1/k},m)\big)\frac{ds}{s}\Big| \le \mathbf{A}_{l,\epsilon_0}\frac{1}{1-(\frac{1}{2})^{k\delta_D}} \\
& \times \sum_{1 \le p \le \delta_l - 1} |A_{\delta_l,p}| \frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \hat{C}_{l,Q,R_D,\delta_D} \Big(\int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^{p+\frac{1}{k}} \frac{dr}{r} \Big) \\
& \times ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,m)}(1+m)^{-\mu}e^{-\beta m}\frac{|u|}{|\epsilon|}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)\frac{1}{1+|\frac{u}{\epsilon}|^{2k}}\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}]
\end{aligned}
$$

for all $u \in D_{\rho_m}$.

D. The second piece (162) is upper bounded on the sector S_d. We get

$$
\begin{aligned}
(164)\quad & \Big| A_l(0,\epsilon)R_l(im) \sum_{1 \le p \le \delta_l - 1} A_{\delta_l,p} \frac{u^k}{P_m(u)\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} \\
& \times \int_0^{u^k} (u^k-s)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}(ks)^p \big(\omega_1(s^{1/k},m) - \omega_2(s^{1/k},m)\big)\frac{ds}{s}\Big| \\
& \le \mathbf{A}_{l,\epsilon_0} \sum_{1 \le p \le \delta_l - 1} |A_{\delta_l,p}| \frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})} 2^{d_{k,l}+k\delta_l} \hat{C}_{l,Q,R_D,\delta_D}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}} \\
& \times ||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,m)}(1+m)^{-\mu}\exp(-\beta m)\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\big(\nu|\frac{u}{\epsilon}|^k\big) \\
& \times \Big(\int_0^1 (1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1} r^{p+\frac{1}{k}} \frac{dr}{r} \Big)
\end{aligned}
$$

as long as $u \in S_d$.

E. The third piece

$$
(165)\quad c_{Q_1,Q_2}\frac{u^k}{P_m(u)}\int_0^{u^k} Q_1(0)\omega_d((u^k-s)^{1/k},0,\epsilon)Q_2(im)\big(\omega_1(s^{1/k},m)-\omega_2(s^{1/k},m)\big)\frac{1}{(u^k-s)s}ds
$$

of the difference $\mathcal{H}_{\epsilon}(\omega_1) - \mathcal{H}_{\epsilon}(\omega_2)$ is examined on the union $S_d \cup D_{\rho_m}$. We reach

$$
\begin{aligned}
(166)\quad & \Big| c_{Q_1,Q_2}\frac{u^k}{P_m(u)}\int_0^{u^k} Q_1(0)\omega_d((u^k-s)^{1/k},0,\epsilon)Q_2(im) \\
& \times \big(\omega_1(s^{1/k},m) - \omega_2(s^{1/k},m)\big)\frac{1}{(u^k-s)s}ds\Big| \\
& \le |c_{Q_1,Q_2}|\frac{|Q_1(0)|}{C_{d,k,\delta_D}}\hat{C}_{Q,Q_2}\varpi||\omega_1 - \omega_2||_{(\nu,\beta,\mu,k,\epsilon,m)}(1+m)^{-\mu}e^{-\beta m}K_k\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)
\end{aligned}
$$

whenever $u \in S_d \cup D_{\rho_m}$.

F. The rear part

$$
\begin{aligned}
(167)\quad & c_{Q_1,Q_2}\frac{u^k}{P_m(u)}\int_0^{u^k} Q_1(im)\big(\omega_1((u^k-s)^{1/k},m) - \omega_2((u^k-s)^{1/k},m)\big) \\
& \times Q_2(0)\omega_d(s^{1/k},0,\epsilon)\frac{1}{(u^k-s)s}ds
\end{aligned}
$$

of $\mathcal{H}_{\epsilon}(\omega_1) - \mathcal{H}_{\epsilon}(\omega_2)$ is perused on the union $S_d \cup D_{\rho_m}$. It follows that

$$
(168) \quad \Big| c_{Q_1,Q_2} \frac{u^k}{P_m(u)} \int_0^{u^k} Q_1(im)\big(\omega_1((u^k-s)^{1/k},m) - \omega_2((u^k-s)^{1/k},m)\big) \times Q_2(0)\omega_d(s^{1/k},0,\epsilon)\frac{1}{(u^k-s)s}ds\Big| \leq |c_{Q_1,Q_2}|\frac{|Q_2(0)|}{C_{d,k,\delta_D}}\hat{C}_{Q,Q_1}\varpi||\omega_1-\omega_2||_{(\nu,\beta,\mu,k,\epsilon,m)}(1+m)^{-\mu}e^{-\beta m}K_k\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)
$$

whenever $u \in S_d \cup D_{\rho_m}$.

We assign $\epsilon_0 > 0$ and $|c_{Q_1,Q_2}| > 0$ small enough enabling the next inequality

$$
(169) \quad \sum_{l=1}^{D-1} |\epsilon|^{\Delta_l - d_l + \delta_l}\Big\{ \max\Big(\mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}\frac{\hat{C}_{l,Q,R_D,\delta_D}}{1-(\frac{1}{2})^{k\delta_D}} \times \big(\int_0^1 (1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r}\big)\frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}], \mathbf{A}_{l,\epsilon_0}\frac{k^{\delta_l}}{\Gamma(\frac{d_{k,l}}{k})}2^{d_{k,l}+k\delta_l}\big(\int_0^1 (1-r)^{\frac{d_{k,l}}{k}-1}r^{\delta_l+\frac{1}{k}}\frac{dr}{r}\big) \times \hat{C}_{l,Q,R_D,\delta_D}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}}\Big) + \max\Big(\mathbf{A}_{l,\epsilon_0}\frac{1}{1-(\frac{1}{2})^{k\delta_D}} \times \sum_{1\leq p\leq \delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}\hat{C}_{l,Q,R_D,\delta_D}\big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\big) \times \frac{1}{|\epsilon|^{2k}}[\rho_0^{2k}+\epsilon_0^{2k}], \mathbf{A}_{l,\epsilon_0}\sum_{1\leq p\leq\delta_l-1}|A_{\delta_l,p}|\frac{k^p}{\Gamma(\frac{d_{k,l}+k(\delta_l-p)}{k})}2^{d_{k,l}+k\delta_l}\hat{C}_{l,Q,R_D,\delta_D}\bar{C}_{k,\delta_D,d_{k,l},\delta_l}\frac{1}{|\epsilon|^{2k}} \times \big(\int_0^1(1-r)^{\frac{d_{k,l}+k(\delta_l-p)}{k}-1}r^{p+\frac{1}{k}}\frac{dr}{r}\big)\Big)\Big\} + |c_{Q_1,Q_2}|\frac{|Q_1(0)|}{C_{d,k,\delta_D}}\hat{C}_{Q,Q_2}\varpi K_k + |c_{Q_1,Q_2}|\frac{|Q_2(0)|}{C_{d,k,\delta_D}}\hat{C}_{Q,Q_1}\varpi K_k \leq 1/2
$$

This latter inequality is warranted from the fact that the coefficient $|\epsilon|^{\Delta_l - d_l + \delta_l - 2k}$ can be taken appropriately small when $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, for $\epsilon_0 > 0$ small enough, according to the condition (16).

Eventually, the gathering of the six above bounds (160), (161), (163), (164), (166) and (168) under the constraint (169) entails

$$
|\mathcal{H}_{\epsilon}(\omega_1(u,m)) - \mathcal{H}_{\epsilon}(\omega_2(u,m))| \leq \frac{1}{2}||\omega_1-\omega_2||_{(\nu,\beta,\mu,k,\epsilon,m)}(1+m)^{-\mu}e^{-\beta m}\frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}}\exp\big(\nu|\frac{u}{\epsilon}|^k\big)
$$

for all $u \in S_d \cup D_{\rho_m}$ and fixed $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. As a result, the shrinking feature (136) holds. □

In the next proposition, we build up a map $\omega_d(u,m,\epsilon)$ that obeys the two items of Proposition 3 and solves the linear convolution equation (48) for the prescribed integer $m \geq 1$ at the beginning of Section 4, after Proposition 3.

Proposition 6 *One can choose a radius $\epsilon_0 > 0$ and a constant $|c_{Q_1,Q_2}| > 0$ close enough to 0 (independently of m) in a way that there exists a unique solution $u \mapsto \omega_d(u,m,\epsilon)$ of the linear convolution equation (48), for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$ fulfilling the next two properties*

- *The map* $u \mapsto \omega_d(u,m,\epsilon)$ *appertains to* $F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$ *and is subjected to the upper bounds*

$$|\omega_d(u,m,\epsilon)| \leq \varpi \frac{|u/\epsilon|}{1+|\frac{u}{\epsilon}|^{2k}} \exp\big(\nu|\frac{u}{\epsilon}|^k\big)(1+m)^{-\mu}\exp(-\beta m) \tag{170}$$

  for all $u \in S_d \cup D_{\rho_m}$*, granted that* $\epsilon \in D_{\epsilon_0} \setminus \{0\}$*, where the radius* ρ_m *is given by (44) and where* $\varpi > 0$ *is given in Proposition 2.*

- *The partial map* $\epsilon \mapsto \omega_d(u,m,\epsilon)$ *stands for an holomorphic function from* $D_{\epsilon_0} \setminus \{0\}$ *into* $\mathbb{C}$*, for any given* $u \in S_d \cup D_{\rho_m}$*.*

Proof We select the constants $\epsilon_0 > 0$, $|c_{Q_1,Q_2}| > 0$ as in Proposition 5 and we consider the radius $\varpi > 0$ singled out in Proposition 2. The space $F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$ is of Banach type for the norm $||.||_{(\nu,\beta,\mu,k,\epsilon,m)}$. It follows that the closed ball $\bar{B}_{\varpi} \subset F^d_{(\nu,\beta,\mu,k,\epsilon,m)}$ represents a complete space for the metric $d(X,Y) = ||X-Y||_{(\nu,\beta,\mu,k,\epsilon,m)}$. The proposition 5 claims that $\mathcal{H}_\epsilon$ induces a $1/2-$Lipschitz map from $(\bar{B}_{\varpi}, d)$ into itself. Then, owing to the classical Banach fixed point theorem, $\mathcal{H}_\epsilon$ possesses a unique fixed point denotes $\omega_d(u,m,\epsilon)$ in $\bar{B}_{\varpi}$, meaning that

$$\mathcal{H}_\epsilon(\omega_d(u,m,\epsilon)) = \omega_d(u,m,\epsilon) \tag{171}$$

provided that $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. Observe that this latter equation (171) states that $u \mapsto \omega_d(u,m,\epsilon)$ solves the linear convolution equation (48) on the domain $S_d \cup D_{\rho_m}$, granted that $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. Furthermore, since $\mathcal{H}_\epsilon$ relies holomorphically on ϵ on $D_{\epsilon_0} \setminus \{0\}$, it follows that $\omega_d(u,m,\epsilon)$ itself depends analytically on ϵ on the same punctured disc. Proposition 6 holds. □

As a result, Proposition 3 follows.

5 Building a finite set of holomorphic solutions to the initial value problem (13)

We first remind the concept of a good covering in $\mathbb{C}^*$ following the definition stated in the reference text book [5], Section XI-2.

Definition 4 *Let* $\varsigma \geq 2$ *be an integer. For all* $0 \leq p \leq \varsigma - 1$*, we consider open sectors* $\mathcal{E}_p$ *centered at 0 with radius* $\epsilon_0 > 0$ *owning the next three qualities*

1. *The intersection* $\mathcal{E}_p \cap \mathcal{E}_{p+1}$ *is not empty for any* $0 \leq p \leq \varsigma - 1$*, with the convention* $\mathcal{E}_\varsigma = \mathcal{E}_0$*.*
2. *The intersection of any three sectors* $\mathcal{E}_p \cap \mathcal{E}_q \cap \mathcal{E}_r$ *is empty for distinct integers* $p,q,r \in \{0,\ldots,\varsigma-1\}$*.*
3. *The union of all the sectors* $\mathcal{E}_p$*,* $0 \leq p \leq \varsigma - 1$*, cups some punctured neighborhood in* $\mathbb{C}^*$*, meaning that*

$$\bigcup_{p=0}^{\varsigma-1} \mathcal{E}_p = U \setminus \{0\}$$

   for some neighborhood U of 0 in $\mathbb{C}$*.*

Such a set $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ *of sectors is called a good covering in* $\mathbb{C}^*$*.*

We further need to define a notion of set of sectors associated to a good covering in $\mathbb{C}^*$.

 doi:10.20944/preprints202206.0224.v1

Definition 5 *We set $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ as a good covering in $\mathbb{C}^*$. We consider a finite set of open unbounded sectors $S_{\mathfrak{d}_p}$ centered at 0 with bisecting direction $\mathfrak{d}_p \in \mathbb{R}$ distinguished in a way that the next two attributes hold*

1. *The next intersection*

$$S_{\mathfrak{d}_p} \cap \big(\cup_{m \geq 0} \cup_{0 \leq l \leq \delta_D k - 1} \{q_l(m)\} \big) = \emptyset \tag{172}$$

is empty for all $0 \leq p \leq \varsigma - 1$.

2. *A bounded sector $\mathcal{T}$ centered at 0, with radius $r_{\mathcal{T}} > 0$, can be singled out that fulfills the next constraint : For any given $0 \leq p \leq \varsigma - 1$, there exists a constant $\Delta_p > 0$ such that for all $\epsilon \in \mathcal{E}_p$, one can select a direction $\gamma_p \in \mathbb{R}$ (that may rely on ϵ) with*
   - $\exp(\sqrt{-1}\gamma_p) \in S_{\mathfrak{d}_p}$
   - $\cos(k(\gamma_p - \arg(\epsilon t))) > \Delta_p$, *for all $t \in \mathcal{T}$.*

The family of sectors $\{S_{\mathfrak{d}_p}\}_{0 \leq p \leq \varsigma-1} \cup \{\mathcal{T}\}$ is called to be associated to $\underline{\mathcal{E}}$.

The next statement represents the first major achievement of our work. Namely, we build up a finite set of holomorphic solutions to the main problem (13) and provide an important qualitative information on the control of their consecutive differences which turns out to be essential for reaching their asymptotic features relatively to the parameter ϵ discussed in the upcoming section 6.

Theorem 1 *We consider a good covering $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ in $\mathbb{C}^*$ to which one associates a family of sectors $\{S_{\mathfrak{d}_p}\}_{0 \leq p \leq \varsigma-1} \cup \{\mathcal{T}\}$ accordingly to Definition 5.*

Then, provided that the radii $\epsilon_0 > 0$, $r_{\mathcal{T}} > 0$ and the constant $c_{Q_1,Q_2} \in \mathbb{C}^*$ are taken close enough to the origin, one can construct, for all $0 \leq p \leq \varsigma - 1$, a genuine solution $u_p(t,z,\epsilon)$ to equation (13) with $u_p(0,z,\epsilon) \equiv 0$, which defines a bounded holomorphic map on the product $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$, for any given $0 < \beta' < \beta$. Furthermore, for each $0 \leq p \leq \varsigma - 1$,

- *The map $u_p(t,z,\epsilon)$ is expressed by means of a Fourier series in z,*

$$u_p(t,z,\epsilon) = \sum_{m \geq 0} u_{p,m}(t,\epsilon) e^{\sqrt{-1}zm} \tag{173}$$

with coefficients $u_{p,m}(t,\epsilon)$ that are represented by Laplace transforms of order k,

$$u_{p,m}(t,\epsilon) = k \int_{L_{\gamma_p}} \omega_{\mathfrak{d}_p}(u,m,\epsilon) \exp\big(-(\frac{u}{\epsilon t})^k\big) \frac{du}{u} \tag{174}$$

along the halfline $L_{\gamma_p} = [0,+\infty)e^{\sqrt{-1}\gamma_p}$, with direction γ_p assigned in Definition 5, where $\omega_{\mathfrak{d}_p}(u,m,\epsilon)$ stands for a map which is holomorphic on the union $S_{\mathfrak{d}_p} \cup D_{\rho_m}$ w.r.t u and holomorphic on $D_{\epsilon_0} \setminus \{0\}$ relatively to ϵ and subjected to the bounds

$$|\omega_{\mathfrak{d}_p}(u,m,\epsilon)| \leq \varpi_p \frac{|u/\epsilon|}{1 + |\frac{u}{\epsilon}|^{2k}} \exp\big(\nu|\frac{u}{\epsilon}|^k\big)(1+m)^{-\mu} \exp(-\beta m) \tag{175}$$

as long as $u \in S_{\mathfrak{d}_p} \cup D_{\rho_m}$, $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, for some constant $\varpi_p > 0$ and where the radius $\rho_m > 0$ is displayed in (44).

- *Let us define*

(176) $$\kappa = \frac{k}{\frac{\deg(R_D)-\deg(Q)}{\delta_D}+1}.$$

The difference of consecutive solutions u_p *and* u_{p+1} *obeys the next bounds : there exist constants* $A_p, B_p > 0$ *with*

(177) $$\sup_{\substack{t\in\mathcal{T}\\ z\in H_{\beta'}}} |u_{p+1}(t,z,\epsilon) - u_p(t,z,\epsilon)| \leq A_p \exp\big(-\frac{B_p}{|\epsilon|^{\kappa}}\big)$$

for all $\epsilon \in \mathcal{E}_{p+1}\cap\mathcal{E}_p$, *for all* $0 \leq p \leq \varsigma - 1$, *where by convention we define* $u_{\varsigma} = u_0$.

Proof According to Propositions 2 and 3 and the construction of the set of sectors $S_{\mathfrak{d}_p}$ submitted to (172), one can prescribe some small radius $\epsilon_0 > 0$ and small constant $c_{Q_1,Q_2} \in \mathbb{C}^*$ such that, for each integer $m \geq 0$ and each sector $S_{\mathfrak{d}_p}$, for $0 \leq p \leq \varsigma - 1$, one can shape a map $u \mapsto \omega_{\mathfrak{d}_p}(u,m,\epsilon)$ which

- is holomorphic on the union $S_{\mathfrak{d}_p} \cup D_{\rho_m}$ and analytic relatively to ϵ on the punctured disc $D_{\epsilon_0} \setminus \{0\}$
- belongs to the Banach space $F^{\mathfrak{d}_p}_{(\nu,\beta,\mu,k,\epsilon,m)}$ and suffers the bounds (175) provided that $u \in S_{\mathfrak{d}_p} \cup D_{\rho_m}$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$
- solves the nonlinear convolution equation (47) when $m = 0$, for all $u \in S_{\mathfrak{d}_p} \cup D_{\rho_0}$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$
- satisfies the linear convolution equation (48) for $m \geq 1$, as long as $u \in S_{\mathfrak{d}_p} \cup D_{\rho_0}$, for any given $\epsilon \in D_{\epsilon_0} \setminus \{0\}$.

We set $p \in \{0,\ldots,\varsigma-1\}$. For all integers $m \geq 0$, all $\epsilon \in \mathcal{E}_p$, one sets the function

(178) $$U_p(T,m,\epsilon) = k\int_{L_{\gamma_p}} \omega_{\mathfrak{d}_p}(u,m,\epsilon)\exp\big(-(\frac{u}{T})^k\big)\frac{du}{u}$$

along the halfline of integration $L_{\gamma_p} = [0,+\infty)e^{\sqrt{-1}\gamma_p}$, with given direction γ_p in accordance with the second requirement 2. of Definition 5. For any fixed $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the partial map $T \mapsto U_p(T,m,\epsilon)$ stands for a bounded holomorphic map on a domain of the form

(179) $$\mathcal{D}_{|\epsilon|,\Delta_p} = \{T \in \mathbb{C}^* / \cos(k(\gamma_p - T)) > \Delta_p \quad , \quad |T| \leq \big(\frac{\Delta_p}{\nu+\hat{\Delta}_p}\big)^{1/k}|\epsilon|\}$$

for some given constant $\hat{\Delta}_p > 0$. Indeed, owing to (175), we reach

(180) $$|\omega_{\mathfrak{d}_p}(u,m,\epsilon)||\exp\big(-(\frac{u}{T})^k\big)|\frac{1}{|u|} \leq \varpi_p\frac{1}{|\epsilon|}(1+m)^{-\mu}\exp(-\beta m)\exp\big(\nu\frac{r^k}{|\epsilon|^k}\big)\exp\big(-\Delta_p\frac{r^k}{|T|^k}\big) \leq \varpi_p\frac{1}{|\epsilon|}(1+m)^{-\mu}\exp(-\beta m)\exp\big(-\frac{\hat{\Delta}_p}{|\epsilon|^k}r^k\big)$$

provided that $u = re^{\sqrt{-1}\gamma_p} \in L_{\gamma_p}$, for $r \geq 0$ and that $T \in \mathcal{D}_{|\epsilon|,\Delta_p}$. We deduce that

$$
(181) \quad |U_p(T,m,\epsilon)| \leq k\varpi_p(1+m)^{-\mu}\exp(-\beta m)\int_0^{+\infty}\frac{1}{|\epsilon|}\exp\big(-\frac{\hat{\Delta}_p}{|\epsilon|^k}r^k\big)dr \\
\leq k\varpi_p(1+m)^{-\mu}\exp(-\beta m)\big(\int_0^{+\infty}e^{-\hat{\Delta}_p s^k}ds\big)
$$

by means of the change of variable $s = r/|\epsilon|$ in the integral, for all integers $m \geq 0$, all $\epsilon \in \mathcal{E}_p$ and $T \in \mathcal{D}_{|\epsilon|,\Delta_p}$.

Since the map $\omega_{\mathfrak{d}_p}(u,0,\epsilon)$ solves (47), for all $u \in S_{\mathfrak{d}_p} \cup D_{\rho_0}$ and $\omega_{\mathfrak{d}_p}(u,m,\epsilon)$ fulfills (48) for $m \geq 1$, whenever $u \in S_{\mathfrak{d}_p} \cup D_{\rho_m}$, provided that $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, we deduce that the sequence of maps $\omega_{\mathfrak{d}_p}(u,m,\epsilon)$, $m \geq 0$, as a whole, is subjected to the convolution relation (40), for all $u \in S_{\mathfrak{d}_p} \cup D_{\rho_m}$ and fixed $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. As a result, the bunch of computations made in Subsection 2.3 leads to the fact that the maps $U_p(T,m,\epsilon)$, $m \geq 0$, solve the differential recursion (36) and then ought to fulfill the sequence of ordinary differential equations (32) whenever $T \in \mathcal{D}_{|\epsilon|,\Delta_p}$, for any given $\epsilon \in D_{\epsilon_0} \setminus \{0\}$.

At last, we set

$$
(182) \qquad u_p(t,z,\epsilon) = \sum_{m \geq 0} u_{p,m}(t,\epsilon)e^{\sqrt{-1}zm}
$$

with coefficients $u_{p,m}(t,\epsilon) = U_p(\epsilon t,m,\epsilon)$, for all $m \geq 0$, which represents a bounded holomorphic function on the product $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$, provided that the radius $r_{\mathcal{T}}$ is restricted to the bounds

$$
(183) \qquad r_{\mathcal{T}} < \big(\frac{\Delta_p}{\nu + \hat{\Delta}_p}\big)^{1/k}.
$$

Namely, according to (181), we get

$$
(184) \quad |u_p(t,z,\epsilon)| \leq k\varpi_p\big(\int_0^{+\infty}e^{-\hat{\Delta}_p s^k}ds\big)\sum_{m\geq 0}(1+m)^{-\mu}e^{-\beta m}e^{-m\mathrm{Im}(z)} \\
\leq k\varpi_p\big(\int_0^{+\infty}e^{-\hat{\Delta}_p s^k}ds\big)\sum_{m\geq 0}(1+m)^{-\mu}e^{-(\beta-\beta')m}
$$

which is a finite quantity, provided that $z \in H_{\beta'}$ with $0 < \beta' < \beta$, for all $t \in \mathcal{T}$ under the condition (183), as long as $\epsilon \in \mathcal{E}_p$. Furthermore, since the coefficients $U_p(T,m,\epsilon)$, $m \geq 0$, conform to the sequence of ordinary differential equations (32), we deduce from straight standard computations that $u_p(t,z,\epsilon)$ obey the initial value problem (13) with vanishing initial condition $u_p(0,z,\epsilon) \equiv 0$ provided that $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}_p$.

In the second part of the discussion, we give attention to the second item of the theorem. We first observe that, owing to the construction of $\omega_{\mathfrak{d}_p}(u,m,\epsilon)$ described at the beginning of the proof, each partial map $u \mapsto \omega_{\mathfrak{d}_p}(u,m,\epsilon)$, for any prescribed integers $0 \leq p \leq \varsigma - 1$, $m \geq 0$, any fixed $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, is the analytic continuation on the unbounded sector $S_{\mathfrak{d}_p}$ of a common analytic map we denote $u \mapsto \omega(u,m,\epsilon)$ on the disc D_{ρ_m}.

Let us fix some integer $0 \leq p \leq \varsigma - 1$. We notice that, for all integers $m \geq 0$, the partial map $u \mapsto \omega_{\mathfrak{d}_p}(u,m,\epsilon)\exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{1}{u}$ is holomorphic on the domain $S_{\mathfrak{d}_p} \cup D_{\rho_m}$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$ and $t \in \mathcal{T}$. The classical Cauchy's theorem enables the warping of the halflines L_{γ_p} and $L_{\gamma_{p+1}}$

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of integration for the difference of Laplace transforms $u_{p+1,m}(t,\epsilon) - u_{p,m}(t,\epsilon)$, for each integer $m \geq 0$, that can be rewritten as a sum of three contributions

$$(185)\quad u_{p+1,m}(t,\epsilon) - u_{p,m}(t,\epsilon) = k\int_{L_{\rho_m/2;\gamma_{p+1}}} \omega_{\mathfrak{d}_{p+1}}(u,m,\epsilon)\exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{du}{u}$$
$$- k\int_{L_{\rho_m/2;\gamma_p}} \omega_{\mathfrak{d}_p}(u,m,\epsilon)\exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{du}{u} + k\int_{C_{\rho_m/2;\gamma_p,\gamma_{p+1}}} \omega(u,m,\epsilon)\exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{du}{u}$$

along an arrangement of integration paths depicted as the union of

- Two halflines
$$L_{\rho_m/2;\gamma_j} = [\rho_m/2,+\infty)e^{\sqrt{-1}\gamma_j}$$
in direction γ_j, laying apart of the origin with distance $\rho_m/2$, for $j=p,p+1$
- An arc of circle
$$C_{\rho_m/2;\gamma_p,\gamma_{p+1}} = \{\frac{\rho_m}{2}e^{\sqrt{-1}\theta}/\theta \in (\gamma_p,\gamma_{p+1})\}$$
with radius $\rho_m/2$ joining the above halflines $L_{\rho_m/2;\gamma_j}$, $j=p,p+1$, endowed with a suitable orientation.

1) Bounds for the first piece of the above decomposition

$$I_1 = \Big|k\int_{L_{\rho_m/2;\gamma_{p+1}}} \omega_{\mathfrak{d}_{p+1}}(u,m,\epsilon)\exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{du}{u}\Big|$$

are provided. Namely, based on the upper estimates (180), we get

$$(186)\quad I_1 \leq k\varpi_{p+1}(1+m)^{-\mu}e^{-\beta m}\frac{1}{|\epsilon|}\int_{\rho_m/2}^{+\infty}\exp\big(-(\frac{\Delta_{p+1}}{|t|^k}-\nu)(\frac{r}{|\epsilon|})^k\big)dr$$
$$\leq k\varpi_{p+1}(1+m)^{-\mu}e^{-\beta m}\frac{1}{|\epsilon|}\int_{\rho_m/2}^{+\infty}\Big\{\frac{|\epsilon|^k}{(\frac{\Delta_{p+1}}{|t|^k}-\nu)k(\rho_m/2)^{k-1}}\Big\}\frac{(\frac{\Delta_{p+1}}{|t|^k}-\nu)kr^{k-1}}{|\epsilon|^k}$$
$$\times\exp\big(-(\frac{\Delta_{p+1}}{|t|^k}-\nu)(\frac{r}{|\epsilon|})^k\big)dr = k\varpi_{p+1}(1+m)^{-\mu}e^{-\beta m}\frac{|\epsilon|^{k-1}}{(\frac{\Delta_{p+1}}{|t|^k}-\nu)k(\rho_m/2)^{k-1}}$$
$$\times\exp\big(-(\frac{\Delta_{p+1}}{|t|^k}-\nu)(\frac{\rho_m/2}{|\epsilon|})^k\big) \leq k\varpi_{p+1}(1+m)^{-\mu}e^{-\beta m}\frac{|\epsilon|^{k-1}}{\hat{\Delta}_{p+1}k(\rho_m/2)^{k-1}}\exp\big(-\hat{\Delta}_{p+1}\frac{(\rho_m/2)^k}{|\epsilon|^k}\big)$$

provided that $t \in \mathcal{T}$ under the constraint (183) and that $\epsilon \in \mathcal{E}_{p+1}\cap\mathcal{E}_p$.

2) The second element

$$I_2 = \Big|k\int_{L_{\rho_m/2;\gamma_p}} \omega_{\mathfrak{d}_p}(u,m,\epsilon)\exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{du}{u}\Big|$$

of the above splitting can be upper estimated in a similar way as done in (186). Indeed,

$$(187)\quad I_2 \leq k\varpi_p(1+m)^{-\mu}e^{-\beta m}\frac{|\epsilon|^{k-1}}{\hat{\Delta}_p k(\rho_m/2)^{k-1}}\exp\big(-\hat{\Delta}_p\frac{(\rho_m/2)^k}{|\epsilon|^k}\big)$$

for all $t \in \mathcal{T}$ subjected to (183) and for all $\epsilon \in \mathcal{E}_{p+1}\cap\mathcal{E}_p$.

3) The third component

$$I_3 = \Big| k \int_{C_{\rho_m/2;\gamma_p,\gamma_{p+1}}} \omega(u,m,\epsilon) \exp\big(-(\frac{u}{\epsilon t})^k \big) \frac{du}{u} \Big|$$

is controled as follows. According to the second item 2. of Definition 5, we can single out a constant $\Delta_{p,p+1} > 0$ (that can be taken as $\min(\Delta_p, \Delta_{p+1})$) with the condition

$$\cos\big(k(\theta - \arg(\epsilon t))\big) > \Delta_{p,p+1}$$

for all $t \in \mathcal{T}$, all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, whenever the angle θ belongs to (γ_p, γ_{p+1}) or (γ_{p+1}, γ_p). Furthermore, bearing in mind the upper bounds (175) for the Borel map ω and the hypothesis (21) granting that the sequence $(\rho_m)_{m\geq 0}$ is decreasing, we come up with a constant $\hat{\Delta}_{p,p+1} > 0$ such that

$$\begin{aligned}(188)\quad I_3 &\leq k \max(\varpi_p, \varpi_{p+1})(1+m)^{-\mu} e^{-\beta m} \\ &\times \Big| \int_{\gamma_p}^{\gamma_{p+1}} \frac{\rho_m/2}{|\epsilon|} \exp\big(\nu \frac{(\rho_m/2)^k}{|\epsilon|^k} \big) \exp\big(-\Delta_{p,p+1} \frac{(\rho_m/2)^k}{|\epsilon t|^k} \big) d\theta \Big| \\ &\leq k \max(\varpi_p, \varpi_{p+1})(1+m)^{-\mu} e^{-\beta m} |\gamma_{p+1} - \gamma_p| \frac{\rho_m}{2} \frac{1}{|\epsilon|} \exp\big(-\frac{1}{2}(\frac{\Delta_{p,p+1}}{|t|^k} - \nu)(\frac{\rho_m/2}{|\epsilon|})^k \big) \\ &\times \exp\big(-\frac{1}{2}(\frac{\Delta_{p,p+1}}{|t|^k} - \nu)(\frac{\rho_m/2}{|\epsilon|})^k \big) \leq k \max(\varpi_p, \varpi_{p+1})(1+m)^{-\mu} e^{-\beta m} |\gamma_{p+1} - \gamma_p| \\ &\times \frac{\rho_m/2}{|\epsilon|} \exp\big(-\frac{1}{2}\hat{\Delta}_{p,p+1}(\frac{\rho_m/2}{|\epsilon|})^k \big) \exp\big(-\frac{1}{2}\hat{\Delta}_{p,p+1}(\frac{\rho_m/2}{|\epsilon|})^k \big) \\ &\leq k \max(\varpi_p, \varpi_{p+1})(1+m)^{-\mu} e^{-\beta m} |\gamma_{p+1} - \gamma_p| \Big(\sup_{x \geq 0} x \exp\big(-\frac{1}{2}\hat{\Delta}_{p,p+1} x^k \big) \Big) \\ &\times \exp\big(-\frac{1}{2}\hat{\Delta}_{p,p+1}(\frac{\rho_m/2}{|\epsilon|})^k \big) \leq k \max(\varpi_p, \varpi_{p+1})(1+m)^{-\mu} e^{-\beta m} \\ &\times |\gamma_{p+1} - \gamma_p| \Big(\sup_{x \geq 0} x \exp\big(-\frac{1}{2}\hat{\Delta}_{p,p+1} x^k \big) \Big) \times \exp\big(-\frac{1}{2}\hat{\Delta}_{p,p+1}(\frac{\rho_m/2}{|\epsilon|})^k \big) (\rho_0/\rho_m)^{k-1}\end{aligned}$$

given that $t \in \mathcal{T}$ under the restriction (183), for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$.

Proceeding from the decomposition (185) and summing up the above bounds (186), (187), (188), we arrive at some constants $C_{p,p+1} > 0$ and $\nabla_{p,p+1} > 0$ (relying on p, ϵ_0, k, ν) such that

$$(189)\qquad |u_{p+1,m}(t,\epsilon) - u_{p,m}(t,\epsilon)| \leq C_{p,p+1}(1+m)^{-\mu} e^{-\beta m} \frac{1}{(\rho_m)^{k-1}} \exp\big(-\nabla_{p,p+1}(\frac{\rho_m}{|\epsilon|})^k \big)$$

granting that $t \in \mathcal{T}$, for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$ and all integers $m \geq 0$.

Departing from the definition (44) of ρ_m and owing to the bounds (19) and (104), we can select a constant $\mathfrak{R} > 0$ such that

$$(190)\qquad \rho_m \geq \frac{1}{2}\Big(\frac{\mathfrak{Q}_1 (1+m)^{\deg(Q)}}{\mathfrak{R}_D (1+m)^{\deg(R_D)} k^{\delta_D}} \Big)^{\frac{1}{k\delta_D}} \geq \frac{\mathfrak{R}}{(1+m)^{\frac{\deg(R_D)-\deg(Q)}{k\delta_D}}}$$

for all integers $m \geq 0$.

From (189) and (190), we deduce a constant $\check{M}_{k,\mu,\delta_D,\beta,\beta'} > 0$ with

$$
(191) \quad |u_{p+1}(t,z,\epsilon) - u_p(t,z,\epsilon)| \leq C_{p,p+1} \sum_{m \geq 0} (1+m)^{-\mu} e^{-\beta m} e^{-m\mathrm{Im}(z)} \frac{1}{(\rho_m)^{k-1}} \exp\big(-\nabla_{p,p+1} (\frac{\rho_m}{|\epsilon|})^k \big)
\leq \frac{C_{p,p+1}}{\mathfrak{R}^{k-1}} \sum_{m \geq 0} (1+m)^{-\mu + \frac{k-1}{k\delta_D}\left(\deg(R_D) - \deg(Q)\right)} e^{-(\beta-\beta')m}
\times \exp\big(-\nabla_{p,p+1}\mathfrak{R}^k \frac{1}{(1+m)^{\frac{\deg(R_D)-\deg(Q)}{\delta_D}}} \frac{1}{|\epsilon|^k} \big) \leq \check{M}_{k,\mu,\delta_D,\beta,\beta'} \frac{C_{p,p+1}}{\mathfrak{R}^{k-1}} \sum_{m \geq 0} e^{-\frac{\beta-\beta'}{2}m}
\times \exp\big(-\nabla_{p,p+1}\mathfrak{R}^k \frac{1}{(1+m)^{\frac{\deg(R_D)-\deg(Q)}{\delta_D}}} \frac{1}{|\epsilon|^k} \big)
$$

for all $t \in \mathcal{T}$, all $z \in H_{\beta'}$ with $0 < \beta' < \beta$ and $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$.

In the last step, we need a technical lemma that has been established and discussed in our former joint work [15] with A. Lastra and J. Sanz.

Lemma 6 *Let $0 < a < 1$ and $\alpha > 0$ be real numbers. There exist three constants $K, M, \delta > 0$ such that*

$$
\sum_{k \geq 0} \exp\big(-\frac{1}{(k+1)^{\alpha}} \frac{1}{\epsilon} \big) a^k \leq K \exp(-M\epsilon^{-\frac{1}{\alpha+1}})
$$

for all $\epsilon \in (0, \delta]$.

As a result of this lemma and the bounds (191) overhead, provided that $\epsilon_0 > 0$ is small enough, one can find two constants $K, M > 0$ with

$$
(192) \quad |u_{p+1}(t,z,\epsilon) - u_p(t,z,\epsilon)| \leq \check{M}_{k,\mu,\delta_D,\beta,\beta'} \frac{C_{p,p+1}}{\mathfrak{R}^{k-1}} K \exp\Big(-M (\frac{|\epsilon|^k}{\nabla_{p,p+1}\mathfrak{R}^k})^{-\frac{1}{\frac{\deg(R_D)-\deg(Q)}{\delta_D}+1}} \Big)
$$

for all $t \in \mathcal{T}$, all $z \in H_{\beta'}$ with $0 < \beta' < \beta$ and $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$. This yields the second item (177) of Theorem 1. □

6 Parametric Gevrey asymptotic expansions for the finite set of holomorphic solutions to (13)

In this section, we show the existence of a common asymptotic expansion in the parameter ϵ of Gevrey type for the set of holomorphic solutions to our main initial value problem constructed in the previous section.

We first call attention to a result known as the Ramis-Sibuya theorem stated in Lemma XI-2-6 in [5].

Theorem (R.S.) *Let $(\mathbb{F}, ||.||_{\mathbb{F}})$ be a Banach space over $\mathbb{C}$ and we consider a good covering $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ in $\mathbb{C}^*$ as described in Definition 4. For all $0 \leq p \leq \varsigma - 1$, we set $G_p : \mathcal{E}_p \to \mathbb{F}$ as holomorphic functions that are subjected to the next two constraints*

1. *The maps G_p are bounded on $\mathcal{E}_p$ for all $0 \leq p \leq \varsigma - 1$.*

2. *The difference* $\Theta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ *defines a holomorphic map on the intersection* $Z_p = \mathcal{E}_{p+1} \cap \mathcal{E}_p$ *which is exponentially flat of order* κ*, for some integer* $\kappa \geq 1$*, meaning that one can select two constants* $C_p, A_p > 0$ *for which*

$$||\Theta_p(\epsilon)||_{\mathbb{F}} \leq C_p \exp(-\frac{A_p}{|\epsilon|^{\kappa}})$$

holds provided that $\epsilon \in Z_p$*, for all* $0 \leq p \leq \varsigma - 1$*. By convention, we set* $G_{\varsigma} = G_0$ *and* $\mathcal{E}_{\varsigma} = \mathcal{E}_0$*.*

Then, one can single out a formal power series $\hat{G}(\epsilon) = \sum_{n\geq 0} G_n \epsilon^n$ *with coefficients* G_n *belonging to* $\mathbb{F}$*, which is the common Gevrey asymptotic expansion of order* $1/\kappa$ *relatively to* ϵ *on* $\mathcal{E}_p$ *for all the maps* G_p*, for* $0 \leq p \leq \varsigma - 1$*. It means that two constants* $K_p, M_p > 0$ *can be pinpointed with the error bounds*

$$||G_p(\epsilon) - \sum_{n=0}^{N} G_n \epsilon^n||_{\mathbb{F}} \leq K_p M_p^{N+1} \Gamma(1 + \frac{N+1}{\kappa})|\epsilon|^{N+1} \tag{193}$$

for all integers $N \geq 0$*, all* $\epsilon \in \mathcal{E}_p$*, all* $0 \leq p \leq \varsigma - 1$*.*

The next claim represents the second salient result of our work.

Theorem 2 *There exist a formal power series* $\hat{u}(t,z,\epsilon) = \sum_{n\geq 0} G_n(t,z)\epsilon^n$ *whose coefficients* $G_n(t,z)$*,* $n \geq 0$*, are bounded holomorphic functions on the product* $\mathcal{T} \times H_{\beta'}$*, which stands for the common asymptotic expansion of Gevrey order* κ^{-1} *for* κ *given in (176) of the partial maps* $\epsilon \mapsto u_p(t,z,\epsilon)$ *on every sectors* $\mathcal{E}_p$*, for* $0 \leq p \leq \varsigma - 1$*, uniformly in* (t,z) *on* $\mathcal{T} \times H_{\beta'}$*. More precisely, one can select two constants* $K_p, M_p > 0$*, for which the next error bounds*

$$\sup_{\substack{t \in \mathcal{T} \\ z \in H_{\beta'}}} |u_p(t,z,\epsilon) - \sum_{n=0}^{N} G_n(t,z)\epsilon^n| \leq K_p M_p^{N+1} \Gamma(1 + \frac{N+1}{\kappa})|\epsilon|^{N+1} \tag{194}$$

hold, for all integers $N \geq 0$*, as long as* $\epsilon \in \mathcal{E}_p$*, for any* $0 \leq p \leq \varsigma - 1$*.*

Proof Let us consider the set of functions $u_p(t,z,\epsilon)$, $0 \leq p \leq \varsigma - 1$, constructed in the first main statement Theorem 1. We set $\mathbb{F}$ as the Banach space of bounded holomorphic functions on the product $\mathcal{T} \times H_{\beta'}$ equipped with the sup norm. For all $0 \leq p \leq \varsigma - 1$, we introduce the maps $G_p : \mathcal{E}_p \to \mathbb{F}$ defined as

$$G_p(\epsilon) := (t,z) \mapsto u_p(t,z,\epsilon).$$

According to Theorem 1, we observe that for each $0 \leq p \leq \varsigma - 1$,

- The map $\epsilon \mapsto G_p(\epsilon)$ is holomorphic and bounded on $\mathcal{E}_p$, since $u_p(t,z,\epsilon)$ is holomorphic and bounded on the product $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$
- The difference $\Theta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ is submitted to the bounds

$$||G_{p+1}(\epsilon) - G_p(\epsilon)||_{\mathbb{F}} \leq A_p \exp\big(-\frac{B_p}{|\epsilon|^{\kappa}}\big)$$

  for the constants $A_p, B_p > 0$ given in (177), κ displayed in (176), provided that $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, where the convention $G_{\varsigma} = G_0$ and $\mathcal{E}_{\varsigma} = \mathcal{E}_0$ holds.

As an outcome, the requirements 1. and 2. of Theorem (R.S.) are fufilled for the set of maps $(G_p)_{0\leq p \leq \varsigma - 1}$ and we deduce the existence of a formal power series $\hat{u}(t,z,\epsilon) = \hat{G}(\epsilon) = \sum_{n\geq 0} G_n(t,z)\epsilon^n$ with coefficients G_n belonging to $\mathbb{F}$ which is the common asymptotic expansion of Gevrey order $1/\kappa$ relatively to ϵ on $\mathcal{E}_p$ for all the maps G_p, $0 \leq p \leq \varsigma - 1$. In other words, the bounds (194) follow. □

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