Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 May 2022 doi:10.20944/preprints202205.0259.v1

Article

Dark matter in a galaxy as vacuum polarization

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Abstract: We considered a vacuum polarization around a massive object in the eikonal approximation and found two types of vacuum polarization. The first type has the equation of state similar to radiation and can produce a halo that increases the rotation velocity of a test particle with a radial distance. The second type of vacuum polarization has a more complicated equation of state. As a static physical effect, it produces renormalization of the gravitational constant. Besides, we demonstrate that a nonstationary polarization of the second type caused by a swift increase of the galactic nuclei mass results in a gravitational potential looking like a dark matter halo.

Keywords: vacuum energy; dark matter; vacuum polarization; active galaxy nuclei

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1. Introduction

Among the various issues of combining general relativity (GR) and quantum mechanics, one encounters the problems of the vacuum energy and the existence of black holes. The first problem is to explain why zero-point vacuum energy, if it is real, does not influence universe expansion (see, e.g., [1] and references herein). The horizon of a black hole leads to loss of unitarity and information (see, e.g., [2,3] and references herein), which prevents the definition of a pure quantum state.

On the other hand, the basis of GR is a notion of manifold [4], i.e., a metric space, which could be covered by the coordinate maps. When a concrete space-time possessing some symmetry is considered, one intends to introduce a system of coordinates allowing maximal covering of this particular manifold. For instance, the Schwarzschild solution describes the only region in front of the horizon, and one has to introduce the Kruscal coordinates to cover the complete domain [5]. Nevertheless, one could admit an opposite point of view: restricting manifold by sewing all the black hole horizons by some coordinate transformation. This approach is similar to a case when a man finds a hole in trousers at a knee. In such a case, he steps back a little from the hole border and then subtends it into a node with the help of sewing. Such a way leads to the conformally-unimodular metric [6], where the compact astrophysical objects look as the nonsingular balls -"eicheons" [7]. At the same time, the vacuum energy problem could be partially solved in the conformally-unimodular metric if one builds a gravity theory admitting an arbitrary choice of the energy density level [6]. That is possible because the equations for evolution of the Hamiltonian $\mathcal{H}$ and the momentum constraints $\mathcal{P}$ admit not only the trivial solution $\mathcal{H}=0$, $\mathcal{P}=0$, but also $\mathcal{H}=const$, $\mathcal{P}=0$. The constant compensates for the main part of the vacuum energy density proportional to the Planck mass in the fourth degree. Residual energy density, remaining after omitting the main part of the vacuum energy density, is some kind of dark energy and results in a cosmological picture containing a period of linear evolution in cosmic time [8,9] changed by the late accelerated expansion.

Both dark energy and dark matter are unknown mystical substances appearing in the modern cosmology and astrophysics [10,11]. Dark matter appears not only at the cosmological scales but also at the galaxy size ones (kPc). The last is the smallest scale at which dark matter was observed. Sometimes the dark energy is associated with the vacuum energy, but the dark matter is expected to be some kind of an actual matter weakly interacting with the known particles of the standard model [12]. Nevertheless, there are attempts to explain the dark matter by a vacuum polarization induced by the gravitational field. Heuristic models of vacuum polarization like [13–15] which demand antisreaning or dipolar fluid [16], or anti-gravitation [17] are of interest. As for the conventional renormalization procedure of the quantum field theory applied to vacuum energy near a massive object [18–22], it leads to modification of the gravitational potential only at small distances that seems unobservable.

The outline of this paper is as follows. In Section 1, we argue the necessity of considering a vacuum polarization from a cosmological point of view and explain that the conformally-unimodular metric is needed to avoid the main part of vacuum energy. Section 2 contains a perturbation formalism in the conformally-unimodular metric, which is required to introduce a vacuum polarization as some media, i.e., "ether". The eikonal approximation is used in section 3 to obtain the vacuum energy density and pressure of a quantum scalar field by summation of the contributions of the virtual distorted plane waves. The expression for a vacuum equation of state is obtained. In Section 4, the first type of vacuum polarization, possessing a radiation equation of state, is used in the Tolman-Volkov-Oppenheimer (TOV) equations for two substances to obtain a dark halo. In Section 5, the second type of vacuum polarization is considered. We considered a vacuum polarization around a massive object in the eikonal approximation and found two types of vacuum polarization. The first type has the equation of state similar to radiation and can produce a halo that increases the rotation velocity of a test particle with a radial distance. The second type of vacuum polarization has a more complicated equation of state. As a static physical effect, it produces renormalization of the gravitational constant. Besides, we demonstrate that a nonstationary polarization of the second type caused by a swift increase of the galactic nuclei mass results in a gravitational potential looking like a dark matter halo.

2. A spatially uniform universe in the conformally-unimodular metric

A class of conformally-unimodular metric is defined as [6]

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = a^2 \left(1 - \partial_m P^m\right)^2 d\eta^2 - \gamma_{ij}(dx^i + N^i d\eta)(dx^j + N^j d\eta), \tag{1}$$

where $x^\mu = \{\eta, \boldsymbol{x}\}$, η is a conformal time, γ_{ij} is a spatial metric, $a = \gamma^{1/6}$ is a locally defined scale factor, and $\gamma = \det \gamma_{ij}$. The spatial part of the interval (1) looks as

$$dl^2 \equiv \gamma_{ij} dx^i dx^j = a^2(\eta, \boldsymbol{x}) \tilde{\gamma}_{ij} dx^i dx^j, \tag{2}$$

where $\tilde{\gamma}_{ij} = \gamma_{ij}/a^2$ is a matrix with the unit determinant. The interval (1) is similar formally to the ADM one [23], but the lapse function is taken in the form $a(1 - \partial_m P^m)$, where P^m is a three-dimensional vector, and ∂_m is a conventional partial derivative. In the gravity theory [6] admitting arbitrary choice of the energy density level there are the Lagrange multipliers $\boldsymbol{P}$, $\boldsymbol{N}$ (shift function), and three triads e^a to parametrize the spatial metric $\gamma_{ij} = e^a_i e^a_j$. Such a theory [6] is known as the five vectors theory (FVT) of gravity. In contrast to GR, where the lapse and shift functions are arbitrary, the restrictions $\partial_n(\partial_m N^m) = 0$ and $\partial_n(\partial_m P^m) = 0$ arise in FVT. The Hamiltonian $\mathcal{H}$ and momentum $\mathcal{P}_i$ constraints in the particular gauge $P^i = 0$, $N^i = 0$ obey the constraint evolution equations [6]:

$$\partial_\eta \mathcal{H} = \partial_i \left(\tilde{\gamma}^{ij} \mathcal{P}_j\right), \tag{3}$$

$$\partial_\eta \mathcal{P}_i = \frac{1}{3} \partial_i \mathcal{H}, \tag{4}$$

which admits adding some constant to $\mathcal{H}$. Thus, the constraint $\mathcal{H}$ is not necessarily to be zero, but $\mathcal{H} = const$ is also allowed.

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Let's consider a spatially uniform, an isotropic and flat universe with the metric

$$ds^2 = a(\eta)^2(d\eta^2 - d\boldsymbol{x}^2). \tag{5}$$

The Friedmann equations takes the form [8,24,25]

$$M_p^{-2}e^{4\alpha}\rho - \frac{1}{2}e^{2\alpha}\alpha'^2 = const, \tag{6}$$

$$\alpha'' + \alpha'^2 = M_p^{-2}e^{2\alpha}(\rho - 3p), \tag{7}$$

where $\alpha(\eta) = \log a(\eta)$. Here and everywhere further, the system of units $\hbar = c = 1$ is used as well as the reduced Planck mass $M_p = \sqrt{\frac{3}{4\pi G}}$ is implied. According to FVT [6], the first Friedmann equation (6) is satisfied up to some constant, and the main parts of the vacuum energy density and pressure

$$\rho_v \approx (N_{boson} - N_{ferm})\frac{k_{max}^4}{16\pi^2 a^4}, \tag{8}$$

$$p_v = \frac{1}{3}\rho_v \tag{9}$$

do not contribute to the universe expansion because the constant in (6) compensates the vacuum energy density, whereas there is no a vacuum contribution in Eq. (7) by virtue of the equation of state (9). In the formula (8), the UV cut-off k_{max} and the difference between bosonic and fermionic degrees of freedom of the quantum fields appear since the zero-point stress-energy tensor is an additive quantity [26]. Here, we do not consider the supersymmetry hypotheses [27] due to absence of evidence of the supersymmetric particles to date.

Other contributors to the vacuum energy density are the terms depending on the derivatives of the universe expansion rate [9,24,25,28]. They have the right order of $\rho_v \sim M_p^2 H^2$, where H is the Hubble constant, and allow explaining the accelerated expansion of universe. Then, the energy density and pressure are [9,24,25,28]:

$$\rho_v = \frac{a'^2}{2a^6}M_p^2(2 + N_{sc})\mathcal{S}_0, \quad p_v = \frac{M_p^2(2 + N_{sc})\mathcal{S}_0}{a^6}\left(\frac{1}{2}a'^2 - \frac{1}{3}a''a\right), \tag{10}$$

where, $\mathcal{S}_0 = \frac{k_{max}^2}{8\pi^2 M_p^2}$. Eqs. (10) include the number of minimally coupled scalar fields N_{sc} plus two, because the gravitational waves give two additional degrees of freedom [24], whereas the massless fermions and photons do not contribute to (10) [24].

The residual vacuum energy density and pressure (10) lead to the accelerated universe expansion, which allows finding a momentum UV cut off

$$k_{max} \approx \frac{12M_p}{\sqrt{2 + N_{sc}}}. \tag{11}$$

from the measured value of the universe decceleration parameter and other cosmological observations [9,24].

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3. Perturbations under uniform background in conformally-unimodular metric

Below, the scalar perturbations[1] [29]

$$ds^2 = a(\eta,\boldsymbol{x})^2 \left(d\eta^2 - \left(\left(1 + \frac{1}{3} \sum_{m=1}^{3} \partial_m^2 F(\eta,\boldsymbol{x}) \right) \delta_{ij} - \partial_i \partial_j F(\eta,\boldsymbol{x}) \right) dx^i dx^j \right), \tag{12}$$

of the conformally-unimodular metric (1) will be considered, where the perturbations of the locally defined scale factor are expressed through a gravitational potential Φ:

$$a(\eta,\boldsymbol{x}) = e^{\alpha(\eta,\boldsymbol{x})} \approx e^{\alpha(\eta)}(1+\Phi(\eta,\boldsymbol{x})). \tag{13}$$

A stress-energy tensor could be written in the hydrodynamic approximation

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - p\, g_{\mu\nu}. \tag{14}$$

The perturbations of the energy density $\rho(\eta,\boldsymbol{x}) = \rho(\eta) + \delta\rho(\eta,\boldsymbol{x})$ and pressure $p(\eta,\boldsymbol{x}) = p(\eta) + \delta p(\eta,\boldsymbol{x})$ are considered around the spatially uniform values. Let us introduce new variables

$$\wp(\eta,\boldsymbol{x}) = a^4(\eta,\boldsymbol{x})\rho(\eta,\boldsymbol{x}), \tag{15}$$

$$\Pi(\eta,\boldsymbol{x}) = a^4(\eta,\boldsymbol{x})p(\eta,\boldsymbol{x}) \tag{16}$$

for the reasons which will be explained below. The perturbations of (15), (16) around the uniform values can be written now as $\wp(\eta,\boldsymbol{x}) = e^{4\alpha(\eta)}\rho(\eta) + \delta\wp(\eta,\boldsymbol{x})$, $\Pi(\eta,\boldsymbol{x}) = e^{4\alpha(\eta)}p(\eta) + \delta\Pi(\eta,\boldsymbol{x})$. The 4-velocity u is represented in the form of

$$u^\mu = e^{-\alpha(\eta)}\{1, \boldsymbol{\nabla}\frac{v(\eta,\boldsymbol{x})}{\rho(\eta)+p(\eta)}\} \approx \{e^{-\alpha(\eta)}(1-\Phi(\eta,\boldsymbol{x})), e^{3\alpha(\eta)}\boldsymbol{\nabla}\frac{v(\eta,\boldsymbol{x})}{\wp(\eta)+\Pi(\eta)}\}, \tag{17}$$

where $v(\eta,\boldsymbol{x})$ is a scalar function. Expanding all perturbations into the Fourier series $\delta\wp(\eta,\boldsymbol{x}) = \sum_{\boldsymbol{k}} \delta\wp_{\boldsymbol{k}}(\eta)e^{i\boldsymbol{k}\boldsymbol{x}}$... etc. results in the equations for the perturbations:

$$-6\Phi_{\boldsymbol{k}}' + 6\alpha'\Phi_{\boldsymbol{k}} + k^2 F_{\boldsymbol{k}}' + \frac{18}{M_p^2}e^{-2\alpha}\sum_i v_{\boldsymbol{k}i} = 0, \tag{18}$$

$$-18\alpha'\Phi_{\boldsymbol{k}}' - 6(k^2+3\alpha'^2)\Phi_{\boldsymbol{k}} + k^4 F_{\boldsymbol{k}} + \frac{18}{M_p^2}e^{-2\alpha}\sum_i \delta\wp_{\boldsymbol{k}i} = 0, \tag{19}$$

$$-12\Phi_{\boldsymbol{k}} - 3(F_{\boldsymbol{k}}'' + 2\alpha' F_{\boldsymbol{k}}') + k^2 F_{\boldsymbol{k}} = 0, \tag{20}$$

$$-9(\Phi_{\boldsymbol{k}}'' + 2\alpha'\Phi_{\boldsymbol{k}}') - 9(2\alpha'' + 2\alpha'^2 + k^2)\Phi_{\boldsymbol{k}} + k^4 F_{\boldsymbol{k}} - \frac{9}{M_p^2}e^{-2\alpha}\left(\sum_i 3\delta\Pi_{\boldsymbol{k}i} - \delta\wp_{\boldsymbol{k}i}\right) = 0, \tag{21}$$

where index i denoting kind of a substance is introduced. It is remarkable that, as a result of the choice of the variables (15), (16), (17), the values ρ and p do not appear in the system (18)-(21). This allows avoiding an influence of the large uniform energy density and pressure (8), (9) on the evolution of perturbation. Eqs. (18), (19) are consequences of the Hamiltonian and momentum constraints, while

[1] We consider only scalar perturbations, because vector and tensor perturbations do not perturb the matter.

Eqs. (20), (21) are equations of motion. For consistency of the constraints with the equations of motion, every kind of fluid has to satisfy continuity and the Euler equations

$$\alpha'(\delta\wp_{\boldsymbol{k}i} - 3\delta\Pi_{\boldsymbol{k}i}) - (3\Pi_i - \wp_i)(\Phi'_{\boldsymbol{k}} - 4\Phi_{\boldsymbol{k}}\alpha') + 4\wp'_i\Phi_{\boldsymbol{k}} - \delta\wp'_{\boldsymbol{k}i} + k^2 v_{\boldsymbol{k}i} = 0, \tag{22}$$

$$\Phi_{\boldsymbol{k}}(\wp_i - 3\Pi_i) + \delta\Pi_{\boldsymbol{k}i} + v'_{\boldsymbol{k}i} = 0. \tag{23}$$

4. Vacuum as a medium: the eikonal approximation for quantum fields

Generally, a vacuum could also be considered as some fluid, i.e., "ether" [25,28], but with some stochastic properties along with the elastic ones [30]. Here we will be interested in its elastic properties only. In Refs. [25,28] sound speed of the scalar waves of vacuum polarization $c_s^2 = \frac{p'_v(\eta)}{\rho'_v(\eta)}$ was introduced, where p_v and ρ_v are given by (10), but it was the too heuristic picture. Here we provide the actual calculations of the vacuum density and pressure on the curved background in the eikonal approximation, which has a straightforward sense: in the Minkowsky's space-time, the virtual plane waves penetrate space-time, and to obtain the vacuum energy density, we must summarize the contributions of the every wave. In the curved space-time, we must summarize the contributions of the distorted waves to obtain the spatially non-uniform energy density and pressure. It should be mentioned that eikonal approximation was successfully used in high energy physics [31] and even in gravity [32], where the small-angle scattering amplitude of two massive particles was calculated in all orders on gravitational constant G.

A massless scalar field in the external gravitational field obeys the equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\phi = 0. \tag{24}$$

Using the gauge $\boldsymbol{N} = 0$, $\boldsymbol{P} = 0$ in (1) reduces the conformally-unimodular metric to

$$ds^2 = a^2(d\eta^2 - \tilde{\gamma}_{ij}dx^i dx^j), \tag{25}$$

so that the equation (24) takes the form

$$\phi'' + 2\frac{a'}{a}\phi' - \frac{1}{a^2}\partial_i\left(a^2\tilde{\gamma}^{ij}\partial_j\right)\phi = 0, \tag{26}$$

which turns to

$$\chi'' - \chi\frac{a''}{a} - \tilde{\gamma}^{ij}\partial_i\partial_j\chi - \partial_i\tilde{\gamma}^{ij}\partial_j\chi + \frac{\chi}{a}\left(\tilde{\gamma}^{ij}\partial_i\partial_j a + \partial_j a\partial_i\tilde{\gamma}^{ij}\right) = 0, \tag{27}$$

after change of the variables $\phi = \chi/a$ or, in terms of metric perturbations Φ and F, becomes

$$\chi'' - \Delta\chi + \hat{V}\chi = 0, \tag{28}$$

where a "potential" $\hat{V}$ has the form

$$\hat{V} = -\alpha'' - \alpha'^2 - 2\alpha'\Phi' - \Phi'' + \Delta\Phi + \frac{1}{3}\Delta F\,\Delta - \frac{\partial^2 F}{\partial x^j\partial x^i}\frac{\partial^2}{\partial x^j\partial x^i} - \frac{2}{3}(\boldsymbol{\nabla}(\Delta F))\cdot\boldsymbol{\nabla}\ . \tag{29}$$

A quantization of the scalar field in terms of creation and annihilation operators implies [33]

$$\hat{\chi}(\eta, \boldsymbol{x}) = \sum_{\boldsymbol{k}} u_{\boldsymbol{k}}(\eta, \boldsymbol{x})\hat{a}_{\boldsymbol{k}} + u^*_{\boldsymbol{k}}(\eta, \boldsymbol{x})\hat{a}^+_{\boldsymbol{k}}, \tag{30}$$

where the function $u_{\boldsymbol{k}}$ satisfies Eq. (27) and the orthogonality condition [33]

$$\int (u_{\boldsymbol{k}}\partial_\eta u^*_{\boldsymbol{q}} - u^*_{\boldsymbol{k}}\partial_\eta u_{\boldsymbol{q}})d^3\boldsymbol{x} = i\delta_{\boldsymbol{k}\boldsymbol{q}}. \tag{31}$$

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Solution of Eqs. (27), (29) for the functions $u_{\boldsymbol{k}}$ can be written in the eikonal approximation

$$u_{\boldsymbol{k}}(\eta,\boldsymbol{x}) = \frac{1}{\sqrt{2k}} \exp\left(-i\eta k + i\boldsymbol{k}\boldsymbol{x} - i\Theta_{\boldsymbol{k}}(\eta,\boldsymbol{x})\right), \tag{32}$$

which leads to the equation for eikonal function

$$2k\Theta_{\boldsymbol{k}}' + \left(2k_m\tilde{\gamma}^{mj} - i\partial_m\tilde{\gamma}^{mj}\right)\partial_j\Theta_{\boldsymbol{k}} + \frac{1}{a}\left(a'' - \tilde{\gamma}^{ij}\partial_i\partial_j a - \partial_j a\partial_i\tilde{\gamma}^{ij}\right) + ik_j\partial_m\tilde{h}^{mj} - k_m k_j\tilde{h}^{mj} = 0, \tag{33}$$

and, according to Eqs. (12), (13), is written in terms of the metric perturbations $\Phi(\eta,\boldsymbol{x})$, $F(\eta,\boldsymbol{x})$:

$$k\Theta_{\boldsymbol{k}}' + \boldsymbol{k}\boldsymbol{\nabla}\Theta_{\boldsymbol{k}}(\eta,\boldsymbol{x}) = \frac{1}{2}V_{\boldsymbol{k}}, \tag{34}$$

where

$$V_{\boldsymbol{k}}(\eta,\boldsymbol{x}) = -2\alpha'\Phi' - \Phi'' + \Delta\Phi + k_i k_j\partial_i\partial_j F - \frac{k^2}{3}\Delta F. \tag{35}$$

Solution of (34) can be obtained in the form

$$\Theta_{\boldsymbol{k}}(\eta,\boldsymbol{x}) = \frac{1}{2k}\int_{\eta_0}^{\eta} V_{\boldsymbol{k}}\left(\tau, \boldsymbol{x} + \frac{\boldsymbol{k}}{k}(\tau-\eta)\right)d\tau, \tag{36}$$

where lower integration limit η_0 depends on the cosmological model. In particular, it could be 0 or $-\infty$. Mean value of the stress-energy tensor of a massless scalar field

$$\hat{T}_{\mu\nu} = \partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\hat{\phi}\partial_\beta\hat{\phi} \tag{37}$$

can be averaged over the vacuum state and compared with the hydrodynamic expression (14). That gives

$$\delta\wp(\eta,\boldsymbol{x}) = e^{2\alpha(\eta)} < 0|\frac{\hat{\phi}'^2}{2} + \frac{(\boldsymbol{\nabla}\hat{\phi})^2}{2}|0> \approx \frac{1}{2}\sum_{\boldsymbol{k}}\frac{\alpha'\Phi'}{k} + \Theta_{\boldsymbol{k}}' - \frac{\boldsymbol{k}\boldsymbol{\nabla}\Theta_{\boldsymbol{k}}}{k}, \tag{38}$$

$$\delta\Pi(\eta,\boldsymbol{x}) = e^{2\alpha(\eta)} < 0|\frac{\hat{\phi}'^2}{2} - \frac{(\boldsymbol{\nabla}\hat{\phi})^2}{6}|0> \approx \frac{1}{2}\sum_{\boldsymbol{k}}\frac{\alpha'\Phi'}{k} + \Theta_{\boldsymbol{k}}' + \frac{\boldsymbol{k}\boldsymbol{\nabla}\Theta_{\boldsymbol{k}}}{3k}, \tag{39}$$

$$\boldsymbol{\nabla}v = -e^{2\alpha(\eta)} < 0|\hat{\phi}'\boldsymbol{\nabla}\hat{\phi}|0> \approx \sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\Theta_{\boldsymbol{k}}'}{k} - \boldsymbol{\nabla}\Theta_{\boldsymbol{k}} - \frac{\alpha'\boldsymbol{\nabla}\Phi}{k}, \tag{40}$$

where only spatially non-uniform parts of the vacuum averages are implied in the second equalities on the right-hand side of (38), (39) and (40), which depends on the metric perturbations $F(\eta,\boldsymbol{x})$ and $\Phi(\eta,\boldsymbol{x})$ contained in Eqs. (12), (13). The final equalities in (38), (39) and (40) are the results of calculations in the eikonal approximation (32). Considering the quantity $\delta\wp(\eta,\boldsymbol{x}) - 3\delta\Pi(\eta,\boldsymbol{x})$ and using equations (34) and (35) one comes to

$$\delta\wp(\eta,\boldsymbol{x}) - 3\delta\Pi(\eta,\boldsymbol{x}) = -\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{\nabla}\Theta_{\boldsymbol{k}}}{k} + \Theta_{\boldsymbol{k}}' + \frac{\alpha'\Phi'}{k} = -\sum_{\boldsymbol{k}}\frac{1}{2k}V_{\boldsymbol{k}} + \frac{\alpha'\Phi'}{k} = \sum_{\boldsymbol{k}}\frac{1}{2k}\left(\Phi'' - \Delta\Phi - k_i k_j\partial_i\partial_j F + \frac{k^2}{3}\Delta F\right) = \frac{N_{sc}}{8\pi^2}k_{max}^2(\Phi'' - \Delta\Phi), \tag{41}$$

where summation has been changed by integration $\sum_{\boldsymbol{k}} \to \int d^3\boldsymbol{k}/(2\pi)^3$ and it is taken into account that $\int_{k<k_{max}}\frac{1}{2k}\left(k_i k_j - \frac{k^2}{3}\delta_{ij}\right)d^3\boldsymbol{k} = 0$. Besides, the number N_{sc} of the scalar fields minimally coupled

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with gravity has been introduced as well as in (10). As it follows from Eq. (41), two types of spatially nonuniform vacuum polarization exist. Namely, F-polarization has a radiation equation of state

$$\delta\Pi_{vF}(\eta,\boldsymbol{x}) = \frac{1}{3}\delta\wp_{vF}(\eta,\boldsymbol{x}), \tag{42}$$

whereas Φ-type has the equation of state

$$\delta\Pi_{v\Phi}(\eta,\boldsymbol{x}) = \frac{1}{3}\delta\wp_{v\Phi}(\eta,\boldsymbol{x}) - \frac{N_{sc}}{24\pi^2}k_{max}^2(\Phi'' - \Delta\Phi). \tag{43}$$

Both types of spatially nonuniform vacuum polarization correspond to the uniform component (8), (9), whereas the uniform polarisation given by (10) has no nonuniform counterpart to an accuracy of our consideration, i.e., in the second order on derivatives.

In principle, the system of equations (18), (19), (20), (21), (22), (23), (42), (43) is a fundamental system allowing to consider a wide range of cosmological and astrophysical phenomena including CMB and BAO. However, below we restrict ourselves to a galactic scale "dark matter".

5. Galactic dark matter as a F-vacuum polarization

As it was shown in the section 4, the F-component of vacuum has the equation of state analogous to radiation. In this sense, it is similar to the uniform part of vacuum energy density in Eq. (8).

At the same time, it is difficult to determine the concrete value of the nonuniform vacuum energy density because according to (38), it contains eikonal function $\Theta_{\boldsymbol{k}}$, which is determined by the integral (36). For instance, from (35),(36) one has $\Theta_{\boldsymbol{k}}(\eta,\boldsymbol{r}) = \sum_{\boldsymbol{q}} \tilde{\Theta}_{\boldsymbol{k},\boldsymbol{q}}(\eta)e^{i\boldsymbol{q}\boldsymbol{r}}$ and

$$\tilde{\Theta}_{\boldsymbol{k},\boldsymbol{q}}(\eta) = \frac{1}{k}\left(\frac{1}{3}k^2q^2 - (\boldsymbol{q}\boldsymbol{k})^2\right)\int_{\eta_0}^{\eta} F_{\boldsymbol{q}}(\tau)e^{i\boldsymbol{k}\boldsymbol{q}(\tau-\eta)/k}d\tau. \tag{44}$$

Calculation of the integral (44) needs to know the full evolution history of $F_{\boldsymbol{q}}(\tau)$. It is simpler to use only the fact that the F-contribution to the vacuum polarization has the equation of state

$$p_{vF} = \rho_{vF}/3. \tag{45}$$

For the static case in the first order on perturbations, the form of the matter-energy density distribution and potential are not determined (see, e.g., Appendix). However, it is possible to consider a heuristic nonlinear model treating the F-vacuum as an abstract substance with the above equation of state. Say that is a core of some incompressible substance on the radiation background, i.e., the F-polarized vacuum or "dark radiation", which does not interact with the ordinary substances. Below we find a spherically symmetric solution for an incompressible substance with the constant energy density ρ_1 on the background of radiation density ρ_2.

5.1. Equations in the conformally-unimodular metric

Conformally-unimodular metric in the case of spherical symmetry acquires the form [7]

$$ds^2 = a^2(d\eta^2 - \tilde{\gamma}_{ij}dx^i dx^j) = e^{2\alpha}\left(d\eta^2 - e^{-2\lambda}(d\boldsymbol{x})^2 - (e^{4\lambda} - e^{-2\lambda})(\boldsymbol{x}d\boldsymbol{x})^2/r^2\right), \tag{46}$$

where $r = |\boldsymbol{x}|$ and $a = \exp\alpha$, λ are the functions of η, r. The matrix $\tilde{\gamma}_{ij}$ with the unit determinant is expressed through $\lambda(\eta,r)$. The interval (46) could be also rewritten in the spherical coordinates:

$$x = r\sin\theta\cos\phi, \;\; y = r\sin\theta\sin\phi, \;\; z = r\cos\theta \tag{47}$$

to give

$$ds^2 = e^{2\alpha}\left(d\eta^2 - dr^2e^{4\lambda} - e^{-2\lambda}r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right). \tag{48}$$

Restricting ourself by static solutions, the equations for the functions $\alpha(r)$ and $\lambda(r)$ are written as [7]

$$\mathcal{H} = e^{2\alpha}\left(-\frac{e^{2\lambda}}{6r^2} + e^{-4\lambda}\left(\frac{1}{6r^2} - \frac{4}{3}\frac{d\alpha}{dr}\frac{d\lambda}{dr} + \frac{1}{6}\left(\frac{d\alpha}{dr}\right)^2 + \frac{2}{3r}\frac{d\alpha}{dr} + \frac{1}{3}\frac{d^2\alpha}{dr^2} + \frac{7}{6}\left(\frac{d\lambda}{dr}\right)^2 - \frac{5}{3r}\frac{d\lambda}{dr} - \frac{1}{3}\frac{d^2\lambda}{dr^2}\right) + \frac{e^{2\alpha}}{M_p^2}\sum_j \rho_j(r)\right) = const, \quad (49)$$

$$\frac{d^2\alpha}{dr^2} = -\frac{3e^{6\lambda}}{r^2} + \frac{3}{r^2} - 8\frac{d\alpha}{dr}\frac{d\lambda}{dr} + 7\left(\frac{d\alpha}{dr}\right)^2 + \frac{10}{r}\frac{d\alpha}{dr} + 3\left(\frac{d\lambda}{dr}\right)^2 - \frac{6}{r}\frac{d\lambda}{dr} + 3\frac{e^{2\alpha+4\lambda}}{M_p^2}\sum_j \rho_j - 3p_j, \quad (50)$$

$$\frac{d^2\lambda}{dr^2} = -\frac{5e^{6\lambda}}{r^2} + \frac{5}{r^2} - 18\frac{d\alpha}{dr}\frac{d\lambda}{dr} + 12\left(\frac{d\alpha}{dr}\right)^2 + \frac{18}{r}\frac{d\alpha}{dr} + 8\left(\frac{d\lambda}{dr}\right)^2 - \frac{14}{r}\frac{d\lambda}{dr} + 6\frac{e^{2\alpha+4\lambda}}{M_p^2}\sum_j \rho_j - 3p_j, \quad (51)$$

where Eq. (49) is the Hamiltonian constraint, which could be rewritten in the form containing no second derivatives using Eqs. (50), (51):

$$\mathcal{H} = \frac{e^{2\alpha-4\lambda}}{2r^2}\left(-3r^2\left(\frac{d\alpha}{dr}\right)^2 + 4r\frac{d\alpha}{dr}\left(r\frac{d\lambda}{dr} - 1\right) - \left(r\frac{d\lambda}{dr} - 1\right)^2 + e^{6\lambda}\right) + \frac{3e^{4\alpha}}{M_p^2}\sum_j p_j = const. \quad (52)$$

Each kind of substance has to satisfy

$$\frac{d\,p_j}{dr} + (p_j + \rho_j)\frac{d\alpha}{dr} = 0. \quad (53)$$

Vacuum solution of the equations (49), (50), (51), corresponding to the point massive particle, was considered in [7] where an absence of evidence for horizon was demonstrated. Let us consider another solution, corresponding to the substance of a radiation type filling all the space. This particular solution is written as

$$\alpha(r) = \ln r - \frac{1}{6}\ln 7, \quad \lambda(r) = \frac{1}{6}\ln 7, \quad (54)$$

and under (45) from (53) it follows

$$\frac{d}{dr}\left(\rho e^{4\alpha}\right) = 0, \qquad \rho = \frac{1}{2}r^{-4}\,7^{-1/3}, \quad (55)$$

if to use (54) and (49) with $const = 0$ in the right hand side of Eq. (49). Here, ρ is measured in terms of $r_g^{-2}M_p^{-2}$, and r is measured in units of r_g, which is not gravitational radius of something, but some arbitrary spatial scale. It should be noted that, for (45), Eqs. (50), (51) look as those for an empty space, whereas Eq. (49) could also be considered as that for an empty space, but with $const \neq 0$. Thus, in conformally-unimodular metric of FVT where the Hamiltonian constraint is satisfied up to some constant, one could alternatively consider the F-vacuum polarization solution like that for an empty space, but with some value of $const$ in Eqs. (49), (52).

Since the solution (55) is singular, it is not related to reality. To obtain a more realistic model, one has to consider at least two substances: a compact object in the center consisting of a substance with constant energy density and a substance with the radiation equation of state (42).

5.2. Equations in the Schwarzschild type metric

It is more convenient to begin consideration from the Schwarzschild type metric [34]

$$ds^2 = B(R)dt^2 - A(R)dR^2 - R^2 d\Omega, \quad (56)$$

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where the solutions (54), (55) correspond to the well-known solution [34]

$$\rho_2(R) = \frac{1}{14R^2}, \tag{57}$$

obeying the TOV equation [35,36] for a radiation fluid

$$\rho_2' = -\frac{3\rho_2\left(m + 4\pi R^3\rho_2/3\right)}{\pi R\left(R - \frac{3m}{2\pi}\right)} \tag{58}$$

in all the spatial region $R \in (0, \infty)$, where $m(R)$ is defined by

$$m'(R) = 4\pi R^2 \rho_2. \tag{59}$$

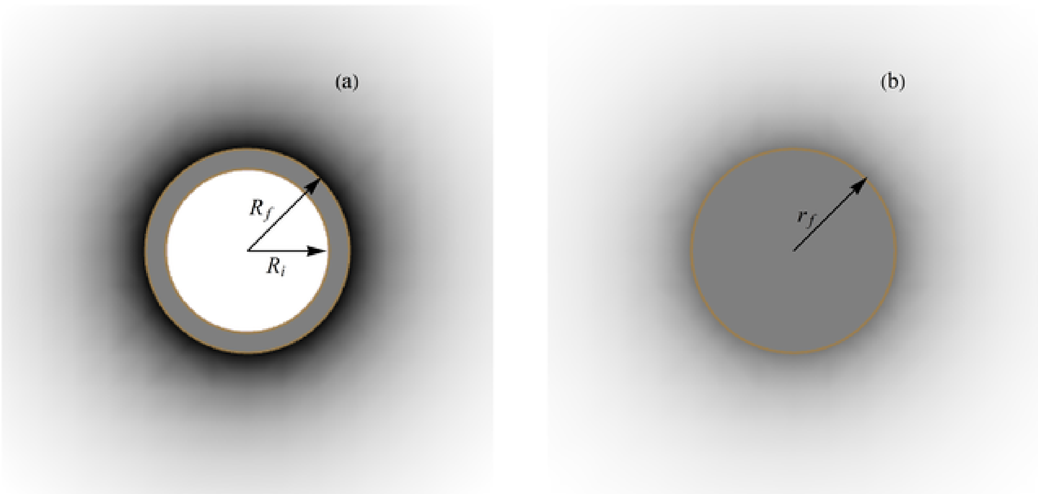

Figure 1. (a) Schematic picture of an eicheon in the metric (56) with taking into account a vacuum polarization in the form of dark radiation, (b) an eicheon in the metric (48) looks like a solid ball with the finite energy density of dark radiation in the center.

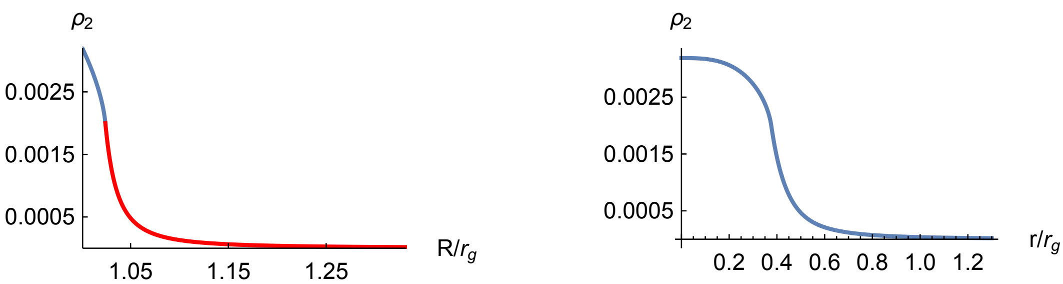

Figure 2. (a) ρ_2 – energy density of the vacuum polarization in a form of dark radiation in coordinates $R > R_i$ calculated for the eicheon parameters $\rho_1 = 7M_p^2 r_g^{-2}$, $R_i = 1.001r_g$, $R_f = 1.024r_g$, $\rho_2(R_f) = 0.002M_p^2 r_g^{-2}$, (b) ρ_2 calculated in the coordinates r of the metric (48).

Again, ρ_2 is measured in terms of $r_g^{-2}M_p^2$, and R is measured in the units of r_g. The solutions (57) and (55) are singular at $R = 0$ and, thereby, nonphysical. The situation changes cardinally in the presence of a core consisting of incompressible matter. More exactly, in a presence of incompressible matter of low density ρ_1, the corresponding solution remains singular, but if $\rho_1 > \frac{1}{2}\left(\frac{8}{9}\right)$, a solid ball in the metric (48) looks like a shell over r_g in the metric (56) [7] shown in Fig. 1 (a). Here we again imply

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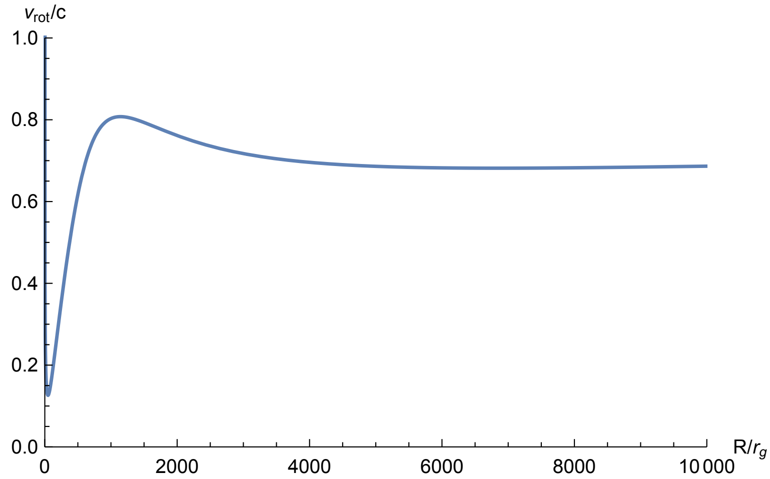

Figure 3. The general form of a mode rotational curve for the eicheon parameters specified below Fig. 2.

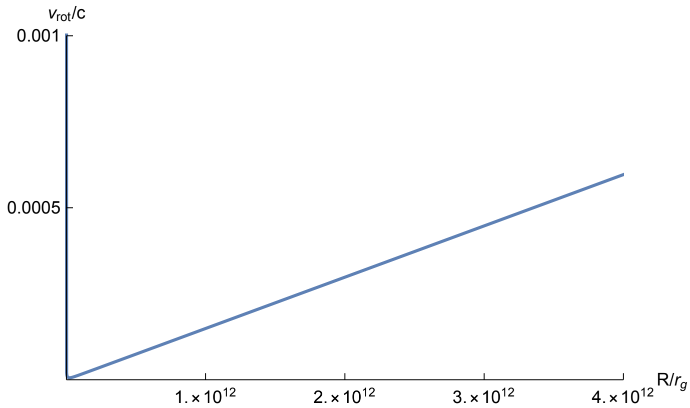

Figure 4. The rotational curve for the eicheon parameters $\rho_1 = 100M_p^2 r_g^{-2}$, $R_i = 1.0001r_g$ and $\rho_2(R_f) = 4 \times 10^{-27} M_p^2 r_g^{-2}$.

the gravitational radius r_g as a measure of the distances, but calculate it taking into account only an incompressible matter. Such a matter occupies a region between R_i and R_f, where

$$R_f = \sqrt[3]{R_i^3 + \frac{1}{2\rho_1}} \tag{60}$$

in the units of r_g. Here the energy density ρ_1 is constant and measured in the terms of $r_g^{-2} M_p^2$, where the gravitational radius is defined as $r_g = \frac{3m_1}{2\pi M_p^2}$ and $m_1 = \frac{4}{3}\pi\rho_1(R_f^3 - R_i^3)$. Compact object of such a type arising in FVT is known as "eicheon" [7] and replaces a black hole of GR. The appearance of eicheon in the center makes the solution (58) to be nonsingular because it allows for setting the finite boundary conditions for radiation.

To explain this, let us consider two fluids in the metric (56) obeying the TOV equations:

$$p_1' = -\frac{(3p_1 + \rho_1)\left(m + 4\pi R^3 \left(p_1 + \frac{\rho_2}{3}\right)\right)}{4\pi R \left(R - \frac{3m}{2\pi}\right)}, \tag{61}$$

$$\rho_2' = -\frac{3\rho_2 \left(m + 4\pi R^3 \left(p_1 + \frac{\rho_2}{3}\right)\right)}{\pi R \left(R - \frac{3m}{2\pi}\right)}, \tag{62}$$

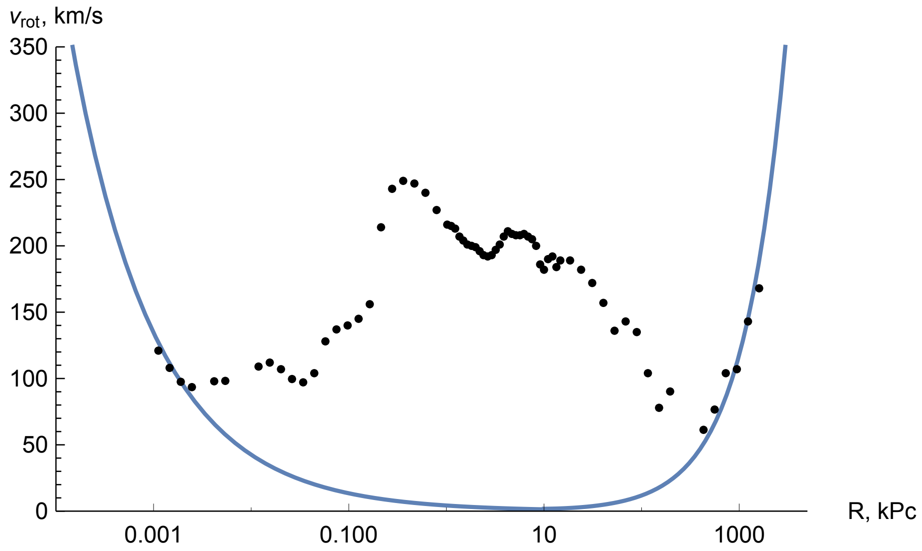

Figure 5. The rotational curve in the logarithmic scale: the points correspond to the Milky Way rotational curve from [37], whereas the calculated curve includes the only contribution of eicheon described by the parameters given under Fig. 4, and the vacuum polarization of $F - type$.

where the function $m(R)$ satisfies to

$$m'(R) = 4\pi R^2(\rho_1 + \rho_2). \tag{63}$$

For $\rho_1 > \frac{1}{2}\left(\frac{8}{9}\right)$ these equations hold for the internal range $R_i < R < R_f$, where $R_i > r_g$ and the border, occupied by ρ_1, is defined through (60).

The pressure of incompressible fluid must turn to zero at the end of the range filled by matter $R = R_f$, and it is a boundary condition for p_1. Then one could set an amount of radiation at $R = R_f$ and solve system of equations in a region $\{R_i, R_f\}$ assuming $m(R_i) = 0$. A solution allows determining $m(R_f)$, and, using this value as an initial condition, one should solve the equation for the radiation fluid (58) in an outer region $\{R_f, \infty\}$. The metric could be obtained by solving the equations [34]

$$\frac{1}{B}\frac{dB}{dR} = -\frac{2}{p_1 + \rho_1}\frac{dp_1}{dR} = -\frac{2}{p_2 + \rho_2}\frac{dp_2}{dR}, \tag{64}$$

$$\frac{d}{dR}\left(\frac{R}{A}\right) = 1 - 2R^2(\rho_1 + \rho_2). \tag{65}$$

Comparing the metric (46) and (56) leads to relation of the radial coordinates R and r [7]

$$\frac{dR}{dr} = \left(\frac{r}{R}\right)^2 \frac{B^{3/2}}{A^{1/2}}, \tag{66}$$

where the dependencies $B(R(r))$ and $A(R(r))$ are implied. Eq. (66) has to be integrated with the initial condition $R(0) = R_i$, which means that R_i in the metric (56) corresponds to $r = 0$ in the metric (46). Knowing $R(r)$ allows ploting $\rho_2(R)$ shown in Fig. 2 (a) as the $r-$ dependent function $\rho_2(R(r))$ (Fig. 2 (b)).

Let us consider the motion of a test particle on a circular orbit in the metric (56). Angular velocity on a circular orbit is calculated as [34]:

$$\frac{d\phi}{dt} = \sqrt{\frac{1}{2R}\frac{dB}{dR}}. \tag{67}$$

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A spatial interval passed by the particle along the circular orbit equals $dl = Rd\phi = R\frac{d\phi}{dt}dt$. To obtain rotation velocity seen by the observer situated in rest near the moving particle, one has to divide the spatial interval over proper time $\sqrt{g_{00}}dt = \sqrt{B}dt$ of such an observer [38]:

$$v_{rot} = \frac{dl}{\sqrt{B}dt} = \sqrt{\frac{R}{2B}\frac{dB}{dR}} = \sqrt{-\frac{R}{p_2+\rho_2}\frac{dp_2}{dR}} = \frac{1}{2}\sqrt{-\frac{R}{\rho_2}\frac{d\rho_2}{dR}}. \tag{68}$$

A qualitative example of a general form of the numerical solution for rotation velocity is shown in Fig. 3. Although the shape of the curve resembles an observed curve, asymptotic of the rotation curve corresponds to $v_{rot} \sim 1/\sqrt{2} \approx 0.71$. This very large velocity (in units of speed of light) corresponds to asymptotic value $\rho_2 \sim R^{-2}$ in (57), whereas, in the reality, the rotation velocities of galaxies are $v_{rot} \sim 100 - 300\ km/s\ \sim 0.001$. To obtain smaller velocities, one has to diminish density of radiation in the center of eicheon, i.e. at $r = 0$ in the metric (48) or $R = R_i$ in the metric (56). For central radiation density of $\rho_2 = 2 \times 10^{-26}$, one has the rotation curve shown in Fig. 4. This is pure "dark radiation" contribution without the galaxy bulge and disk. It increases linearly with the distance and corresponds to the rising part of the general curve shown in Fig. 3. In the logarithmic scale, one could see (Fig. 5) the contribution of the eicheon of the mass of $4.2 \times 10^6\ M_\odot$ in the center of the Milky Way (the left side of the curve) and the impact of the dark radiation (the right side of the curve), whereas the effects of galactic bulge and disk responsible for the intermediate region are not taken into account. However, it is expected that bulge and disk attraction will influence the F-type vacuum polarization in such a way that the curve in Fig. 4 will be not linear but bend. We do not gain insight into such details because our goal is to show that the F-type vacuum polarization could arise only around a "sewed" black hole, i.e., around eicheon.

We emphasize that the presented consideration is heuristic because, although the linear system for the perturbation and the eikonal approximation for vacuum polarization seem trustable, we use its results in the nonlinear TOV model. Another thing is that we set the density of radiation (F-type vacuum polarization) in the center of eicheon, i.e., at $r = 0$, of $R = R_i$ empirically but not calculate it from the first principles, i.e., we use only the equation of state from the eikonal calculations.

6. Vacuum polarization of Φ-type

In the sections 3, 4 the linear system of equation (20),(21),(22),(23),(41) was deduced, which describes the evolution of perturbation by taking into account vacuum polarization (see Eq. (41)). Galaxy formation is a complex nonlinear process that develops during cosmological time scales. Generally, the linear system is insufficient to describe the galaxy. However, one could create a heuristic picture setting an approximate profile of matter near the galaxy center and obtain a gravitational potential produced by vacuum polarization obeying the linear equations. Below we will discuss that the observed galaxy halo could originate from a very fast, compared to the cosmological times, process of mass increasing the galaxy nuclei. We will neglect cosmological evolution assuming $\alpha(\eta) = 0$. That reduces the above system of the equations to

$$-12\Phi_q - 3F_q'' + q^2 F_q = 0, \tag{69}$$

$$-9\Phi_q'' - 9q^2\Phi_q + q^4 F_q + \frac{9}{M_p^2}\left(\sum_i \delta\wp_{ki} - 3\delta\Pi_{qi}\right) = 0. \tag{70}$$

$$\delta\wp_{qv} - 3\delta\Pi_{qv} = \frac{N_{sc}}{8\pi^2}k_{max}^2(\Phi_q'' + q^2\Phi_q), \tag{71}$$

where the last equation holds out for the vacuum polarisation of Φ–type and is denoted by $i = v$. Substituting Φ_q from Eq. (69) and $\delta\wp_{qv} - 3\delta\Pi_{qv}$ from Eq. (71) into Eq. (70) gives the equation

$$3\left({k_{max}}^2 - 8\pi^2{M_p}^2\right)\left(3F_q'''' + 2q^2F_q''\right) - q^4F_q\left(3N_{sc}{k_{max}}^2 + 8\pi^2{M_p}^2\right) = 288\pi^2\delta\wp_{q\,eff}(\eta), \tag{72}$$

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where an effective "density" of all the substances except vacuum is denoted as

$$\delta\wp_{q\,eff}(\eta) = \sum_{i\neq v} \delta\wp_{ki} - 3\delta\Pi_{qi}. \tag{73}$$

Eq. (72) allows developing an empirical model: setting profile and time dependence of the quantity $\wp_{q\,eff}(\eta)$ empirically makes possible to find the metric perturbation F_q and calculate Φ_q using (69), i.e., to determine the gravitational field corresponding to $\wp_{q\,eff}(\eta)$.

Let us for simplicity, take $\wp_{q\,eff}(\eta)$ in the form

$$\wp_{q\,eff}(\eta) = m\,Z(q)e^{\eta/T}, \tag{74}$$

where m is a "mass" of the object at $\eta = 0$, $Z(q)$ is a form-factor and T is a some typical time of increasing of the "mass". That is a heuristic model implying that some rapid processes like accretion occurs around the massive object, i.e., around the galaxy nuclei. Substitution of the expression (74) into Eq. (72) allow finding $F_q(\eta) = \tilde{F}_q e^{\eta/T}$, where

$$\tilde{F}_q = -\frac{288\pi^2 T^4 mZ(q)}{3N_{sc}{k_{max}}^2\left(q^4T^4 - 2q^2T^2 - 3\right) + 8\pi^2{M_p}^2\left(q^2T^2+3\right)^2}, \tag{75}$$

and Eq. (69) gives $\Phi_q(\eta) = \tilde{\Phi}_q e^{\eta/T}$

$$\tilde{\Phi}_q = -\frac{24\pi^2T^2\left(q^2T^2-3\right)mZ(q)}{3N_{sc}{k_{max}}^2\left(q^4T^4 - 2q^2T^2 - 3\right) + 8\pi^2{M_p}^2\left(q^2T^2+3\right)^2}. \tag{76}$$

At $T \to \infty$ the corresponding static limit is

$$\tilde{\Phi}_q = -\frac{24\pi^2 mZ(q)}{(3N_{sc}{k_{max}}^2 + 8\pi^2{M_p}^2)q^2}, \tag{77}$$

which implies that the vacuum polarization leads to renormalization (increasing) of the Planck mass, i.e., decreasing the gravitational constant. In particular, using the value (11) obtained from the cosmological observations [9] gives

$$M^2_{p\,ren} = \left(1 + \frac{54N_{sc}}{\pi^2(2+N_{sc})}\right)M_p^2, \qquad G_{ren} = G/\left(1 + \frac{54N_{sc}}{\pi^2(2+N_{sc})}\right). \tag{78}$$

It seems that the vacuum polarization, in some sense, acts like antigravitation, and the gravitational constant G_{ren} appearing in Newton's law has to differ from the gravitational constant G in the Friedmann equations for the uniform universe. Although the gravitational constant's renormalization does not influence the cosmological balance of the different kinds of matter expressed in units of the critical density $M_p^2H^2$, it should be taken into account in comparison with the directly measured (for instance, utilizing luminosity) density. Numerically $N_{sc} = 2$ gives $G_{ren} \approx 0.27\,G$.

6.1. Invariant potentials and rotational curve

Astrophysicists are trying to express the results of observations in terms of gauge-invariant quantities, which are not dependent on system of coordinates. Potentials $\Phi(\eta, \boldsymbol{x})$ and $F(\eta, \boldsymbol{x})$ are not invariant relatively the infinitesimal transformations of coordinates and time of the following type

$$t = \eta + \xi_1(\eta, \boldsymbol{x}), \qquad \boldsymbol{r} = \boldsymbol{x} + \boldsymbol{\nabla}\xi_2(\eta, \boldsymbol{x}), \tag{79}$$

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where $\xi_1(\eta, \boldsymbol{x})$ and $\xi_2(\eta, \boldsymbol{x})$ are some small functions. Usually the potentials $\Phi_{inv}(\eta, \boldsymbol{x})$ and $\Psi_{inv}(\eta, \boldsymbol{x})$ are introduced [39–41] which are invariant relatively transformations (79). The potentials correspond to the metric

$$ds^2 = a^2(\eta)\left((1 + 2\Phi_{inv}(\eta, \boldsymbol{x}))d\eta^2 - (1 - 2\Psi_{inv}(\eta, \boldsymbol{x}))\,\delta_{ij}dx^i dx^j\right) \tag{80}$$

and are expressed though Φ and F as

$$\Phi_{\boldsymbol{q}\,inv}(\eta) = \Phi_{\boldsymbol{q}}(\eta) + \frac{a'(\eta)F'_{\boldsymbol{q}}(\eta) + a(\eta)F''_{\boldsymbol{q}}(\eta)}{2a(\eta)} = \Phi_{\boldsymbol{q}} + \frac{F_{\boldsymbol{q}}}{2T^2}, \tag{81}$$

$$\Psi_{\boldsymbol{q}\,inv}(\eta) = -\frac{a'(\eta)F'_{\boldsymbol{q}}(\eta)}{2a(\eta)} - \Phi_{\boldsymbol{q}}(\eta) + \frac{1}{6}q^2 F_{\boldsymbol{q}}(\eta) = -\Phi_{\boldsymbol{q}}(\eta) + \frac{1}{6}q^2 F_{\boldsymbol{q}}, \tag{82}$$

where the final equalities at the right hand side of (81), (82) hold for our case $a = const$, $\Phi, F \sim \exp(\eta/T)$. Using (75), (76) gives

$$\tilde{\Phi}_{\boldsymbol{q}\,inv} = -\frac{24\pi^2 T^2\,(q^2T^2 + 3)\,m\,Z(q)}{3N_{sc}\,{k_{max}}^2\,(q^4T^4 - 2q^2T^2 - 3) + 8\pi^2 {M_p}^2\,(q^2T^2 + 3)^2}, \tag{83}$$

and $\tilde{\Psi}_{\boldsymbol{q}\,inv} = \tilde{\Phi}_{\boldsymbol{q}\,inv}$. Thus, we obtained the Fourier transformation of the time-dependent gravitational potential $\Phi_{\boldsymbol{q}\,inv} = \tilde{\Phi}_{\boldsymbol{q}\,inv}e^{\eta/T}$ allowing to establish

$$\Phi_{inv}(\boldsymbol{x}, \eta) = \frac{e^{\eta/T}}{(2\pi)^3}\int \tilde{\Phi}_{\boldsymbol{q}\,inv}\,e^{i\boldsymbol{q}\boldsymbol{x}}d^3\boldsymbol{q}. \tag{84}$$

To obtain a concrete empirical formula, one has to set the form factor $Z(q)$, for instance, using the Gaussian profile $\delta\tilde{\wp}_{eff}(\boldsymbol{x}) = \pi^{-3/2}m\,D^{-3}\,e^{-x^2/D^2}$. The spatial dependence of potential (84) at present time, i.e., $\eta = 0$ allows finding the rotational velocity dependence on the spatial coordinate

$$v_{rot}(r) = \sqrt{-r\frac{d\Phi_{inv}(r)}{dr}}. \tag{85}$$

Here potential (84) is time-dependent, and actually, there are no pure rotational curves because the radial velocities are presented, as well, but for the estimation, we discuss only tangential velocity. The rotational velocity dependence on radial coordinate is shown in Fig. 6. The rotational curve has some similarities with the conventional picture at $N_{sc} = 2$, but in the conventional picture, the contribution of the galactic nuclei, bulge, and disk are taken into account. We include all these components into the Gaussian form factor of galactic nuclei in our oversimplified picture. Then we permit it to increase (or decrease) with time and obtain vacuum polarization caused by this process.

7. Conclusion

We have considered two types of vacuum polarization corresponding to the F and Φ metric perturbations in the conformally-unimodular frame.

The F-type spatially-nonuniform vacuum polarization has the equation of state the same radiation. In the first order on perturbations, one could not determine the form of the static gravitational potential around an astrophysical object. Nevertheless, we propose a nonlinear heuristic model considering the TOV equations for matter and radiation. It was found that the solution, which is nonsingular at $r = 0$, arises only if an eicheon arises. Eicheon is an analog of the black hole in GR and looks like an empty nut in the Schwarzschild type metric. From this point of view, we assume that the dark matter, as a vacuum polarization, arises only in the galaxies having an eicheon (i.e., a "black hole" in the old terminology) in the center. Namely, the eicheon conjecture allows converting a singular solution for pure radiation into a nonsingular physical one. Galaxies without an eicheon in the center (e.g., diffuse galaxies [42]) have not a dark matter halo. One more conclusion is that the dark halo in terms of a test

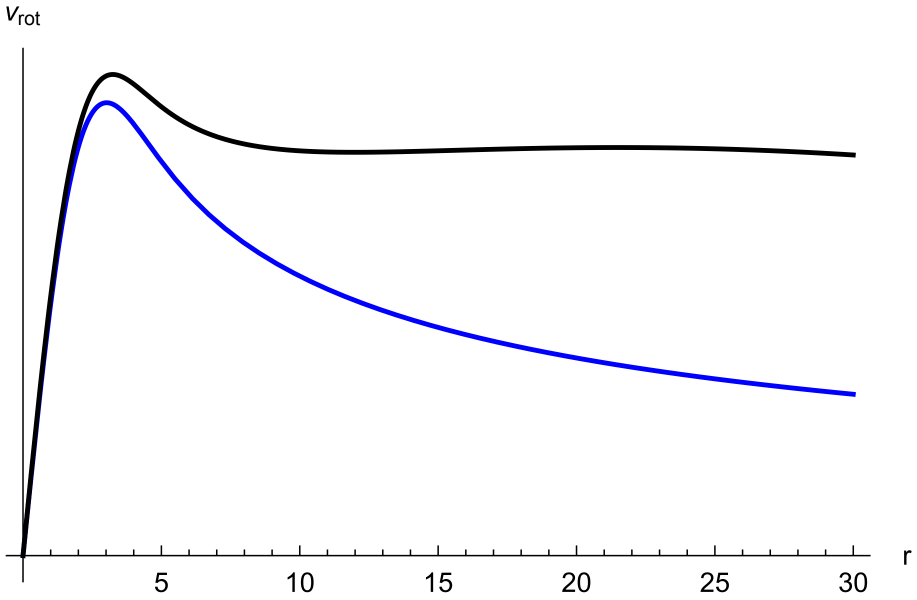

Figure 6. The rotational curves are in arbitrary units, but r-axes units could be associated with kPc. The form factor of Galaxy nuclei is taken as $Z(q) - \exp\left(-\lambda q^2\right)$, $\lambda = 1$, accretion rate is $T = 10$ i.e. 10 kPc, which corresponds to 32000 years. Number of the minimally coupled scalar fields set to $N_{sc} = 2$, and $k^2_{max} = 8M_p^2\pi^2\frac{98}{100}$ is assumed. The lower blue curve corresponds to the contribution of galactic nuclei, the upper curve takes the vacuum Φ-polarization into account.

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particle's rotation velocity always increases with the distance from the center if an isolated galaxy is considered. Decreasing the halo could occur only due to a violation of the isolation of a galaxy, i.e., at a distance $\sim$ 2 MPc. Thus, the dark halo mass must be infinite for an isolated galaxy.

For the Φ-type vacuum polarization, the renormalization of the gravitational constant (or Planck mass) has been found. That means that the gravitational constant found in the Earth, the Solar system, and galaxy observations is not equal (approximately four times less) to the gravitational constant used in cosmology to describe a spatially uniform universe. This fact does not influence the balance of the different kinds of matter in cosmology if one measures them in $M_p^2H^2$, but fourfold increases for the directly counted matter contribution, i.e. the luminous baryonic matter has to contribute 3.7 times stronger into the cosmological Friedmann equations. The second effect of the Φ-type polarization is the creation of the dark halo in the nonstationary process. It is found that the time-dependent evolving mass of the galaxy nuclei produces the gravitational potential of the dark halo type. This point urges a more careful observational investigation of the possibility of the nonstationary origin of the dark halo. However, the required time for the galaxy nuclei mass growth seems very small $\sim$ 32000 years. In such a situation, clarifying the physical status of the possible accretion of vacuum energy and vacuum condensates discussed in [43–45] is very desirable.

To summarize, it is possible to obtain an equation of the state of vacuum polarization, which is some kind of "ether". A result seems trustworthy, but it is challenging to find the "amount" of ether because it depends on the object's entire history due to the nonlocality of the vacuum state on the curved background. Here we have adjusted the "amount" to astrophysical observations. Thus, the final results obtained have, in some sense, a heuristic nature.

Author Contributions: Concepts and methodology are developed by S.C. and V.K.; software, S.C.; validation, writing and editing, S.C. and V.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Appendix A

We emphasize that the presented consideration is heuristic because, although the linear system for the perturbation and the eikonal approximation for vacuum polarization seem trustable, we use its results in the nonlinear TOV model. Another point is that we empirically set the density of radiation (F-type of the vacuum polarization) in the center of an eicheon, i.e., at $r = 0$, of $R = R_i$. That is, we do not calculate it from the first principles, i.e., we use only the equation of state from the eikonal calculations. Let us consider the system of equations (18), (19), (20), (21), (22), (23), for an empty space-time with the vacuum polarization of F-type in the form of radiation fluid. For $e^{4\alpha}\rho = const$ the constant in Eq. (6) could be chosen in such a way, that there is no evolution of the scale factor, i.e., $\alpha = 0$ (a static universe).

For the substance obeying (45), Eqs. (23), (22) are reduced to

$$-\delta\wp'_{\boldsymbol{q}\,vF} + q^2 v_{\boldsymbol{q}\,vF} = 0, \tag{A1}$$

$$\delta\Pi_{\boldsymbol{q}\,vF} + v'_{\boldsymbol{q}\,vF} = 0, \tag{A2}$$

and have the solution

$$\delta\wp_{\boldsymbol{q}\,vF} = c_1 \sin\frac{q\eta}{\sqrt{3}} + c_2\cos\frac{q\eta}{\sqrt{3}}, \tag{A3}$$

$$v_{\boldsymbol{q}\,vF} = \frac{c_2\cos\left(\frac{\eta q}{\sqrt{3}}\right) - c_1\sin\left(\frac{\eta q}{\sqrt{3}}\right)}{\sqrt{3}q}. \tag{A4}$$

Let us also to place into this universe some amount of a dust matter $\delta\wp_{\boldsymbol{q}\,m}$ obeying $\delta\Pi_{\boldsymbol{q}\,m} = 0$ and without uniform component, i.e., $\Pi_m = 0$, $\wp_m = 0$. The complete solution of the system (18), (19), (20), (21), (22), (23) takes the form

$$\delta\wp_{\boldsymbol{q}\,m}(\eta) = -\frac{1}{36}q^4 {M_p}^2(c_6\eta + c_5), \tag{A5}$$

$$v_{\boldsymbol{q}\,m} - \frac{c_6}{36}q^2{M_p}^2, \tag{A6}$$

$$F_{\boldsymbol{q}}(\eta) = c_6\eta + c_5 - 3q^{-4}{M_p}^{-2}\left(\sin\left(\frac{\eta q}{\sqrt{3}}\right)\left(c_4q^2{M_p}^2 + 2\sqrt{3}c_1\eta q + 15c_2\right) + \cos\left(\frac{\eta q}{\sqrt{3}}\right)\left(q\left(c_3q{M_p}^2 - 2\sqrt{3}c_2\eta\right) + 15c_1\right)\right), \tag{A7}$$

$$\Phi_{\boldsymbol{q}}(\eta) = \frac{q^2}{12}(c_6\eta + c_5) - \frac{1}{2M_p^2q^2}\left(6\sin\left(\frac{\eta q}{\sqrt{3}}\right)\left(c_4{M_p}^2q^2 + 2\sqrt{3}c_1\eta q + 9c_2\right) + \cos\left(\frac{\eta q}{\sqrt{3}}\right)\left(q\left(c_3{M_p}^2q - 2\sqrt{3}c_2\eta\right) + 9c_1\right)\right). \tag{A8}$$

Then, in accordance with (38) the energy density for a F-vacuum polarization is expressed approximately as

$$\delta\wp_{vF}(\eta,\boldsymbol{x}) = \frac{1}{2}\sum_{\boldsymbol{k}} - \frac{\boldsymbol{k}\boldsymbol{\nabla}\Theta_{\boldsymbol{k}}}{k} + \Theta'_{\boldsymbol{k}}\ , \tag{A9}$$

which gives

$$\delta\wp_{\boldsymbol{q}\,vF}(\eta) = -2\pi {k_{max}}^4\int_{-\infty}^{\eta}\left(\left(9 - 4q^2(\eta-\tau)^2\right)\sin(q(\eta-\tau)) + q(\eta-\tau)\left(q^2(\eta-\tau)^2 - 9\right)\cos(q(\eta-\tau))\right)\frac{F_{\boldsymbol{q}}(\tau)}{3q(\eta-\tau)^4}d\tau. \tag{A10}$$

If to consider this equation as an additional equation to the system (18), (19), (20), (21), (22), (23), one finds that the constants c_1, c_2, c_3, c_4, c_6 have to be zero and only c_5 term is permitted because

$$\int_{-\infty}^{\eta} \frac{\left(\left(9-4q^2(\eta-\tau)^2\right)\sin(q(\eta-\tau))+q(\eta-\tau)\left(q^2(\eta-\tau)^2-9\right)\cos(q(\eta-\tau))\right)}{3q(\eta-\tau)^4}d\tau=0.$$

Thus, the static gravitational potential

$$\Phi_q = \frac{q^2 c_5(q)}{12} \tag{A11}$$

of arbitrary form (because c_5 could be function of q) is permitted in the framework of a linear system of equations considered.

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