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Article

A Study of McVittie Spacetime in Stationary and Dynamical Regime

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Abstract: One of the solutions of the Einstein equations, called McVittie solution, signifying a black hole embedded by the dynamic spacetime is studied. In the stationary spacetime the Mcvittie metric becomes the Schwarzschild-de Sitter metric (SdS). The geodesic of a freely falling test particle towards the black hole is examined in the SdS spacetime. It is found that unlike Schwarzschild case the potential of such particle becomes maximum at a point where it eventually stops to follow an unstable circular motion and then resumes its motion towards black hole center. It is shown that an observer or system of particles is spaghettified near the black hole singularity in the SdS spacetime. The dynamic of the universe in the framework of McVittie metric, being a generalized time dependent SdS solution, is represented in terms of that point, called stationary or turning point. The motion of the stationary point is studied in various regimes of the expanding universe and the possible outcomes are discussed in brief.

Keywords: Einstein equations; McVittie solution; black hole; dark energy; embedded spacetime

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1. Introduction

Finding solutions to Einstein's field equations that describes a physically viable system has always been challenging. One of the main reasons for it is the highly non-linear nature of these equations. Describing the dynamics of objects like compact stars require interior solutions of Einstein's field equations, as well as for finding black hole solutions, physically acceptable solutions for black holes are essential. If we look into the history of Einstein's theory of general relativity (GR), there are two exact solutions obtained that have been studied in various contexts for several years. One is the Schwarzschild solution that can describe the nature of the gravitational field residing outside a chargeless spherical mass with rotational velocity and cosmological constant taken to be zero. For several years this solution was vastly used to theoretically probe the nature of spacetime outside compact stars and black holes. But, the problem was that this solution doesn't take into account the expanding nature of the universe in the background with the existence of mass. However, the other exact solution named Friedmann Robertson Walker Lemaìtre (FRWL) metric [1,2] assumes the spatial component of the metric to be time-dependent and the space to be isotropic and homogeneous. Faraoni [3] and Jacques [4] took the effects of the expanding universe and the cosmological impact of this expansion on the local system. For gravitationally bound systems, they also studied the nature of local attraction and analyzed black holes embedded in the spherical cosmological background in the framework of general relativity. Using a time-dependent spacetime, Arakida [5] analyzed the dominant effects that come from the cosmological expansion.

Before coming to the methodology applied in this paper for studying a black hole embedded by the dynamic spacetime, let us briefly shed light on some different approaches that black holes have been studied. With the rapid strides made by gravitational wave astronomy, the way we understand black holes have been revolutionized [6]. But, in order to interpret the gravitational wave signals, we must be adept with efficient theoretical models to predict the waveforms observed. These theoretical models are obtained by solving Einstein's field equations, which is a very difficult task as discussed earlier. Moreover, for the upgraded detectors, the existing models or methodologies might not be efficient enough [7]. In recent times, some researchers have thought of unique ideas to come up with efficient theoretical models which can match up with the evolving observational techniques. One such way is the high-energy physics approach. In high-energy physics, most of the processes are considered as some form of scattering. Here, the two black holes are treated as quantum particles that interact with each other by exchanging gravitons, just like electron interactions happen via photon exchange. In this light, Damour and his collaborators [8–10] made remarkable progress using this scattering analogy. In their recent work, it was shown that there can be a computational shortcut to treat the generic scattering problem by taking a limiting value, where one black hole is way heavier than the other. One great advantage of this method is to combine all possible interactions and come up with extremely accurate results and several research groups are playing with this idea and combining several other techniques of quantum field theory in order to calculate the gravitational scattering amplitude between the socalled "black hole particles". Some recent works in this regard to study scattering black holes can be found in the following Refs. [8–13].

Another methodology, which is one of the most general ways to analyze the black holes embedded in the expanding universe is using the McVitte metric. In the current work, we'll focus on using this methodology. In the year 1933 McVittie found a class of solutions [14,15] whose physical descriptions are still going on. With a time dependent model McVittie proposed a spherical symmetric metric describing a central massive object embedded in an expanding spacetime. Essentially both the Schwarzschild [16] as well as spatially flat FRW [1] solutions are incorporated in that metric. McVittie's work was remarkable in the sense of exploring the cosmological effects in terms of expansion on local system. The researchers and scientists have studied [17–24] the McVittie solution under the back-drop of massive object in an expanding universe. A generalized McVittie metric has also been taken into account and studied by some authors [25–27]. In 2018 Nolan [28] studied the particle and photon orbits in McVittie spacetime. That is the motivation of the present article, but slightly a different manner to study the freely falling body towards the central mass. In the Schwarzschild spacetime if a body is thrown towards the black hole then it continues to follow the geodesic of free fall and ultimately gets destroyed by the tidal effect near the singularity.

Now the question may arise if such motion of a body is studied carefully in the framework of McVittie spacetime, then whether it will follow the same path as that of the Schwarzschild case, and if not then what will be that new feature in the motion? If such free fall is considered both in stationary and dynamical scenario then what will be the possible picture? We have discussed the possible answers to these queries in the present investigation. Our scheme is as follows: In Section 2 such analysis is carried out for the stationary metric (Schwarzschild-de Sitter), while in Section 3 the McVittie spacetime is studied how it shapes the dynamic of the universe, especially how it can handle the singularity especially in the late stages, that is in the phantom energy regime.

2. McVittie metric in stationary spacetime

If the McVittie metric under the stationary spacetime is studied in a particular regime, say in the present age of the evolution of universe, it is sufficient to study Schwarzschild de Sitter (SdS) metric [29] in which a positive cosmological constant is taken into account. Its global structure have been studied by a number of authors [30–36]. Podolsky [37] analyzed the structure of extreme SdS spacetime in details. In the McVittie scenario the study is manifested by the focus on spacetime originated from a black hole; whereas in the SdS the stationary as well as static solution of Einstein

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equation in vacuum is taken into account. Although their physical perspective may be slightly different, but their mathematical structure is exactly same and therefore, there is no harm to consider the SdS metric in the present scenario.

2.1. Path of the test particle

The path of a test particle in Schwarzschild spacetime is widely discussed. Now the question may arise whether there any change is observed if such particle is considered in SdS spacetime. Without carrying out any analysis it may safely be predicted that for the planetary motion such difference, if exists, is not much significant. But when a massive test particle is assumed to fall freely towards a black hole, then there may be a significant difference between the results obtained in SdS spacetime compared to that of Schwarzschild. It is to be explored in more details.

The said SdS metric is given as follows:

$$ds^2 = -(1 - \frac{2m}{r} - H^2r^2)dt^2 + (1 - \frac{2m}{r} - H^2r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

Here H the Hubble parameter, is considered to be independent of time since the study is carried out in a specific age, may be the present universe. One may now study the geodesic of a test particle falling freely towards the center of the black hole. That is just equivalent to the path of a particle without any angular momentum in the SdS spacetime. Now, with a non-zero angular momentum, say l, in the plane $\theta = \frac{\pi}{2}$ the expressions of $\dot{t}$ and $\dot{\phi}$ can be obtained as

$$\dot{t} = \frac{e}{(1 - \frac{2m}{r} - H^2r^2)}, \qquad \dot{\phi} = \frac{l}{r^2}, \tag{2}$$

where e represents energy of the test particle per unit of its mass. Such energy may be negative, but must be conserved. Imposing the timelike constraint on the geodesic it is found that

$$\dot{r}^2 - \frac{2m}{r} + \frac{l^2}{r^2} - \frac{2ml^2}{r^3} - H^2(r^2 + l^2) = -2|\mathcal{E}| = e^2 - 1. \tag{3}$$

If the particle is considered under the free fall then the angular momentum vanishes and one may find the change of r coordinate relative to the proper time as

$$\dot{r} = -\left(\frac{2m}{r} + H^2r^2 - 2|\mathcal{E}|\right)^{\frac{1}{2}}. \tag{4}$$

The negative sign is taken since the particle moves towards the center of the black hole.

The above expression may also be re-expressed in the following manner

$$\frac{1}{2}\dot{r}^2 + V(r) = -|\mathcal{E}|, \tag{5}$$

where

$$V(r) = -\frac{m}{r} - H^2r^2. \tag{6}$$

Equation (6) represents the potential of the particle depending on the areal coordinate r. It is to be noted carefully that in absence of angular momentum under the free fall of a particle towards a massive celestial body (may or may not be a black hole) there is no difference between Newtonian and potential and the same in the framework of Schwarzschild solution. As a result once a particle begins to fall towards the center of black hole, in both Newtonian as well as Schwarzschild frames, it falls uninterruptedly. In the SdS the story is quite different due to the presence of an extra $-H^2r^r$ term in the expression of potential. It is going to be explored how such extra term affects the dynamic of test particle in the apparently stationary spacetime.

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It is evident from Eq. (6) that for SdS the potential $V(r)$, at infinity, behaves quite differently compared to that in Newtonian or Schwarzschild case, where the potential tends to vanish at infinity. In the present scenario the potential, both at large r and near the singularity at $r = 0$, becomes infinitely large with the negative value. When the test particle starts from large r (but not from infinity) and goes towards the black hole then, with decreasing r, the potential increases (or in other words its absolute value decreases) and at $r = r_{max}$ (say) the potential attains its maximum value, still in the negative regime. This picture is well observed in the Fig. 1 where a comparison between the potential term in the SdS and normal Schwarzschild spacetime is carried out. Clearly such figure indicates the above phenomenon.

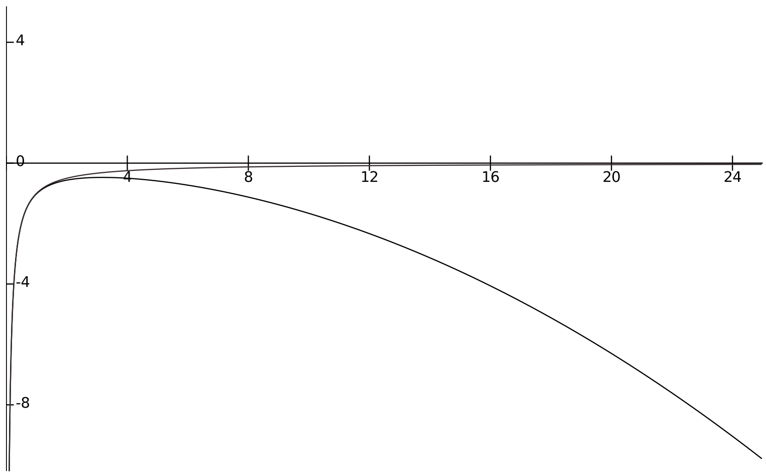

Figure 1. The change of potential $V(r)$ with the radial coordinate r, both for Schwarzschild (light curve) and SdS (dark curve) metric. The comparison shows that unlike Schwarzschild spacetime the potential term in SdS spacetime is extremely large (with negative value) for r tending to infinity, becomes maximum at some finite r and then coincides with the curve in Schwarzschild spacetime for small r.

Equating $\frac{dV}{dr} = 0$ such maximum point is found as

$$r = r_{max} = \frac{m}{H^2}. \tag{7}$$

By calculating $\frac{d^2V}{dr^2}$ it can easily be verified that $r = r_{max}$ is the maximum point and thus unstable. It signifies that the magnitude of the potential becomes minimum at that point. Therefore, according to Eq. (5) the particle must retard and stop somewhere in the mid way. If $r = r_0$ denotes the point where the particle comes to rest, then from Eqs. (5) and (6) one may obtain

$$r\dot{r} = H^2(r - r_0)(r - r_1)(r - r_2), \tag{8}$$

where

$$r_{1,2} = \frac{r_0}{2}\left(-1 \pm \sqrt{1 + 8\frac{r_{max}^3}{r_0^3}}\right). \tag{9}$$

The negative root r_2 of the equation $\dot{r} = 0$ should not be taken into account because of being unphysical. Now, the following two cases may be considered for further studies.

Case I: $r_{max} < r_0$.
That shows $r_1 < r_0$ in Eq. (9). Now, the equation indicates $\dot{r}^2 > 0$ outside the open interval (r_1, r_0), but inside the interval $\dot{r}^2 < 0$. Such unphysical condition leads to the fact that $r_{max} < r_0$ cannot hold.

Case II: $r_{max} < r_0$.
With the similar argument of Case I one may show the unphysical situation in terms of $\dot{r}^2 < 0$ inside $0 < r_0 < r_1$, which indicates that the condition $r_{max} > r_0$ is also a wrong one.

Both the above cases conditions indicate that r_{max} must coincides with r_0. In other words, the point of turning is same as that where the potential of the test particle attains the maximum value. With one step ahead it can be said that the particle rushing towards the center of black hole stops at the point $r = r_0$ with maximum potential (or minimum $|V(r)|$) and then starts to revolve round the black hole in a unstable circular orbit. The orbit is unstable since the particle stays on the maximum potential. Subsequently the particle starts to move again towards the black hole singularity. Next, it is to calculate the position of the apparent horizons relevant to the SdS spacetime.

2.2. Apparent horizons and the stationary or turning point r_0

In the framework of SdS metric one may find the Kretschmann scalar as $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6} + 24H^4$ and therefore, the singularity at $r = 0$ is also present as that in Schwarzschild spacetime. To locate the position of the apparent horizon it is necessary to solve the equation $1 - \frac{2m}{r} - H^2r^2 = 0$, which is nothing but a third degree algebraic equation expressed as

$$r^3 - \frac{r}{H^2} + \frac{2m}{H^2} = 0. \tag{10}$$

In the previous subsection it has been obtained that $r_{max} = r_0$. Using the result of Eq. (7) the discriminant of the above cubic equation can be expressed as

$$\triangle = r_0^6\left[1 - \left(\frac{r_0}{3m}\right)^3\right]. \tag{11}$$

The nature of the solution to the cubic equation (10) is decided by the sign of $\triangle$; whereas the sign of $\triangle$ depends on the relation between r_0 and $3m$. Under the circumstances three different cases are discussed below.

Case I: $r_0 > 3m$.
It shows the sign of $\triangle$ is always positive. That implies the cubic equation has two distinct positive and one negative real roots as follows:

$$\begin{aligned} r &= \frac{2r_0^{\frac{3}{2}}}{\sqrt{3m}}\cos\alpha \\ &= \frac{r_0^{\frac{3}{2}}}{\sqrt{m}}\left[\sin\alpha - \frac{\cos\alpha}{\sqrt{3}}\right] \\ &= -\frac{r_0^{\frac{3}{2}}}{\sqrt{m}}\left[\sin\alpha + \frac{\cos\alpha}{\sqrt{3}}\right], \end{aligned} \tag{12}$$

where $\alpha = \frac{1}{3}\arccos[-(\frac{3m}{r_0})^{\frac{3}{2}}]$. The negative root must be ignored due to its unphysical nature.

Two positive roots are to be considered as the point of discussion. Out of those the $\frac{2r_0^{\frac{3}{2}}}{\sqrt{3m}}\cos\alpha$ is identified as the apparent horizon or event horizon of the SdS black hole. On the other hand, $\frac{r_0^{\frac{3}{2}}}{\sqrt{m}}[\sin\alpha - \frac{\cos\alpha}{\sqrt{3}}]$ represents the cosmological horizon. These two horizons are well separated if and only if $r_0 \gg 3m$, that is, the point of turning is well ahead of $3m$. It is important to note that the black hole horizon depends on the Hubble parameter H.

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Case II: $r_0 = 3m$.
It leads to $\triangle = 0$ and there will be a positive real double root along with a negative one. Those can be obtained as

$$\begin{aligned} r &= r_0 \ (double) \\ &= -2r_0, \end{aligned} \tag{13}$$

which essentially indicates that not only the black hole horizon and the cosmological horizon become same, but the point of turning (which is also the stationary point in terms of the potential) also does coincide with those two horizons. It is known that the coincidence of two horizons corresponds to the extreme Nariai solution [38]. Since the point of turning lies on the black hole horizon the test particle is supposed to revolve along that horizon. That is quite peculiar because no timelike particle moves along a null surface. Therefore, the situation $r_c = 3m$ should not be a feasible one.

Case III: $r_0 < 3m$.
Here $\triangle$ becomes negative and hence the cubic equation has three unphysical (one negative and two complex) roots. Therefore, such condition may be dropped. It is worth mentioning that Faraoni et al. [23]. pointed out there would be a possibility of naked singularity associated to that case, but that is beyond the scope of discussion of the present article.

The Fig. 2 represents the Penrose diagram of the maximal SdS spacetime containing both the black hole horizon r_{BH} as well as cosmological horizon represented by r_{CH}. Now, it is required to locate the stationary point r_0, where the particle stops and follows unstable circular path.

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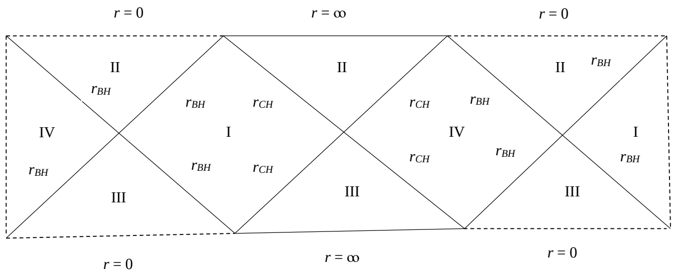

Figure 2. Penrose diagram for Schwarzschild-de Sitter (SdS) spacetime.

To find it the condition $V(r) = 2\mathcal{E}$ is imposed and it is straight forward to solve another third degree algebraic equation given hereunder

$$r^3 - 2 \mid \mathcal{E} \mid \frac{r}{H^2} + \frac{2m}{H^2} = 0. \tag{14}$$

It has already been shown that the solution of the equation has a positive double root r_0 and a negative root. Such type of roots exist only when the discriminant of the cubic equation (14) vanishes and the following condition is satisfied:

$$\frac{2|\mathcal{E}|r_0}{3m} = 1. \tag{15}$$

The energy condition $|\mathcal{E}| < \frac{1}{2}$ (or $e^2 > 0$) ensures the restriction $r_0 > 3m$, which is necessary for the existence of two distinct horizons. To obtain a condition that the stationary point $r = r_0$ is situated well ahead the black hole horizon one may compare both of the points $r = r_0$ and $r = r_{BH} = \frac{2r_0^{\frac{3}{2}}}{\sqrt{3m}} \cos\alpha$ under the condition $r_0 > 3m$. It is presumed that the point of turning r_0 must lie outside the horizon of the black hole as it is relevant only in the said outside region.

Under the circumstances one may find r_0 to be larger than r_{BH} subject to the following restriction:

$$\alpha \geq \arccos\left[\frac{1}{2}\left(\frac{3m}{r_0}\right)^{\frac{1}{2}}\right] \tag{16}$$

The factor $\frac{3m}{r_0}$ determines how the angle α is close to $\frac{\pi}{2}$. With the restriction given by the equation (16) the cosmological horizon $r = r_{CH} = \frac{r_0^{\frac{3}{2}}}{\sqrt{m}}(\sin\alpha - \frac{\cos\alpha}{sqrt3}) \approx \frac{r_0^{\frac{3}{2}}}{\sqrt{m}}$ must be larger than the stationary point. On the nutshell one may infer the following:

$$r_{BH} < r_0 < r_{CH} \tag{17}$$

When $r_0 = 3m$ all three points coincide and as a result not only black hole horizon coincides with cosmological horizon, but the particle circulates along it. It contradicts the timelike nature of the test particle. Later it will be shown that when two horizons coincide, the tidal effect near the black hole singularity disappears.

2.3. Tidal effect in the Schwarzschild de-Sitter spacetime

The generation of tidal effect is inevitable when the gravitational field becomes highly non-homogeneous. To calculate such tidal phenomenon one must consider a system of test particles falling freely towards black hole singularity. It has been observed from the previous Sub-section that unlike the ordinary Schwarzschild case here the freely falling system gets halted after some time and executes circular motion round SdS black hole in a quite unstable manner, although soon the stability is broken and it continues to maintain its movement in the earlier direction. In the freely falling frame under the current scenario the tidal equations take the form as follows:

$$\frac{d^2\xi^{\hat{r}}}{d\tau^2} = \frac{2m + H^2r^3}{r^3}\xi^{\hat{r}}, \tag{18}$$

$$\frac{d^2\xi^{\hat{l}}}{d\tau^2} = -\frac{m - H^2r^3}{r^3}\xi^{\hat{l}}, \tag{19}$$

where $\hat{l} = \hat{\theta}, \hat{\phi}$.

If the space inside the black hole is taken to be empty, or more precisely, inside the movement of the system is restricted only in the empty portion then the above pair of equations may also hold therein. The special interest is focused on the study of those equations in a small neighbourhood of the singularity. If somebody consider the change of the deviation vector relative to the arial coordinate r, instead of the proper time τ, the Eqs. (18) and (19) become

$$\left(2 + \frac{r^3}{r_0^3} - \frac{3m}{r_0}\right)\frac{d^2\xi^{\hat{r}}}{dr^2} - \frac{1}{r^2}\left(1 - \frac{r^3}{r_0^3} - \frac{3m}{2r_0}\right)\frac{d\xi^{\hat{r}}}{dr} - \frac{1}{r^3}\left(2 + \frac{r^3}{r_0^3}\right)\xi^{\hat{r}} = 0, \tag{20}$$

$$\left(2 + \frac{r^3}{r_0^3} - \frac{3m}{r_0}\right)\frac{d^2\xi^{\hat{l}}}{dr^2} - \frac{1}{r^2}\left(1 - \frac{r^3}{r_0^3} - \frac{3m}{2r_0}\right)\frac{d\xi^{\hat{l}}}{dr} - \frac{1}{r^3}\left(1 - \frac{r^3}{r_0^3}\right)\xi^{\hat{l}} = 0. \tag{21}$$

Since the tidal effect plays a significant role near the black hole singularity the current study should be focused therein. In the neighborhood of singularity at $r = 0$ one may simply omit the third order terms $\frac{r^3}{r_0^3}$ present in the expressions of the above equations as the stationary point $r = r_0$ lies sufficiently outside the black hole. Then the set of equations may be simplified to form a couple of Cauchy-Riemann equations that are easily solvable subject to the suitable initial conditions. Such initial conditions must be introduced carefully. It may be assumed that the system of particles is not supposed experience the tidal effect right from the beginning of its motion. The point at which the tidal effect in all three spatial directions begins may be denoted as $r = b$. At that initial point the terms $\frac{d\xi^{\hat{r}}}{dr}$ and $\frac{d\xi^{\hat{l}}}{dr}$ must vanish, whereas tidal deviation vectors $\xi^{\hat{r}}(b)$ and $\xi^{\hat{r}}(b)$ are non-vanishing. Here Eqs. (20) and (21) may be solved subject to those initial conditions in addition to the approximation $\frac{r^3}{r_0^3} \ll 1$.

Under such framework Eqs. (20) and (21) can be solved analytically to obtain the solutions given as follows:

$$\xi^{\hat{r}} = \left[\frac{5\kappa_1 + 3}{10\kappa_1}\left(\frac{b}{r}\right)^{\frac{5\kappa_1 - 3}{4}} + \frac{5\kappa_1 - 3}{10\kappa_1}\left(\frac{r}{b}\right)^{\frac{5\kappa_1 + 3}{4}}\right]\xi^{\hat{r}}(b), \tag{22}$$

where

$$\kappa_1 = \sqrt{\frac{1 - (\frac{3}{5})^2 \frac{3m}{r_0}}{1 - \frac{3m}{r_0}}} > 1$$

and

$$\begin{aligned} \xi^{\hat{l}} &= \left[\frac{3 + \kappa_2}{2\kappa_2}\left(\frac{r}{b}\right)^{\frac{3-\kappa_2}{2\kappa_2}} - \frac{3 - \kappa_2}{2\kappa_2}\left(\frac{r}{b}\right)^{\frac{3+\kappa_2}{2\kappa_2}}\right]\xi^{\hat{l}}(b) && for\ \frac{3m}{r_0} < \frac{1}{9} \\ &= \sec\left[\arctan\left(\frac{3}{\kappa_2}\right) - \frac{\kappa_2}{4}\ln b\right]\cos\left[\frac{\kappa_2}{4}\ln\left(\frac{r}{b}\right) + \arctan\left(\frac{3}{\kappa_2}\right)\right]\xi^{\hat{l}}(b), && for\ \frac{3m}{r_0} > \frac{1}{9} \end{aligned} \tag{23}$$

where

$$\begin{aligned} \kappa_2 &= \sqrt{\frac{1 - 3^2\frac{3m}{r_0}}{1 - \frac{3m}{r_0}}} && for\ \frac{3m}{r_0} < \frac{1}{9} \\ &= \sqrt{\frac{3^2\frac{3m}{r_0} - 1}{1 - \frac{3m}{r_0}}} && for\ \frac{3m}{r_0} > \frac{1}{9}. \end{aligned} \tag{24}$$

The above solutions indicate when r is sufficiently close to zero the $\xi^{\hat{r}}$ diverges, but $\xi^{\hat{l}}$ diminishes to zero. The second part $\xi^{\hat{l}}$, for $\frac{3m}{r_0} < \frac{1}{9}$, contains a term $\cos[\frac{\kappa_2}{4}\ln(\frac{r}{b}) + \arctan(\frac{3}{\kappa_2})]$, which remains finite except at $r = 0$. Therefore, such term does not disturb $\xi^{\hat{l}}$ to be diminishing near the singularity as long as the system does not reach at $r = 0$. That is the famous spaghettification effect in which length of an object becomes infinitely large and the width gets crunched, jut like a spaghetti. In other words an observer will be torn apart by the tidal effect near the singularity.

If r_0 is very close to $3m$, implying two horizons and the stationary point tend to coincide, then it is observed from both of Eqs. (20) and (21) that

$$2 + \frac{r^3}{r_0^3} - \frac{3m}{r_0} \approx 0. \tag{25}$$

That makes $\xi^{\hat{r}} = \xi^{\hat{l}} = 0$ signifying the vanishing tidal effect at the vicinity of singularity.

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3. Exploring McVittie spacetime

3.1. McVittie as a generalization of SdS spacetime

McVittie metric is taken into consideration for the expanding spacetime embedding a central massive body, likely to be a black hole. In such dynamical scenario the isotropic form of the McVittie metric is given by

$$ds^2 = -\frac{(1-\frac{\overline{m}(t)}{2\overline{r}})^2}{(1+\frac{\overline{m}(t)}{2\overline{r}})^2}dt^2 + a^2(t)\left(1+\frac{\overline{m}(t)}{2\overline{r}}\right)^4(d\overline{r}^2+\overline{r}^2 d\Omega^2), \tag{26}$$

where $\overline{m}(t)$ represents the variable mass of the centralized massive body depending on time in cosmic scale. The no accretion condition of such massive object results $G^1_0 = 0 \Rightarrow \frac{\dot{\overline{m}}}{\overline{m}} + \frac{\dot{a}}{a} = 0$.

Integrating one may find $\overline{m}(t) = \frac{m}{a(t)}$, where m, the integration constant, may be interpreted as the central mass. Using a transformation $R = a(t)\overline{r}(1+\frac{\overline{m}(t)}{2\overline{r}})^2$ the metric (25) takes the form as given below:

$$ds^2 = -(1-\frac{2m}{R} - H^2R^2)dt^2 + \frac{1}{1-\frac{2m}{R}}dR^2 - \frac{2HR}{\sqrt{1-\frac{2m}{R}}}dtdR + R^2 d\Omega^2. \tag{27}$$

That metric is very useful in many calculations involving McVittie spacetime. For the present purpose, to eliminate the cross term from the metric, one may introduce a new time coordinate defined as $dT = \frac{1}{F}(dt + \beta dR)$, where F is a integrating factor. With a choice of $\beta = \frac{HR}{\sqrt{1-\frac{2m}{R}}(1-\frac{2m}{R}-H^2R^2)}$ the cross term is eliminated and the metric becomes $ds^2 = -(1-\frac{2m}{R} - H^2R^2)F^2dT^2 + (1-\frac{2m}{R} - H^2R^2)^{-1}dR^2 + R^2 d\Omega^2$.

In the present article McVittie metric is simply considered as a minimal generalization of SdS metric in which H depends on time. Therefore, out of a class of McVittie metrics characterized by F only one with $F = 1$ is taken into account for the present purpose. Now, replacing R by r and T by t the metric in (26) can be expressed as

$$ds^2 = -(1-\frac{2m}{r} - H^2r^2)dt^2 + (1-\frac{2m}{r} - H^2r^2)^{-1}dr^2, \tag{28}$$

where it is assumed $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$ and $\dot{\phi} = 0$. The above metric exactly looks like the SdS metric, but to be remembered that the Hubble parameter H, in the dynamical spacetime, must depend on the time, obviously in the cosmic scale. As in the static case the horizon is obtained by equating $1 - \frac{2m}{r} - H^2r^2 = 0$. It gives

$$\begin{aligned} r &= \frac{2r_0^{\frac{3}{2}}}{\sqrt{3m}}\cos\psi = r_{BH} \\ &= \frac{r_0^{\frac{3}{2}}}{\sqrt{m}}\left[\sin\psi - \frac{\cos\psi}{\sqrt{3}}\right] = r_{CH}, \end{aligned} \tag{29}$$

with the restriction $\psi \geq \arccos[-\frac{1}{2}(\frac{3m}{r_0})^{\frac{3}{2}}]$. Here essentially the turning or stationary point $r_0 = (\frac{m}{H^2})^{\frac{3}{2}}$, depends on time. Under the backdrop of dynamical spacetime the following three cases may be considered.

Case I:
In the cosmic time scale when $t > t_*$, as in the present universe, the condition $r_0 > 3m$ ensures the distinct horizons r_{BH} and r_{CH} remain far apart, and get separated by the stationary point r_0.

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Case II:
When $t = t_*$, the condition $r_0 = 3m$ implies r_{BH} and r_{CH} coincide and there exist a single horizon on which r_0 is located. That has already been established in the previous section that there is no tidal effect on any system of timelike test particles at that point.

Case III:
Before the instant t_*, i.e., when $t < t_*$ the condition $r_0 < 3m$ holds and it makes all three roots of the equation $1 - \frac{2m}{r} - H^2r^2 = 0$ unphysical. In that case no apparent horizon is formed and there exists a naked singularity.

However, the present article restricts the discussion only in the $t > t_*$ regime.

3.2. Energy density and stationary point

In order to generalize the SdS spacetime from static to dynamic the McVittie metric, given in Eq. (27), depends on time coordinate. Therefore, no timelike Killing vector exists here. Taking the Lagrangian $\mathcal{L}_M = \frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ one may calculate and find the following terms:

$$\frac{\partial \mathcal{L}_M}{\partial t_\tau} = -(1 - \frac{2m}{r} - H^2r^2)t_\tau, \tag{30}$$

$$\frac{\partial \mathcal{L}_M}{\partial t} = Hr^2\dot{H}t_\tau^2 + \frac{Hr^2\dot{H}}{(1 - \frac{2m}{r} - H^2r^2)^2}r_\tau^2. \tag{31}$$

In the above equations t_τ and r_τ represent the derivatives of time and radial coordinates respectively with respect to the proper time τ, whereas $\dot{H}$ stands as the derivative of Hubble parameter relative to coordinate time t. In analogy with the static spacetime one may simply take $\frac{\partial \mathcal{L}_M}{\partial t_\tau} = e(t)$ which determines t_τ as

$$t_\tau = -\frac{e}{1 - \frac{2m}{r} - H^2r^2}. \tag{32}$$

Here e is considered to be the function of t only, not the function of r. With the aid of timelike nature of the geodesic one may now establish

$$e^2 - r_\tau^2 = 1 - \frac{2m}{r} - H^2r^2. \tag{33}$$

The Euler Lagrangian equation, along with the relations (29) to (32) may find a relation between energy density e and the stationary or turning point r_0 as follows:

$$\dot{e} = \frac{3m}{2}\frac{r^2}{r_0^2}\frac{e^2 + r_\tau^2}{e(e^2 - r_\tau^2)}\dot{r}_0. \tag{34}$$

To establish the above equation the relation between H and r_0 has been recalled and the derivative of H has been expressed in that of r_0, i.e., $H\dot{H} = -\frac{3m}{r_0^4}\dot{r}_0$. It is worth noting that the radial coordinate r does not change with time, but r_0 does. Now, the following two cases are significant in the current scenario.

Case I:
On the black hole horizon as well as cosmological horizon one may show in the equation (33) if $e = r_\tau$ then $\dot{e} \to \infty$. In other words the rate of energy density becomes infinite on both the horizons.

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Case II:
One may be interested how the energy density changes with the movement of the turning point r_0. The $\dot{e}$ may be evaluated at $r = r_0$ (where r_τ vanishes) and the following relation can be obtained:

$$r_0 = \frac{3m}{r_{0d}(e_d^2 - e^2)} r_{0d}, \tag{35}$$

where r_{0d} represents the turning point in static SdS spacetime and e_d is the corresponding energy density. Here Eq. (34) shows a relation between energy density and the turning point in the dynamical spacetime.

3.3. Equation of state, dark energy and singularity

The relation between pressure p and the density ρ, termed as equation of state, plays a significant role in different cosmological regimes. Such relation in the framework of McVittie spacetime has already been carried out analytically under the backdrop of isotropic metric. Using generalized SdS form of McVittie metric, given by Eq. (27) the equation of state may be expressed as

$$p = \rho \left[-1 - \frac{2\dot{H}}{3H^2} \frac{1}{\sqrt{1 - \frac{2m}{r}}} \right], \tag{36}$$

in analogy with $p = w\rho$, where w is equation of state parameter or barotropic index. The Hubble parameter denotes the expansion of the present universe. The relation between H and r_0 makes the relation (35) into the same in terms of r_0 and $\dot{r}_0$ as given below:

$$p = \rho \left[-1 + \dot{r}_0 \sqrt{\frac{r_0}{m(1 - \frac{2m}{r})}} \right]. \tag{37}$$

An important point to be noted that since both the horizons are proportional to $r_0^{\frac{3}{2}}$ both of them must move along with the point r_0, which indicates the dynamic nature of the spacetime. The equation (36) is very important in the sense that the dynamic term $\dot{r}_0$ may signify the equation of state parameter in different regimes. Under the scenario the following cases signifying different era of the universe are discussed hereunder.

Case I:
In the matter dominated and radiation era the equation of state becomes $p = 0$ and $p = \frac{\rho}{3}$ respectively. Imposing that condition in the equation (36) it is found

$$r_0^{\frac{3}{2}} = r_{0I}^{\frac{3}{2}} - \kappa \sqrt{m\left(1 - \frac{2m}{r}\right)}\, t \tag{38}$$

with

$$\begin{aligned} \kappa &= \frac{3}{2} \qquad (\textit{matter dominated era}) \\ &= 2 \qquad (\textit{radiation era}), \end{aligned}$$

where r_{0I} represents the value of r_0 at the beginning of the regime (matter dominated or radiation). Here an interesting phenomenon may be worth noting. At the radial point $r = 2m$ the factor within the square root in Eq. (37) vanishes. It signifies that r_0 stays at the fixed value r_{0I} from the beginning of that era if observed from the point $r = 2m$. Therefore, it is inferred that no expansion of spacetime is observed in both matter dominated and radiation era if the observer stays at $r = 2m$, although it is not

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guaranteed that the said point lies outside the black hole.

Case II:
The study of dark energy era in McVittie universe has also brought the interests to researchers [39]. If in the equation (36) it is found that $\dot{r}_0 > 0$, then w must be greater than -1 and it indicates the regime is dark energy dominated as in the present universe. Therefore, it may be said that the dark energy dominated era arises when the stationary point moves away of the black hole, or in other words, when the r_0 is shifted towards the cosmological horizon. It is to be noted that both the horizons move simultaneously in the same direction as that of r_0. If the motion of r_0 comes to rest, i.e., $\dot{r}_0$ for an instant or for an interval of time, it is found from Eq. (37) $p = -\rho$, which points towards an ideal dark energy era with dust free universe.

Case III:
The scenario becomes more interesting when $\dot{r}_0 < 0$, or in other words the turning point is shifted to move towards the black hole horizon. It is then clearly implied from equation (36) that $w < -1$, which is nothing but a phantom energy regime. Exploring the McVittie metric in the phantom energy background [40] has already been carried out. To examine the possible singularity predicted in the phantom age it is to be verified whether $r = 2m$ is located inside or outside the black hole horizon r_{BH}. In order to fix its location the following restrictions can be obtained from the expression of r_{BH}, given by Eq. (28).

$$\begin{aligned} &\tfrac{1}{3}\left(\tfrac{3m}{r_0}\right)^{\frac{3}{2}} < \cos\psi < \tfrac{1}{2}\left(\tfrac{3m}{r_0}\right)^{\frac{1}{2}} && r_{BH} > 2m \\ &\cos\psi < \tfrac{1}{3}\left(\tfrac{3m}{r_0}\right)^{\frac{3}{2}} && r_{BH} < 2m \end{aligned} \tag{39}$$

The condition $r_{BH} > 2m$ implies the point $r = 2m$ is covered by the black hole horizon. But for $r_{BH} < 2m$ the turning point r_0 moves towards r_{BH} and it is quite inevitable that it coincides with $2m$ and at that instant, say $t = t_c$ the $|p| \to \infty$, while both ρ and a remain finite. In the phantom energy regime that situation is termed as 'sudden rip'. Such kind of singularity may be avoided only when $r = 2m$ lies inside the black hole horizon, that is when the angle ψ is more than $\arccos[\frac{1}{3}(\frac{3m}{r_0})^{\frac{3}{2}}]$.

4. Discussion

In this article the McVittie metric is analyzed in the stationary as well as dynamical spacetime. In the stationary spacetime such metric is identified as SdS one, which exhibits a remarkably difference to that compared to the Schwarzschild metric. In the Schwarzschild spacetime a system of test particles or an observer falling freely towards a black hole must follows an undisturbed motion along the radial direction until and unless gets spaghettified by the tidal force near singularity. In contrast to that picture a freely falling body towards the center of a black hole in the SdS spacetime must get slow down and starts orbiting the said black hole in an unstable circular orbit, then again resumes its motion towards singularity. It has been established that such turning point r_0, where the radial velocity vanishes, is also the stationary point where the potential having negative value becomes maximum (and therefore unstable). If $r_0 > 3m$, not only the black hole horizon and cosmological horizon remains separated, but such turning or stationary point lies in between the two horizons.

The whole analysis of SdS as well as dynamical McVittie spacetime is carried out in terms of that point $r_0(t)$, instead of the Hubble parameter $H(t)$. It is worth noting that in the present age r_0, as the Hubble parameter H, may be considered to be constant. In the ordinary Schwarzschild black hole it is well established that a freely falling observer towards the black hole fails to reach the spacelike singularity, because near that location it is torn apart by the tidal effect and thus the fate of the said observer is uniquely determined. In the field theoretic language it has been shown that the Schwarzschild spacetime is inextendible as C^0 metric in the larger spacetime and thus the 'strong cosmic censorship conjecture' is restored. In the Sub-section 2.3 the tidal effect near black hole in the SdS spacetime is calculated to show that the observer would be spaghettified by the tidal force near the singularity; which is the same inference that can be drawn in case of Schwarzschild black hole. One minor generalization is the expression arising in the angular component of deviation vector for a special case. Physically such term is not quite relevant as it cannot alter the spaghettification effect.

In analyzing the dynamical scenario the Hubble parameter characterizing the expanding universe is replaced by the stationary or turning point r_0 whose motion is studied in different perspectives. In such analytical study an important point be noted in the matter and radiation dominated era is that the universe seems to be stationary if observed from the location $r = 2m$, if lies outside the black hole horizon i.e., if $2m > r_{BH}$. It is a remarkable feature in the sense that the homogeneity may be broken, at least at the said point in these regimes. Such violation of homogeneity is restored if the point $r = 2m$ lies inside the black hole horizon at those stages. The most important aspect of the current analysis arises in the dark and phantom energy regime. It is shown that r_0 moves away of the black hole horizon in the dark energy era, while moves back towards it in the phantom era. It may apparently give a notion that the universe expands in the dark energy era, but in the phantom era it contracts. That may not be true. In the dark energy age the universe not only expands, but it also accelerates. Such phenomenon is characterized by the relation $\dot{r}_{CH} > \dot{r}_0 > \dot{r}_{BH} > 0$. If the universe enters into the phantom energy regime, which is predicted by some observations [41,42], r_0 reverses its direction, while r_{BH} continues to move in the same direction. As a result the expansion of the universe still continues, but in some sense, non-uniformly. It is also shown that if $r < 2m$, the universe encounters 'sudden rip' singularity [43,44].

Lastly, the present model is able to fine tune the vacuum energy density as small as possible. The key relation behind such fine tuning is $r_0 > 3m$, which is inevitable for the black hole and cosmological horizons to be kept far apart. In the present universe, which is dominated by the dark energy, one may consider $H^2 \approx \frac{8\pi\rho_\Lambda}{3}$, where ρ_Λ represents the vacuum energy density. It imposes a restriction on that vacuum energy density in terms of black hole mass as $\rho_\Lambda m^2 < \frac{1}{72\pi}$. If the mass of black hole becomes very high then the value of ρ_Λ may be considerably low to obey the present experimental value.

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References

1. Friedmann, A.A. On the curvature of space. *Z. Phys.* **1922**, *10*, 377–386. [CrossRef]
2. Lemaître, A.G.; Eddington, A.S. The Expanding Universe. *Mon. Not. R. Astron. Soc.* **1931**, *91*, 490–501. [CrossRef]
3. Faraoni, V. An analysis of the Sultana-Dyer cosmological black hole solution of the Einstein equations. *Phys. Rev. D* **2009**, *80*, 044013. [CrossRef]
4. Jacques A.; Faraoni, V. Cosmological expansion and local physics. *Phys. Rev. D* **2007**, *76*, 063510. [CrossRef]
5. Arakida, H. Time delay in Robertson-McVittie spacetime. *New Astron.* **2008**, *14*, 264–268. [CrossRef]
6. Abbott B.P. et al. (LIGO Scientific and Virgo Collaborations), Observation of gravitational waves from a binary black hole merger, *Phys. Rev. Lett.* **2016**, *116*, 061102. [CrossRef]
7. P'urrer M.; Haster, C.-J. Gravitational waveform accuracy requirements for future ground-based detectors, *Phys. Rev. Res.* **2020**, *2*, 023151. [CrossRef]
8. Damour, T. Classical and quantum scattering in post-Minkowskian gravity. *Phys. Rev. D* **2020**, *102*, 024060. [CrossRef]
9. Bini, D. et al. Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders. *Phys. Rev. D* **2020**, *102*, 024062. [CrossRef]
10. Bini, D. et al. Sixth post-Newtonian local-in-time dynamics of binary systems. *Phys. Rev. D* **2020**, *102*, 024061. [CrossRef]
11. Bern, Z. et al. Scattering amplitudes and the conservative Hamiltonian for binary systems at third post-Minkowskian order. *Phys. Rev. Lett.* **2019**, *122*, 201603. [CrossRef]

12. Bini, D. et al. Novel approach to binary dynamics: Application to the fifth post-Newtonian level. *Phys. Rev. Lett.* **2019**, *123*, 231104. [CrossRef]
13. Antonelli, A. et al. Gravitational spin-orbit coupling through third-subleading post-Newtonian order: From first-order self-force to arbitrary mass ratios. *Phys. Rev. Lett.* **2020**, *125*, 011103. [CrossRef]
14. McVittie, G.C. The Mass-Particle in an Expanding Universe. *Mon. Not. R. Astron. Soc.* **1933**, *93*, 325–339. [CrossRef]
15. McVittie, G.C. *General Relativity and Cosmology*; University of Illenois Press, 1965.
16. Schwarzschild, K. On the Gravitational Field of a Mass Point according to Einstein's Theory. *Sitzungsber. Preuss. Akad. Wess. Berlin (Math. Phys.)* **1916**, *1916*, 189–196; (translated by S. Antoci and A. Loinger, 1999). [CrossRef]
17. Sussman, R. Conformal structure of a Schwarzschild black hole immersed in a Friedman universe. *Gen. Relativ. Gravit.* **1985**, *17*, 251–291. [CrossRef]
18. Nolan, B.C. A point mass in an isotropic universe: Existence, uniqueness, and basic properties. *Phys. Rev. D* **1998**, *58*, 064006. [CrossRef]
19. Nolan, B.C. A point mass in an isotropic universe: II. Global properties. *Class. Quantum Grav.* **1999**, *16*, 1227–. [CrossRef]
20. Nolan, B.C. A point mass in an isotropic universe: III. The region $R = 2m$. *Class. Quantum Grav.* **1999**, *16*, 3183–3191. [CrossRef]
21. Kaloper, N.; Kleban M.; Martin, D. McVittie's legacy: Black holes in an expanding universe. *Phys. Rev. D* **2010**, *81*, 104044. [CrossRef]
22. Lake K.; Abdelqader, M. More on McVittie's Legacy: A Schwarzschild - de Sitter black and white hole embedded in an asymptotically LCDM cosmology. *Phys. Rev. D* **2011**, *84*, 044045. [CrossRef]
23. Faraoni, V.; Moreno, A.F.Z.; Nandra, R. Making sense of the bizarre behavior of horizons in the McVittie spacetime. *Phys. Rev. D* **2012**, *85*, 083526. [CrossRef]
24. Guariento, D.C.; da Silva, A.M.; Fontanini, M. Sourcing a dynamically accreting black hole in an expanding background with imperfect fluids. *Phys. Rev. D* **2013**, *87*, 064030.
25. Faraoni, V.; Jacques, A. Cosmological expansion and local physics. *Phys. Rev. D* **2007**, *76*, 063510. [CrossRef]
26. Gao, C.; Chen, X.; Faraoni, V.; Shen, Y.-G. Does the mass of a black hole decrease due to the accretion of phantom energy. *Phys. Rev. D* **2008**, *78*, 024008. [CrossRef]
27. Faraoni, V.; Gao, C.; Chen, X.; Shen, Y.-G. What is the fate of a black hole embedded in an expanding universe? *Phys. Lett. B* **2009**, *671*, 7–9. [CrossRef]
28. Nolan, B.C. Particle and photon orbits in McVittie spacetimes. *Class. Quantum Grav.* **2014**, *31*, 235008–[CrossRef]
29. D. Kramer, H. Stephani, M.A.H. MacCallum and H. Herlt, *Exact Solutions of the Einstein's Field Equations*; Cambridge University Press, Cambridge, 1980.
30. Gibbons, G.W.; Hawking, S.W. Action integrals and partition functions in quantum gravity. *Phys. Rev. D* **1977**, **15**, 2738. [CrossRef]
31. Lake, K.; Roeder, R.C. Effects of a nonvanishing cosmological constant on the spherically symmetric vacuum manifold. *Phys. Rev. D* **1977**, *15*, 3513–3519. [CrossRef]
32. Laue, H.; Weiss, M. Maximally extended Reissner-Nordstr´om manifold with cosmological constant. *Phys. Rev. D* **1977**, *16*, 3376–3379. [CrossRef]
33. Geyer, K.H. Space-time geometry of the metric tensor of von Kottler, Weyl and Trefftz. *Astron. Nachr.* **1980**, *301*, 135–149. [CrossRef]
34. Bazanski, S.L.; Ferrari, V. Analytic extension of the Schwarzschild-de Sitter metric. *Nuo. Cim. B* **1986**, *91*, 126–142. [CrossRef]
35. Nakao, K.; Maeda, K.; Nakamura, T.; Oohara, K. Numerical study of cosmic no-hair conjecture: Formalism and linear analysis. *Phys. Rev. D* **1991**, *44*, 1788–1797. [CrossRef]
36. Curry, C.; Lake, K. Vacuum solutions of Einstein's equations in double-null coordinates. *Class. Quan. Grav.* **1991**, *8*, 237–243. [CrossRef]
37. Podolsky, J. The Structure of the Extreme Schwarzschild-de Sitter Space-time. *Gen. Relativ. Gravit.* **1999**, *31*, 1703–1725. [CrossRef]

38. Nariai H., On some static solutions of Einstein's gravitational field equations in a spherically symmetric case. *Scientific Reports of the Tohoku University* **34**, *1950* 160; (Reprinted in *Gen. Relativ. Gravit.* **1999**, *31*, 951–961). [CrossRef]
39. Hadi, H.; Heydarzade, Y.; Darabi, F.; Atazadeh, K. D-bound and the Bekenstein bound for the surrounded Vaidya black hole. *Eur. Phys. J. Plus* **2020**, *135*, 584. [CrossRef]
40. Antoniou, I.; Perivolaropoulos, L. Geodesics of McVittie Spacetime with a Phantom Cosmological Background. *Phys. Rev. D* **2016**, *93*, 123520. [CrossRef]
41. Caldwell, R.R. A phantom menace? Cosmological consequences of a dark energy component with super-negative equation of state. *Phys. Lett. B* **2002**, *545*, 23–29. [CrossRef]
42. Caldwell, R.R.; Kamionkowski, M.; Weinberg, N.N. Phantom Energy: Dark Energy with $\omega < -1$ Causes a Cosmic Doomsday. *Phys. Rev. Lett.* **2003**, *91*, 071301. [CrossRef]
43. Barrow, J.D.; Galloway, G.J.; Tipler, F.J. The closed-universe recollapse conjecture. *Mon. Not. R. Astron. Soc.* **1986**, *223*, 835–844. [CrossRef]
44. Nojiri, S.; Odintsov, S.D. Quantum escape of sudden future singularity. *Phys. Lett. B* **2020**, *595*, 1–8. [CrossRef]