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Analytical solutions to unsteady unidirectional flows of a generalized Oldroyd-B fluid

Fang Wang [1,*], Yu Wang [2]

1 School of Civil Engineering, North Minzu University, Yinchuan, 750021, China; fangw1211@163.com
2 School of Aeronautics, Northwestern Polytechnical University, Xi'an, 710072; fangw1211@163.com
* Correspondence: fangw1211@163.com

Abstract: For solving the fractional differential equations in computational fluid dynamics (CFD), it's complicated and difficult by the Laplace and Fourier transforms. Based on the Caputo fractional derivative, the analytical solutions for unsteady unidirectional flows of a generalized Oldroyd-B fluid are deduced by the separation of variables method. Results show that the analytical solutions are given easily, and have good university. For some specific parameter values, the well-known analytical solutions for the generalized second grade fluid, the generalized Upper-Convected Maxwell (UCM) fluid as well as the ordinary Oldroyd-B fluid can be obtained.

Keywords: Fractional calculus; Generalized Oldroyd-B fluid; separation of variables method; Unidirectional flow; Caputo fractional derivative

MSC:

1. Introduction

In the past several years, the fractional calculus has already been found quite flexible and efficient in the description of the constitutive relations for the viscoelastic fluids. Friedrich[1] showed that the fractional Maxwell model is consistent with the law of thermodynamic. Fetecau C et al[2] investigated the decay of a potential vortex in a fractional Oldroyd-B fluid. Nonnenmacher[3] and Gockle[4] established the stress relaxation modulus of fractional Maxwell and fractional Zener model by employing Fox functions. Jaishankar and McKinley[5] revisited the concept of quasi-property and its connection to the fractional Maxwell model and successfully simulated Scott-Blair's experimental data. Wang et al[6] studied the unsteady Poiseuille flow of fractional Oldroyd-B viscoelastic fluid between two parallel plates by the numerical inversion of Laplace transforms, and further validated the wider scope of application for the fractional constitutive equations.

Among the traditional constitutive models for the viscoelastic fluids, the Oldroyd-B model presents a typical constitutive law which does not obey the Newtonian law, and such non-Newtonian flow could describe a class of some viscoelastic fluids, such as the system coupling fluids and polymers[7]. Furthermore, the Oldroyd-B model contains as special cases some of the previous models such as the Maxwell model. Consequently，a generalized Oldroyd-B model is established[8] and many papers about the mathematical computation and physical analysis for the model have been subsequently published. Tong[9] obtained the exact solutions of some unsteady helical flows of Oldroyd-B fluid in an annular pipe by using Hankel transform and Laplace transform for fractional calculus. Qi[10] obtained the analytical solutions of Poiseuille flow and Couette flow of generalized Oldroyd-B fluid with Riemann-Liouville fractional derivative by using Fourier sine transform and discrete Laplace transform. Zheng et al[11] studied the magnetohydrodynamic flow of an incompressible generalized Oldroyd-B fluid due to an infi-

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nite accelerating plate, and obtained the exact solutions by means of Fourier sine and Laplace transforms. Ming et al. [12] derives analytical solutions for a class of new multi-term fractional-order partial differential equations, and considered different situations for the unsteady flows of generalized Oldroyd-B fluid and Burgers fluid. Chen et al. [13] presented two types of multi-term fractional differential equations in high dimensions, which are used to describe the nonlinear relationship between the shear stress and the shear rate of generalized Oldroyd-B fluid. Song et al. [14] investigated the mixed initial value problem for the incompressible fractionalized Oldroyd-B fluid by utilizing the integral transforms.

In the previous studies, the exact solutions for the flows of viscoelastic fluids with fractional constitutive model were obtained generally by the Fourier and Laplace transforms, but the method makes the solving process more complicated. In addition, the most fractional partial differential equations were established by the Riemann- Liouville fractional derivative[15,16]. The defect of the Riemann-Liouville fractional derivative is that its initial conditions are in the form of fractional derivative, which is too difficult to demonstrate their physical significance to fulfill the applied value in engineering and physics. While another definition, namely the Caputo fractional derivative, demands ordinary integer order derivatives on initial conditions, and the initial conditions based on the definition are corresponding with that under the classical differential equation theory[17]. Chen[18] studied analytical solution for the time-fractional telegraph equation by the separation of variables method (SVM). Zhang[19] obtained the analytical solution for a two-dimensional multi-term time- fractional Oldroyd-B equation on a rectangular domain by the SVM, based on the Caputo time-fractional derivative. Consequently, the purpose of this paper is to consider the fractional constitutive equation with the Caputo fractional derivative, and present the analytical solutions corresponding to the two types of unsteady unidirectional flows of a generalized Oldroyd-B fluid between two parallel plates. Through some specific parameter values, the analytical solutions for the generalized Maxwell fluid, the generalized second grade fluid as well as the ordinary Oldroyd-B fluid could be obtained.

2. Governing equations

For the unsteady incompressible flow, the governing equations are as follows:

$$div\boldsymbol{V} = 0 \tag{1}$$

$$\rho \frac{d\boldsymbol{V}}{dt} = div\boldsymbol{T} \tag{2}$$

in which $\boldsymbol{V}$ is the velocity field, ρ is the uniform density of the fluid, and $\boldsymbol{T}$ is the Cauchy stress tensor.

The Cauchy stress tensor $\boldsymbol{T}$ for a fractional Oldroyd-B fluid can be described as

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{\sigma}$$

The fractional Oldroyd-B constitutive equation [8,20] is written as

$$(1+\lambda_1^\alpha \frac{\delta^\alpha}{\delta t^\alpha})\boldsymbol{\sigma} = \mu(1+\lambda_2^\beta \frac{\delta^\beta}{\delta t^\beta})\boldsymbol{A}_1 \tag{3}$$

where $p\boldsymbol{I}$ denotes the indeterminate spherical stress, $\boldsymbol{\sigma}$ is the extra stress tensor, μ is the viscosity, λ_1 and λ_2 are respectively the relaxation and retardation times, and $0 \le \lambda_1 \le \lambda_2$. α and β are fractional calculus parameters such that $0 \le \alpha \le \beta \le 1$. $\boldsymbol{A}_1$ is the first Rivlin– Ericksen tensor given by $\boldsymbol{A}_1 = \boldsymbol{L} + \boldsymbol{L}^T$ with $\boldsymbol{L} = grad\boldsymbol{V}$. And

$$\begin{aligned} \frac{\delta^\alpha \boldsymbol{\sigma}}{\delta t^\alpha} &= \frac{\partial^\alpha \boldsymbol{\sigma}}{\partial t^\alpha} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{\sigma} - \boldsymbol{L}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{L}^T \\ \frac{\delta^\beta \boldsymbol{A}_1}{\delta t^\beta} &= \frac{\partial^\beta \boldsymbol{A}_1}{\partial t^\beta} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{A}_1 - \boldsymbol{A}_1\boldsymbol{L}^T \end{aligned} \tag{4}$$

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$\frac{\partial^\alpha}{\partial t^\alpha}$ and $\frac{\partial^\beta}{\partial t^\beta}$ are fractional differential operators of α and β order with respect to t, respectively , and based on Caputo's definition is defined as [16]:

$$\frac{\partial^\alpha f(x)}{\partial t^\alpha}=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau,\quad n-1<\alpha<n \tag{5}$$

in which $\Gamma(\cdot)$ is Gamma function.

For the unidirectional flows, we consider the velocity and the stress taking the form of

$$\boldsymbol{V}=u(y,t)\boldsymbol{i},\ \boldsymbol{\sigma}=\boldsymbol{\sigma}(y,t) \tag{6}$$

Where $\boldsymbol{i}$ is the unit vector along the x-direction of the Cartesian coordinate system, u is velocity component along the x-direction.

Thus using Eq. (6), the continuity Eq. (1) is satisfied identically and Eq. (3) and (5), having in mind the initial condition $\boldsymbol{\sigma}(y,0)=0$, yields $\sigma_{xz}=\sigma_{yy}=\sigma_{yz}=\sigma_{zz}=0$ and

$$(1+\lambda_1^\alpha\frac{\partial^\alpha}{\partial t^\alpha})\sigma_{xy}=\mu(1+\lambda_2^\beta\frac{\partial^\beta}{\partial t^\beta})(\frac{\partial u}{\partial y}) \tag{7}$$

$$(1+\lambda_1^\alpha\frac{\partial^\alpha}{\partial t^\alpha})\sigma_{xx}-2\lambda_1^\alpha\sigma_{xy}\frac{\partial u}{\partial y}=-2\mu\lambda_2^\beta(\frac{\partial u}{\partial y})^2 \tag{8}$$

where σ_{xy} is shearing stress.

In the absence of body forces, the motion equation (2) for the unidirectional flow of the generalized Oldroyd-B fluid is written as

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y} \tag{9}$$

3. Basic concepts and theorem

Here, we introduce the following definitions and theorem, which are used further in this paper.

Definition 3.1[21]. A real or complex-valued function $f(x), x>0$, is said to be in the space $C_\alpha, \alpha\in R$, if there exists a real number $p>\alpha$ such that $f(x)=x^p f_1(x)$ for a function $f_1(x)$ in $C[0,\infty]$.

Definition 3.2[22]. A function $f(x), x>0$, is said to be in the space C_α^m, $m\in N_0=N\bigcup\{0\}$ if and only $f^m\in C_\alpha$.

Definition 3.3[22]. A multivariate Mittag-Leffler function is defined as

$$E_{(a_1,\ldots,a_n),b}(z_1,\cdots,z_n)=\sum_{k=0}^{\infty}\sum_{\substack{l_1+\cdots+l_n=k\\ l_1\geq0,\cdots,l_n\geq0}}\frac{k!}{l_1!\times\cdots\times l_n!}\frac{\Pi_{i=1}^n z_i^{l_i}}{\Gamma(b+\sum_{i=1}^n a_i l_i)} \tag{10}$$

in which $b>0, a_i>0, |z_i|<\infty, i=1,\cdots,n$.

In particularly, if $n=1$, the multivariate Mittag-Leffler function is reduced to the Mittag-Leffler function

$$E_{a_1,b}(z_1)=\sum_{k=0}^{\infty}\frac{z_1^k}{\Gamma(b+ka_1)},\quad a_1,b>0, |z_1|<\infty.$$

Theorem 3.4[22]. Let $\mu>\mu_1>\cdots>\mu_n\geq0, m_i-1<\mu_i\leq m_i, m_i\in N_0=N\bigcup\{0\}$, $\lambda_i\in R, i=1,\cdots,n$, the initial value problem

$$\begin{cases} (D^{\mu}y)(x) - \sum_{i=1}^{n} \lambda_i (D^{\mu_i} y)(x) = g(x), \\ y^k(0) = c_k \in R, \quad k = 0, \cdots, m-1, m-1 < \mu \le m, \end{cases} \tag{11}$$

where the function $g(x)$ is assumed to lie in C_{-1} if $\mu \in N$, in C_{-1}^1 if $\mu \notin N$, and the unknown function $y(x)$ is to be determined in the space C_{-1}^m, has the solution

$$y(x) = y_g(x) + \sum_{k=0}^{m-1} c_k u_k(x), \quad x \ge 0, \tag{12}$$

where

$$y_g(x) = \int_0^x t^{\mu-1} E_{(\cdot),\mu}(t) g(x-t) dt,$$

and

$$u_k(x) = \frac{x^k}{k!} + \sum_{i=l_k+1}^{n} \lambda_i x^{k+\mu-\mu_i} E_{(\cdot),k+1+\mu-\mu_i}(x), \quad k = 0, \cdots, m-1,$$

fulfills the initial conditions $u_k^{(l)}(0) = \delta_{kl}, k, l = 0, \cdots, m-1$. And the function

$$E_{(\cdot),\beta}(x) = E_{\mu-\mu_1,\cdots,\mu-\mu_n,\beta}(\lambda_1 x^{\mu-\mu_1}, \cdots, \lambda_n x^{\mu-\mu_n}).$$

The natural numbers $l_k, k = 0, \cdots, m-1$, are determined from the condition

$$\begin{cases} m_{l_k} \ge k+1, \\ m_{l_k+1} \le k. \end{cases}$$

In the case $m_i \le k, i = 1, \cdots, m-1$, we set $l_k = 0$, and if $m_i \ge k+1, i = 1, \cdots, m-1$, then $l_k = n$.

4. Unsteady Poiseuille flow

Flow of a fluid between two parallel plates which are stationary is set in motion due to sudden application of a constant pressure gradient is termed as the plane Poiseuille flow. Suppose that the fluid is bounded by two parallel plates at $y = 0$ and $y = d$, d is the width between the two parallel plates. And it is initially at rest and the motion starts suddenly due to a constant pressure gradient. Through Eq. (7) and Eq. (9), the governing equation is

$$\frac{\partial u}{\partial t} + \lambda_1^{\alpha} \frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}} = \nu \frac{\partial^2 u}{\partial y^2} + \nu \lambda_2^{\beta} \frac{\partial^{\beta}}{\partial t^{\beta}} (\frac{\partial^2 u}{\partial y^2}) + A \tag{13}$$

The initial and boundary conditions are

$$u(y,0) = u_t(y,0) = 0, \quad 0 \le y \le d, \tag{14}$$

$$u(0,t) = u(d,t) = 0, \quad t \ge 0. \tag{15}$$

where $A = -(\partial p / \partial x) / \rho$ is the constant pressure gradient that acts on the liquid in the x-direction and $\nu = \mu / \rho$ is the kinematical viscosity.

We solve the corresponding homogeneous equation in Eq. (13) with the boundary conditions Eq. (15) by the method of separation of variables firstly.

If we let $u(y,t) = Y(y)T(t)$ and substitute for $u(y,t)$ in Eq. (13), we obtain an ordinary linear differential equation for $Y(y)$:

$$Y''(y) + \eta Y(y) = 0, \quad Y(0) = Y(d) = 0 \tag{16}$$

and a fractional ordinary linear differential equation with the Caputo derivative for $T(t)$:

$$\frac{dT(t)}{dt} + \lambda_1^{\alpha} \frac{d^{\alpha+1} T(t)}{dt^{\alpha+1}} = -\nu\eta T(t) - \nu\eta\lambda_2^{\beta} \frac{d^{\beta} T(t)}{dt^{\beta}} \tag{17}$$

where η is a positive constant.

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The Sturm–Liouville problem given by Eq. (16) has eigenvalues

$$\eta = \frac{n^2\pi^2}{d^2}, \quad n = 1, 2, \cdots$$

and corresponding eigenfunctions

$$Y_n(y) = \sin\frac{n\pi y}{d} \quad n = 1, 2, \cdots$$

Now we seek a solution of the nonhomogeneous problem in Eq. (13) with the form

$$u(y,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi y}{d} \tag{18}$$

We assume that the series can be differentiated term by term. And we expand constant term A in Eq. (13) as a Fourier series by the Eigen functions $\sin(n\pi y)/d$:

$$A = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{d} \tag{19}$$

where

$$A_n = \frac{2}{d}\int_0^d A\sin\frac{n\pi y}{d}dy = \frac{2A}{n\pi}(1-(-1)^n) \tag{20}$$

Substituting Eq. (18) and Eq. (19) into Eq. (13) yields

$$\sum_{n=1}^{\infty}\sin\frac{n\pi y}{d}\frac{dT_n(t)}{dt} + \lambda_1^{\alpha}\sum_{n=1}^{\infty}\sin\frac{n\pi y}{d}\frac{d^{\alpha+1}T_n(t)}{dt^{\alpha+1}}$$

$$= -\nu(\frac{n\pi}{d})^2\sum_{n=1}^{\infty}\sin\frac{n\pi y}{d}T_n(t) - \nu(\frac{n\pi}{d})^2\lambda_2^{\beta}\sum_{n=1}^{\infty}\sin\frac{n\pi y}{d}\frac{d^{\beta}T_n(t)}{dt^{\beta}} + \sum_{n=1}^{\infty}\sin\frac{n\pi y}{d}A_n$$

Because of orthogonality of trigonometric function and by equating the coefficients of both members, we get

$$\frac{dT_n(t)}{dt} + \lambda_1^{\alpha}\frac{d^{\alpha+1}T_n(t)}{dt^{\alpha+1}} = -\nu(\frac{n\pi}{d})^2T_n(t) - \nu(\frac{n\pi}{d})^2\lambda_2^{\beta}\frac{d^{\beta}T_n(t)}{dt^{\beta}} + A_n \tag{21}$$

Through sorting Eq. (21), we obtain

$$\frac{d^{\alpha+1}T_n(t)}{dt^{\alpha+1}} + \frac{1}{\lambda_1^{\alpha}}\frac{dT_n(t)}{dt} + \nu\frac{\lambda_2^{\beta}}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2\frac{d^{\beta}T_n(t)}{dt^{\beta}} + \frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2T_n(t) = \frac{A_n}{\lambda_1^{\alpha}} \tag{22}$$

Based on Eq. (14) and Eq. (16), we gain

$$T_n(0) = 0, \quad T_n'(0) = 0 \tag{23}$$

According to Theorem 3.4, the fractional initial value problem (22)-(23) has the soltion

$$T_n(t) = \int_0^t x^{\alpha}E_{(\alpha,\alpha+1-\beta,\alpha+1),\alpha+1}(-\frac{1}{\lambda_1^{\alpha}}x^{\alpha}, -\frac{\nu\lambda_2^{\beta}}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2x^{\alpha+1-\beta}, -\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2x^{\alpha+1})A_n dx \tag{24}$$

where the multivariate Mittag-Leffler function is given in Definition 3.3. Hence, we get the solution of the initial-boundary value problem Eq. (13) in the form

$$u(y,t) = \sum_{n=1}^{\infty}\int_0^t x^{\alpha}E_{(\alpha,\alpha+1-\beta,\alpha+1),\alpha+1}(-\frac{1}{\lambda_1^{\alpha}}x^{\alpha}, -\frac{\nu\lambda_2^{\beta}}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2x^{\alpha+1-\beta}, -\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2x^{\alpha+1}) \times A_n dx\sin\frac{n\pi y}{d} \tag{25}$$

Inserting the expression for the velocity given by Eq.(25) into Eq. (9), the shearing stress is obtained in the following form:

$$\sigma_{xy}(y,t) = -\sum_{n=1}^{\infty}t^{\alpha}E_{(\alpha,\alpha+1-\beta,\alpha+1),\alpha+1}(-\frac{1}{\lambda_1^{\alpha}}t^{\alpha}, -\frac{\nu\lambda_2^{\beta}}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2t^{\alpha+1-\beta}, -\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2t^{\alpha+1}) \times\frac{dA_n}{n\pi}\cos\frac{n\pi y}{d} - A\rho y \tag{26}$$

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Particularly, we obtain the following limiting cases:

The first case: For $\lambda_1 = 0$ and $\alpha \neq 0$ in Eq. (21), we can get the velocity distribution and the shearing stress for a generalized second grade fluid

$$u(y,t) = \sum_{n=1}^{\infty} \int_0^t E_{(1-\beta,1),1}(-\nu\lambda_2^{\beta}(\frac{n\pi}{d})^2 x^{1-\beta}, -\nu(\frac{n\pi}{d})^2 x) A_n dx \sin\frac{n\pi y}{d} \quad (27)$$

$$\sigma_{xy}(y,t) = -\sum_{n=1}^{\infty} E_{(1-\beta,1),1}(-\nu\lambda_2^{\beta}(\frac{n\pi}{d})^2 t^{\alpha+1-\beta}, -\nu(\frac{n\pi}{d})^2 t^{\alpha+1}) \frac{dA_n}{n\pi}\cos\frac{n\pi y}{d} - A\rho y \quad (28)$$

Further, if $\beta = 1$, we can get the velocity distribution and the shearing stress which are identical to the result in Ref.[23].

The second case: For $\lambda_2 = 0$ and $\beta \neq 0$ in Eq. (25), we can obtain a similar solution to a generalized UCM fluid and the shearing stress

$$u(y,t) = \sum_{n=1}^{\infty} \int_0^t x^{\alpha} E_{(\alpha,\alpha+1),\alpha+1}(-\frac{1}{\lambda_1^{\alpha}} x^{\alpha}, -\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 x^{\alpha+1}) A_n dx \sin\frac{n\pi y}{d} \quad (29)$$

$$\sigma_{xy}(y,t) = -\sum_{n=1}^{\infty} t^{\alpha} E_{(\alpha,\alpha+1),\alpha+1}(-\frac{1}{\lambda_1^{\alpha}} t^{\alpha}, -\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 t^{\alpha+1}) \frac{dA_n}{n\pi}\cos\frac{n\pi y}{d} - A\rho y \quad (30)$$

The third case: For $\alpha = \beta = 1$, the solution (25) reduces to a similar solution to an Oldroyd-B fluid performing the same motion.

$$u(y,t) = \sum_{n=1}^{\infty} \int_0^t x E_{(1,1,2),2}(-\frac{1}{\lambda_1} x, -\frac{\nu\lambda_2}{\lambda_1}(\frac{n\pi}{d})^2 x, -\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 x^2) A_n dx \sin\frac{n\pi y}{d} \quad (31)$$

The shearing stress reduces to

$$\sigma_{xy}(y,t) = -\sum_{n=1}^{\infty} t E_{(1,1,2),2}(-\frac{1}{\lambda_1} t, -\frac{\nu\lambda_2}{\lambda_1}(\frac{n\pi}{d})^2 t, -\frac{\nu}{\lambda_1}(\frac{n\pi}{d})^2 t^2) \frac{dA_n}{n\pi}\cos\frac{n\pi y}{d} - A\rho y \quad (32)$$

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation, as well as the experimental conclusions that can be drawn.

5. Unsteady Couette flow

Consider the flow of a generalized Oldroyd-B fluid with fractional derivative between two parallel plates at $y = 0$ and $y = d$ and is initially at rest. Then the fluid starts suddenly due to a constant velocity of the upper plate in its own plane, the lower plate being always at rest. This flow is termed as the plane Couette flow. The governing equation is obtained from Eq. (13) for $A = 0$ and the initial-boundary conditions are

$$u(y,0) = u_t(y,0) = 0, \quad 0 \le y \le d, \quad (33)$$

$$\begin{matrix} u(0,t) = 0 \\ u(d,t) = U \end{matrix}, \quad t \ge 0 \quad (34)$$

In order to solve the problem with nonhomogeneous boundary, we firstly transform the non- homogeneous boundary into a homogeneous boundary condition. Let

$$u(y,t) = V(y,t) + W(y,t) \quad (35)$$

where $V(y,t)$ is a new unknown function, and

$$W(y,t) = Uy/d \quad (36)$$

satisfies the boundary conditions

$$W(0,t) = 0, \quad W(d,t) = U$$

The function $V(y,t)$ then satisfies the problem with homogeneous boundary conditions:

$$\begin{cases} \frac{\partial V}{\partial t}+\lambda_1^{\alpha}\frac{\partial^{\alpha+1}V}{\partial t^{\alpha+1}}=\nu\frac{\partial^2 V}{\partial y^2}+\nu\lambda_2^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}(\frac{\partial^2 V}{\partial y^2}), & 0\le y\le d, t\ge 0 \\ V(y,0)=-Uy/d, & 0\le y\le d \\ V_t(y,0)=0, & 0\le y\le d \\ V(0,t)=V(d,t)=0, & t\ge 0 \end{cases} \tag{37}$$

We solve the problem with homogeneous boundary conditions in the same way as before. Here we only present the final results as

$$\frac{u(y,t)}{U}=\frac{y}{d}+\sum_{n=1}^{\infty}\left(1-\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 t^{\alpha+1}\right. \\ \left.\times E_{(\alpha,\alpha+1-\beta,\alpha+1),\alpha+2}(-\frac{1}{\lambda_1^{\alpha}}t^{\alpha},-\frac{\nu\lambda_2^{\beta}}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 t^{\alpha+1-\beta},-\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 t^{\alpha+1})\right) \\ \times\frac{2(-1)^n}{n\pi}\sin\frac{n\pi y}{d} \tag{38}$$

$$\frac{\sigma_{xy}(y,t)}{\rho U}=\sum_{n=1}^{\infty}\frac{\nu}{\lambda_1^{\alpha}}\frac{2(-1)^n}{d}t^{\alpha} \\ \times E_{(\alpha,\alpha+1-\beta,\alpha+1),\alpha+1}(-\frac{1}{\lambda_1^{\alpha}}t^{\alpha},-\frac{\nu\lambda_2^{\beta}}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 t^{\alpha+1-\beta},-\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 t^{\alpha+1}) \\ \times\cos\frac{n\pi y}{d} \tag{39}$$

Particularly, we obtain the following special cases:

The first case: For $\lambda_1=0$ and $\alpha\neq 0$ in Eq.(37), we can get the velocity distribution and the shearing stress for a generalized second grade fluid

$$\frac{u(y,t)}{U}=\frac{y}{d}+\sum_{n=1}^{\infty}(1-\nu(\frac{n\pi}{d})^2 tE_{(1-\beta,1),2}(-\nu\lambda_2^{\beta}(\frac{n\pi}{d})^2 t^{1-\beta},-\nu(\frac{n\pi}{d})^2 t)) \\ \times\frac{2(-1)^n}{n\pi}\sin\frac{n\pi y}{d} \tag{40}$$

$$\frac{\sigma_{xy}(y,t)}{\rho U}=\sum_{n=1}^{\infty}\nu\frac{2(-1)^n}{d}E_{(1-\beta,1),1}(-\nu\lambda_2^{\beta}(\frac{n\pi}{d})^2 t^{1-\beta},-\nu(\frac{n\pi}{d})^2 t)\cos\frac{n\pi y}{d} \tag{41}$$

Further, if $\beta=1$, we can get the velocity distribution and the shearing stress which are identical to the result in Ref.[23].

The second case: For $\lambda_2=0$ and $\beta\neq 0$ in Eq.(37), we can obtain a solution to a generalized UCM fluid

$$\frac{u(y,t)}{U}=\frac{y}{d}+\sum_{n=1}^{\infty}(1-\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 t^{\alpha+1}E_{(\alpha,\alpha+1),\alpha+2}(-\frac{1}{\lambda_1^{\alpha}}t^{\alpha},-\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 t^{\alpha+1})) \\ \times\frac{2(-1)^n}{n\pi}\sin\frac{n\pi y}{d} \tag{42}$$

$$\frac{\sigma_{xy}(y,t)}{\rho U}=\sum_{n=1}^{\infty}\frac{\nu}{\lambda_1^{\alpha}}\frac{2(-1)^n}{d}t^{\alpha}E_{(\alpha,\alpha+1),\alpha+1}(-\frac{1}{\lambda_1^{\alpha}}t^{\alpha},-\frac{\nu}{\lambda_1^{\alpha}}(\frac{n\pi}{d})^2 t^{\alpha+1})\cos\frac{n\pi y}{d} \tag{43}$$

For $\alpha=\beta=1$, the solution (38) and (39) reduces to a similar solution or an Oldroyd-B fluid performing the same motion.

$$\frac{u(y,t)}{U}=\frac{y}{d}+\sum_{n=1}^{\infty}(1-\frac{\nu}{\lambda_1}(\frac{n\pi}{d})^2t^2E_{(1,1,2),3}(-\frac{1}{\lambda_1}t,\frac{\nu\lambda_2}{\lambda_1}(\frac{n\pi}{d})^2t,\frac{\nu}{\lambda_1}(\frac{n\pi}{d})^2t^2)) \times\frac{2(-1)^n}{n\pi}\sin\frac{n\pi y}{d} \tag{44}$$

$$\frac{\sigma_{xy}(y,t)}{\rho U}=\sum_{n=1}^{\infty}\frac{\nu}{\lambda_1}\frac{2(-1)^n t}{d}E_{(1,1,2),2}(-\frac{1}{\lambda_1}t,-\frac{\nu\lambda_2}{\lambda_1}(\frac{n\pi}{d})^2t,-\frac{\nu}{\lambda_1}(\frac{n\pi}{d})^2t^2)\cos\frac{n\pi y}{d} \tag{45}$$

6. Conclusion

For the Poiseuille and Couette flows, the corresponding analytical solutions about the velocity and the shearing stress with a generalized Oldroyd-B model are obtained by using the separation of variables method. Results show that the SVM simplifies the solution procedure without regard to the Laplace and Fourier transforms, and some solutions are identical to those in the previous papers. Furthermore, some well-known solutions the generalized second grade fluid, the generalized Maxwell fluid as well as the ordinary Oldroyd-B fluid can also be obtained as the limiting cases of the presented results.

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