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Shifted Chebyshev spectral method for two–dimensional space–time fractional partial differential equations

Shina D. Oloniiju*, Nolwazi S. Nkomo, Sicelo P. Goqo, Precious Sibanda
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X01, Scottsville, Pietermaritzburg, 3209, South Africa.

Abstract. In this study, we present a numerical method for solving two–dimensional space–time fractional partial differential equations (FPDEs), where the solutions of the FPDEs are expanded in terms of the shifted Chebyshev polynomials. The numerical approximations are evaluated at the Chebyshev–Gauss–Lobatto points. In the case when the FPDE is nonlinear, we employ a Newton–Raphson approach to linearize the equation. Both the linear and nonlinear cases lead to a consistent system of linear algebraic equations. The scheme is tested on selected FPDEs and the numerical results show that the proposed numerical scheme is accurate and computationally efficient in terms of CPU times. To establish the accuracy of the method, we also present an error analysis which shows the convergence of the numerical method. These positive attributes make the proposed method a good approach for solving two–dimensional fractional partial differential equations with both space and time fractional orders.

Key Words and Phrases: Shifted Chebyshev polynomials, Chebyshev–Gauss–Lobatto quadrature, space–time FPDE

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1. Introduction

Fractional partial differential equations (FPDEs) are a generalization of classical partial differential equations to include derivatives of arbitrary order [27]. For the past three centuries, FPDEs were considered to be of little mathematical or practical interest [21, 23]. However, in the last few years, they have been used in applications such as fluid dynamics [5, 34], finance [26, 33] and physics [3, 20]. Fractional partial derivatives provide an excellent instrument for the description of physical systems with inherent memory [2, 8]. Fractional partial derivatives are more flexible in modelling real world dynamical systems.

In many problems of interest, FPDEs have no recognized analytical solution, so approximate and numerical techniques must be used [15]. Several numerical methods such as the finite

*Corresponding author.

Email addresses: shina@aims.edu.gh, mankomolwazinkomo.ln@gmail.com, goqos@ukzn.ac.za, sibandap@ukzn.ac.za

difference method [24, 25], wavelet methods [17, 18], adomian decomposition method [11, 19], predictor–corrector method [22] and spectral methods [7, 32] have been developed for FPDEs. One of the earliest contributions to numerical solutions of FPDEs was by Lynch et al.[24], where they applied an approach based on the $L2$ method proposed by Oldham and Spanier[29] to solve an anomalous diffusion equation. In their study, the FPDEs were fractional in the spatial dimension and the second derivative was approximated by the standard three-point centred finite difference formula. Meerschaert and Tadjeran[28] gave a detailed study of a one-dimensional fractional diffusion equations using the finite difference method. They used a truncated Grunwald–Letnikov derivative evaluated on a shifted grid. Meerschaert and Tadjeran[27] later extended the method to one-dimensional space FPDEs.

The literature on spectral methods for solving FPDEs is comparatively short, although, interest has grown steadily in recent years. The spectral methods are an excellent tool for computing approximate solutions of differential equations because of high-order accuracy (see for instance Bhrawy[6] and Doha et al.[14]). With this high-order accuracy, spectral methods are useful for both temporal and spatial discretizations of FPDEs. Using spectral methods may significantly reduce the storage requirement because fewer time and space grid points are needed to compute smooth solutions. The basic concept of spectral methods is to express the solution of the differential equation as a sum of basis functions and estimate the coefficients of expansion such that approximation error is minimized [7]. Doha et al.[12] derived an operational matrix of the fractional derivatives and used the spectral tau method. The study of Doha et al.[12] was later extended to the more general Jacobi polynomials by Doha et al.[13], but was only applied to fractional ordinary differential equations.

In this study, a purely spectral based method is introduced and applied to solve $(2+1)$ dimensional FPDEs of the initial–boundary value problems type. The solution method involve approximating the variable and its fractional order partial derivatives in terms of the first kind shifted Chebyshev polynomials constructed on the Chebyshev–Gauss–Lobatto quadrature. In the case where the FPDE is nonlinear, we simplify using the quasilinearization method of Bellman and Kalaba[4]. The FPDE is discretized to yield a consistent system of linear algebraic equations. The method is tested using some FPDEs of the initial boundary value type and the numerical results obtained are compared with the exact solutions.

2. Preliminaries and Notations

In this section, we provide some preliminary results on fractional operators and the shifted Chebyshev polynomials of the first kind.

2.1. The fractional operators

Definition 2.2. *The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined as [31]*

$$ {}_0I_x^{\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tilde{x})^{\alpha-1} g(\tilde{x}) d\tilde{x}, \quad \alpha > 0,\ x > 0, \tag{1a}$$

$$ {}_0I_x^{0} g(x) = g(x), \tag{1b}$$

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where Γ *is the Euler gamma function.*

The fractional integral of a power function x^j is given as

$$I^{\alpha} x^{j} = \frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)} x^{j+\alpha}. \tag{2}$$

Definition 2.3. *We defined the corresponding differential operator of order* α *in the Caputo sense as [31]*

$$ {}_0D_x^{\alpha} g(x) = {}_0I_x^{n-\alpha}\left(\frac{d^n}{dx^n} g(x)\right) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{d^n g(\tilde{x})}{d\tilde{x}^n} \frac{d\tilde{x}}{(x-\tilde{x})^{\alpha+1-n}}, & n-1 < \alpha < n, \\ \frac{d^n g(x)}{dx^n}, & \alpha = n. \end{cases} \tag{3}$$

We also define the fractional derivative of order α of x^j, which will be useful later in this study as

$$ {}_0D_x^{\alpha} x^j = \begin{cases} 0 & j \in \mathbb{N}_0,\ j < \lceil \alpha \rceil, \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} x^{j-\alpha}, & j \in \mathbb{N}_0,\ j \geq \lceil \alpha \rceil, \end{cases} \tag{4}$$

where $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α.

2.4. The shifted Chebyshev polynomials

Consider the Sturm–Liouville eigenvalue problem

$$\left(\sqrt{1-x^2} T_n'(x)\right)' + \frac{n^2}{\sqrt{1-x^2}} T_n(x) = 0, \quad -1 \leq x \leq 1,\ n = 0, 1, \ldots. \tag{5}$$

The Chebyshev polynomials are eigenvalue solutions of the eigenvalue problem, defined as [1]

$$T_n(x) = \cos(n \arccos x), \quad n = 0, 1, \ldots, \quad x \in [-1, 1], \tag{6}$$

and are defined recurrently as

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x), \quad n = 1, 2, \ldots, \tag{7}$$

where the zeroth and first order polynomials are respectively defined as $T_0(x) = 1$ and $T_1(x) = x$. We shall consider the interval $\tilde{x} \in [0, 1]$, hence we define the shifted Chebyshev polynomials by using the affine mapping $x = 2\hat{x} - 1$. Therefore the shifted Chebyshev polynomials can be generated through the recurring formula

$$\hat{T}_{n+1}(\hat{x}) = 2(2\hat{x} - 1)\hat{T}_n(\hat{x}) - \hat{T}_{n-1}(\hat{x}), \quad 0 \leq \hat{x} \leq 1, \quad n = 1, 2, \ldots, \tag{8}$$

where $\hat{T}_0(\hat{x}) = 1$ and $\hat{T}_1(\hat{x}) = 2\hat{x} - 1$, and the polynomials can be expanded in series form as (dropping the hat for brevity)

$$T_n(x) = n \sum_{j=0}^{n} \frac{(-1)^{n-j}(n+j-1)!2^{2j}}{(n-j)!(2j)!} x^j. \tag{9}$$

The shifted Chebyshev polynomials of the first kind satisfy the orthogonality condition

$$\int_0^1 T_n(x)T_m(x)w(x)dx = \delta_{mn}h_n, \tag{10}$$

where $w(x)$ is a weight function defined as $1/\sqrt{x - x^2}$ and $h_n = c_n\pi/2$, with $c_0 = 2$ and $c_n = 1$ for $n \geq 1$. The shifted Chebyshev–Gauss–Lobatto points on which the interpolation is performed are extrema of $T_n(x)$ on the interval $x \in [0, 1]$ defined as

$$x_j = \frac{1}{2}\left(1 - \cos\left(\frac{j\pi}{N}\right)\right), \quad 0 \leq j \leq N. \tag{11}$$

The corresponding Christoffel numbers are the same as those of the standard Chebyshev–Gauss–Lobatto quadrature and are defined as $\varpi_j = \pi/c_jN$, $0 \leq j \leq N$, where $c_0 = c_N = 2$ and $c_j = 1$ for $j = 1, 2, \ldots, N-1$.

3. Solution method

In this section, we describe the scheme for solving a $(2 + 1)$ dimensional fractional partial differential equations. For conveniency, we divide this section into three subsections. In Section 3.1, we propose the arbitrary derivative of a square integrable function in terms of the shifted Chebyshev polynomials. The resulting fractional differentiation matrix is used to approximate the derivatives of the dependent variable using Chebyshev–Gauss–Lobatto quadrature. In Section 3.4, we construct a scheme for solving linear fractional partial differential equations and in Section 3.5 we describe an iterative scheme for finding the solution of nonlinear fractional partial differential equations based on quasilinearization and Chebyshev collocation approaches.

3.1. Function and derivatives approximations

Consider a smooth function $u(x)$ defined on the interval $[0, 1]$, then $u(x)$ can be approximated in terms of the truncated shifted Chebyshev polynomials as

$$u(x) \approx U_N(x) = \sum_{n=0}^{N} U_nT_n(x), \tag{12}$$

where the coefficients U_n are given to satisfy the orthogonality condition, and we write in discrete form as

$$U_n = \frac{1}{h_n}\sum_{j=0}^{N} U(x_j)T_n(x_j)\varpi_j, \quad n = 0, \ldots, N. \tag{13}$$

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Therefore,

$$U_N(x_k) = \sum_{j=0}^{N} \left[\varpi_j \sum_{n=0}^{N} \frac{1}{h_n} T_n(x_j) T_n(x_k) \right] U(x_j), \qquad k = 0, 1, \ldots, N. \tag{14}$$

In order to approximate the arbitrary order derivative of $U_N(x)$, we first obtain the fractional derivative of the shifted Chebyshev polynomials.

Lemma 3.2. *The fractional derivative of the shifted Chebyshev polynomial is (see [12, 30])*

$${}_0D_x^{\alpha} T_n(x) = \sum_{k=0}^{N} D_{n,k}^{(\alpha)} T_k(x), \tag{15}$$

where

$$D_{n,k}^{(\alpha)} = n \sum_{j=0}^{n} \frac{(-1)^{n-j}(n+j-1)!2^{2j}}{(n-j)!(2j)!} \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} q_{j,k}, \tag{16}$$

and $q_{j,k}$ is given as

$$q_{j,k} = \begin{cases} 0 & j = 0, 1, \ldots \lceil \alpha \rceil - 1, \\ \frac{k\sqrt{\pi}}{h_k} \sum_{r=0}^{k} \frac{(-1)^{k-r}(k+r-1)!2^{2r}}{(k-r)!(2r)!} \frac{\Gamma(j-\alpha+r+1/2)}{\Gamma(j-\alpha+r+1)}, & j = \lceil \alpha \rceil, \lceil \alpha \rceil + 1, \ldots, N, \\ & k = 0, 1, \ldots, N. \end{cases} \tag{17}$$

Theorem 3.3. *The arbitrary order derivative of $u(x)$ is given as*

$${}_0D_x^{\alpha} u(x_l) \approx D^{\alpha} U_N(x_l) = \sum_{j=0}^{N} \mathcal{D}_{j,l}^{\alpha} U(x_j) = \mathcal{D}^{\alpha} U, \tag{18}$$

where the entries of $\mathcal{D}^{\alpha}$ are defined as

$$\mathcal{D}_{j,l}^{\alpha} = \varpi_j \sum_{n=0}^{N} \sum_{k=0}^{N} \frac{1}{h_n} T_n(x_j) D_{n,k}^{(\alpha)} T_k(x_l), \quad j, l = 0, 1, \ldots, N. \tag{19}$$

Proof. If we use the result of Lemma 3.2 and the expression in Equation (13), together with the expansion in terms of the shifted Chebyshev polynomials given in Equation (12), the theorem is proved.

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3.4. Linear FPDEs

In this section, we describe the scheme for solving linear FPDEs. We consider a linear FPDE with variable coefficients of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u + \sum_{d=1} f_d(x,y)\frac{\partial^{\beta_d} u}{\partial x^{\beta_d}} + \sum_{e=1} g_e(x,y)\frac{\partial^{\gamma_e} u}{\partial y^{\gamma_e}} + q(x,y,t), \quad (x,y) \in (0,1)\times(0,1),\ t \in (0,1], \tag{20}$$

where $0 < \alpha < 1$, $\beta_1 < \beta_2 < \ldots$, $\gamma_1 < \gamma_2 < \ldots$ and $d-1 < \beta_d < d$ $(d = 1, 2, \ldots,)$, $e - 1 < \gamma_e < e$ $(e = 1, 2, \ldots)$. Moreover, $f_d(x, y)$ $(d = 1, 2, \ldots,)$ and $g_e(x, y)$ $(e = 1, 2, \ldots,)$ are functions of x and y and $q(x, y, t)$ is dependent on all three variables. Equation (20) is solved subject to the initial condition

$$u(x,y,0) = \mathcal{I}(x,y), \tag{21}$$

and boundary conditions

$$\begin{aligned} u(0,y,t) &= \mathcal{B}_0^x(y,t), \quad u(1,y,t) = \mathcal{B}_1^x(y,t),\\ u(x,0,t) &= \mathcal{B}_0^y(x,t), \quad u(x,1,t) = \mathcal{B}_1^y(x,t), \end{aligned} \tag{22}$$

where $\mathcal{I}(x,y), \mathcal{B}_0^x(y,t), \mathcal{B}_1^x(y,t), \mathcal{B}_0^y(x,t), \mathcal{B}_1^y(x,t)$ are known functions. If $\alpha > 1$, additional initial condition will be needed to make the differential equation well–posed. We define the fractional order derivative with respect to x approximated at the interpolation points (x_i, y_j, t_k) for $i = 0, 1, \ldots, N_x$, $j = 0, 1, \ldots, N_y, k = 0, 1, \ldots, N_t$ as follows

$$\frac{\partial^\beta}{\partial x^\beta} u(x,y,t) \approx \frac{\partial^\beta}{\partial x^\beta} U(x_i,y_j,t_k) = \mathcal{D}^\beta U_{j,k}, \tag{23}$$

where $U_{j,k} = [U(x_0, y_j, t_k), U(x_1, y_j, t_k), \ldots, U(x_{N_x}, y_j, t_k)]^T, j = 0, 1, \ldots, N_y,\ k = 0, 1, \ldots, N_t$. Fractional order derivatives with respect to y and t are approximated in similarly manner. Therefore, we can approximate Equation (20) in terms of the shifted Chebyshev polynomial as

$$\begin{aligned} &\sum_{k=1}^{N_t}\sum_{j=1}^{N_y-1} \mathcal{D}^\alpha_{k,n} U_{j,k} - \sum_{j=1}^{N_y-1}\left[\sum_{e=1} G_e \mathcal{D}^{\gamma_e}_{j,m}\right] U_{j,k} - \left[\sum_{d=1} F_d \mathcal{D}^{\beta_d}\right] U_{j,k} - U_{j,k} = Q_{j,k},\\ &\quad n = 0, 1, \ldots, N_t,\ m = 0, 1, \ldots, N_y, \end{aligned} \tag{24}$$

where $Q_{j,k} = [Q(x_0, y_j, t_k), Q(x_1, y_j, t_k), \ldots, Q(x_{N_x}, y_j, t_k)]^T, j = 0, 1, \ldots, N_y,\ k = 0, 1, \ldots, N_t$ is the approximation of $q(x, y, t)$ on the Chebyshev–Gauss–Lobatto points (x_i, y_j, t_k). The above is a linear algebraic system which when combined with the initial and boundary conditions evaluated at the interpolation points as

$$U(x_0,y_j,t_k) = \mathcal{B}_0^x(y_j,t_k), \quad U(x_{N_x},y_j,t_k) = \mathcal{B}_1^x(y_j,t_k), \tag{25}$$

$$U(x_i,y_0,t_k) = \mathcal{B}_0^y(x_i,t_k), \quad U(x_i,y_{N_y},t_k) = \mathcal{B}_1^y(x_i,t_k), \tag{26}$$

and

$$U(x_i,y_j,t_0) = \mathcal{I}(x_i,y_j) \tag{27}$$

yields a consistent system.

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3.5. Nonlinear FPDEs

In this section, we detail how to use the proposed method to solve nonlinear FPDEs. Consider the nonlinear FPDE

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = F\left(u, \frac{\partial^{\beta_1} u}{\partial x^{\beta_1}}, \frac{\partial^{\beta_2} u}{\partial x^{\beta_2}}, \ldots, \frac{\partial^{\gamma_1} u}{\partial y^{\gamma_1}}, \frac{\partial^{\gamma_2} u}{\partial y^{\gamma_2}}, \ldots\right) + q(x, y, t), \tag{28}$$

with the initial and boundary conditions in Equations (21) and (22) and F is a nonlinear operator. In order to solve Equation (28), we first linearize using the quasilinearization technique of [4]. This quasilinearization method is based on the Newton–Raphson approach and developed from the linear expansion in terms of the Taylor's series about an initial solution. Applying the approach on Equation (28), we obtain the iterative scheme

$$\begin{aligned} &\frac{\partial^{\alpha} u_{r+1}}{\partial t^{\alpha}} - \left(\frac{\partial F_r}{\partial u}\right) u_{r+1} - \left(\frac{\partial F_r}{\partial D_x^{\beta_1} u}\right) \frac{\partial^{\beta_1} u_{r+1}}{\partial x^{\beta_1}} - \left(\frac{\partial F_r}{\partial D_x^{\beta_2} u}\right) \frac{\partial^{\beta_2} u_{r+1}}{\partial x^{\beta_2}} - \ldots \\ &\quad - \left(\frac{\partial F_r}{\partial D_y^{\gamma_1} u}\right) \frac{\partial^{\gamma_1} u_{r+1}}{\partial y^{\gamma_1}} - \left(\frac{\partial F_r}{\partial D_y^{\gamma_2} u}\right) \frac{\partial^{\gamma_2} u_{r+1}}{\partial y^{\gamma_2}} - \ldots = R_r + q(x, y, t), \end{aligned} \tag{29}$$

where

$$\begin{aligned} R_r = &\left(\frac{\partial F_r}{\partial u}\right) u_r - \left(\frac{\partial F_r}{\partial D_x^{\beta_1} u}\right) D_x^{\beta_1} u_r - \left(\frac{\partial F_r}{\partial D_x^{\beta_2} u}\right) D_x^{\beta_2} u_r - \ldots - \left(\frac{\partial F_r}{\partial D_y^{\gamma_1} u}\right) D_y^{\gamma_1} u_r \\ &- \left(\frac{\partial F_r}{\partial D_y^{\gamma_2} u}\right) D_y^{\gamma_2} u_r - \ldots - F_r, \end{aligned} \tag{30}$$

and r in this case signifies the previous iteration. Equation (29) can then be expanded in terms of shifted Chebyshev polynomials, when combined with the initial and boundary conditions form a consistent linear algebraic system.

4. Space of fractional derivative and convergence analysis

In this section, we establish some functional spaces of fractional derivatives and then present a convergence analysis of the spectral method for a space–time fractional partial differential equation of the form Equation (20). We introduce the domain $\Omega = \Phi \times \Upsilon$, where $\Phi = (0, 1]$ and $\Upsilon = [0, 1] \times [0, 1]$. Define $(\cdot, \cdot)_{\Omega}$ and $\|\cdot\|_{0,\Omega}$ the inner product and norm $L^2(\Omega)$ respectively and assume that $u(x, y, t), f_d(x, y)|_{d\geq 1}, g_e(x, y)|_{e\geq 1}$ and $q(x, y, t)$ are defined in the space of smooth functions.

Definition 4.1. *Define the space $H^m(\Phi), m \geq 0$, the Sobolev space [9, 10]*

$$H^m(\Phi) = \left\{u \in L^2(\Phi) : \frac{\partial^j u}{\partial t^j} \in L^2(\Phi), 0 \leq j \leq m\right\}, \tag{31}$$

endowed with the respective weighted semi–norm and norm

$$|u|_{m,\Phi} = \left(\left\| \frac{\partial^m u}{\partial t^m} \right\|^2_{L^2(\Phi)} \right)^{1/2}, \qquad \|u\|_{m,\Phi} = \left(\sum_{j=0}^{m} \left\| \frac{\partial^j u}{\partial t^j} \right\|^2_{L^2(\Phi)} \right)^{1/2}. \tag{32}$$

Definition 4.2. *We define the fractional Sobolev space $H^\alpha(\Phi)$ for any $\alpha > 0$ as [9, 16]*

$$H^\alpha(\Phi) = \left\{ u \in L^2(\Phi) : {}_0D^\alpha u \in L^2(\Phi) \right\}, \tag{33}$$

such that the semi–norm and norm associated with the space are defined respectively as

$$|u|_{\alpha,\Phi} = \|{}_0D^\alpha u\|_{L^2(\Phi)}, \qquad \|u\|_{\alpha,\Phi} = \left(\|u\|^2_{L^2(\Phi)} + |u|^2_{\alpha,\Phi} \right)^{1/2}. \tag{34}$$

Lemma 4.3. *For $u \in H^\alpha(\Phi)$, if $0 < p < \alpha$, then there is a non–negative constant C such that [16]*

$$\|u\|_{p,\Phi} \leq C\|u\|_{\alpha,\Phi}. \tag{35}$$

Lemma 4.4. *Let $u \in H^m(\Phi)$ and T_N be the expansion in terms of $(N+1)$ Chebyshev–Gauss–Lobatto nodes, then the error is estimated as [10]*

$$\|u - T_N u\|_{l,\Phi} \leq CN^{2l-m}\|u\|_{m,\Phi}, \; l \leq m \tag{36}$$

$$\|u - T_N u\|_{L^2_w(\Phi)} \leq CN^{-m}\|u\|_{m,\Phi}. \tag{37}$$

Lemma 4.5. *Assume that U_{N,N_t}, with $N = (N_x + N_y + 2)$ be the orthogonal projection operator of $L^2(\Omega))$ onto $T_{N,N_t}(\Omega)$, then for all $m, n \geq 0$, there exist a positive constant C not dependent on N and N_t such that [9, 10]*

$$\|u - T_{N,N_t} u\|_{L^2_w(\Omega)} \leq C\left(N^{-m}\|u\|_{m,0} + N_t^{-n}\|u\|_{0,n} \right). \tag{38}$$

Theorem 4.6 (Convergence)**.** *Assume that u is the exact solution of Equation (20) and its approximation is given as U, and $q \in C(\Omega) \cap H^{0,p_2}(\Phi; H^{p_1,0}(\Upsilon))$, then*

$$\|u - U\|_{L^2_w(\Omega)} \to 0. \tag{39}$$

Proof. Considering the integration of Equation (20), we have

$$u = {}_0I_t^{\alpha} u + \sum_{d=1} f_d \, {}_0I_t^{\alpha} \frac{\partial^{\beta_d} u}{\partial x^{\beta_d}} + \sum_{e=1} g_e \, {}_0I_t^{\alpha} \frac{\partial^{\gamma_e} u}{\partial y^{\gamma_e}} + {}_0I_t^{\alpha} q, \tag{40}$$

and let U represents the approximation of u which uses N_x truncated Chebyshev approximation in x, N_y truncated Chebyshev expansion in y and N_t truncated expansion in t is given as

$$U = {}_0I_t^{\alpha} U + \sum_{d=1} f_d \, {}_0I_t^{\alpha} \frac{\partial^{\beta_d} U}{\partial x^{\beta_d}} + \sum_{e=1} g_e \, {}_0I_t^{\alpha} \frac{\partial^{\gamma_e} U}{\partial y^{\gamma_e}} + {}_0I_t^{\alpha} Q. \tag{41}$$

In view of Equations (40) and (41), we obtain

$$\begin{aligned} \|u - U\|_{L^2_w(\Omega)} \leq \|{}_0I_t^{\alpha}(u-U)\|_{L^2_w\Omega} + \left\| \sum_{d=1} f_d \, {}_0I_t^{\alpha} \frac{\partial^{\beta_d}(u-U)}{\partial x^{\beta_d}} \right\|_{L^2_w(\Omega)} + \left\| \sum_{e=1} g_e \, {}_0I_t^{\alpha} \frac{\partial^{\gamma_e}(u-U)}{\partial y^{\gamma_e}} \right\|_{L^2_w(\Omega)} \\ + \|{}_0I_t^{\alpha}(q-Q)\|_{L^2_w(\Omega)}. \end{aligned} \tag{42}$$

For a Chebyshev–Gauss–Lobatto quadrature and using the properties of a Sobolev norm and Young's inequality, we have

$$\begin{aligned} |({}_0I_t^{\alpha}(q-Q))|_{L^2_w(\Omega)} &= |({}_0I_t^{\alpha}(q - T_{N,N_t}q + T_{N,N_t}q - Q))| \\ &\leq |({}_0I_t^{\alpha})||(q - T_{N,N_t}q + T_{N,N_t}q - Q)| \\ &\leq C|(q - T_{N,N_t}q + T_{N,N_t}q - Q)| \\ &\leq C(|(q - T_{N,N_t}q)| + |(T_{N,N_t}q - Q)|) \\ &\leq C(\|(q - T_{N,N_t}q\|_0 + \|T_{N,N_t}q - Q)\|_0) \\ &\leq C(N^{-p_1}\|q\|_{p_1,0} + N_t^{-p_2}\|q\|_{0,p_2}). \end{aligned} \tag{43}$$

Therefore, Equation (42) becomes

$$\begin{aligned} \|u - U\|_{L^2_w(\Omega)} &\leq \left(|{}_0I_t^{\alpha}||u-U|_0 + \left| \sum_{d=1} f_d \, {}_0I_t^{\alpha} \right| |u-U|_{\beta_d} + \left| \sum_{e=1} g_e \, {}_0I_t^{\alpha} \right| |u-U|_{\gamma_e} \right) \\ &\quad + C(N^{-p_1}\|q\|_{p_1,0} + N_t^{-p_2}\|q\|_{0,p_2}) \qquad (44) \\ &\leq C\left(\|u-U\|_0 + \|u-U\|_{\beta_d} + \|u-U\|_{\gamma_e} + N^{-p_1}\|q\|_{p_1,0} + N_t^{-p_2}\|q\|_{0,p_2} \right). \qquad (45) \end{aligned}$$

In order to estimate $\|u-U\|_\beta$, we have

$$\begin{aligned}\|u-U\|_\beta &\leq \|u-T_{N,N_t}u\|_{\beta,0}+\|T_{N,N_t}u-U\|_{\beta,0}\\ &\leq N^{2\beta-m}\|u\|_{m,0}+N_t^{-n}\|u\|_{0,n}.\end{aligned} \tag{46}$$

For any $\beta_d(d=1,2,\ldots)$ or $\gamma_e(e=1,2,\ldots)$, there exist non negative constants C_d or C_e such that (by Lemma 4.3)

$$\|u-U\|_{\beta_d}\leq C_d\|u-U\|_{\beta,0} \quad \text{or} \quad \|u-U\|_{\gamma_e}\leq C_e\|u-U\|_{\gamma,0}. \tag{47}$$

Therefore, we have

$$\begin{aligned}\|u-U\|_{L^2_w(\Omega)}\leq C\Big(&N^{-m}\|u\|_{m,0}+N_t^{-n}\|u\|_{0,n}+N^{2\beta-m}\|u\|_{m,0}+N_t^{-n}\|u\|_{0,n}\\ &+N^{2\gamma-m}\|u\|_{m,0}+N_t^{-n}\|u\|_{0,n}+N^{-p_1}\|q\|_{p_1,0}+N_t^{-p_2}\|q\|_{0,p_2}\Big),\end{aligned} \tag{48}$$

where C is non–negative and not dependent on N_x, N_y and N_t.

5. Numerical examples

In this section, we applied the numerical scheme to selected linear and nonlinear two–dimensional time–space fractional partial differential equations of the initial–boundary value type. The accuracy and efficiency of the method is demonstrated by comparing the results with exact solutions.

Example 5.1. *We consider the two–dimensional time–space fractional diffusion equation with variable coefficients on a finite domain*

$$\frac{\partial^{0.8}u}{\partial t^{0.8}}=f(x,y)\frac{\partial^{1.8}u}{\partial x^{1.8}}+g(x,y)\frac{\partial^{1.5}u}{\partial y^{1.5}}+q(x,y,t), \tag{49}$$

where $f(x,y)$ and $g(x,y)$ are the diffusion coefficients. We investigate the case where

$$f(x,y)=\frac{x^{1.8}}{\Gamma(3.8)}, \quad g(x,y)=\frac{\Gamma(2.5)}{6}y^{1.5}, \tag{50}$$

and subject to the boundary conditions

$$u(0,y,t)=0, \quad u(1,y,t)=y^3(1+t^2), \quad 0\leq y\leq 1,\ 0<t\leq 1 \tag{51}$$

$$u(x,0,t)=0, \quad u(x,1,t)=x^{2.8}(1+t^2), \quad 0\leq x\leq 1,\ 0<t\leq 1. \tag{52}$$

The initial condition for this FPDE is given as

$$u(x,y,0)=x^{2.8}y^3, \quad 0\leq x\leq 1,\ 0\leq y\leq 1, \tag{53}$$

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and the source term $q(x, y, t)$ is defined as

$$q(x, y, t) = -2\left(t^2 - \frac{t^{1.2}}{\Gamma(2.2)} + 1\right) x^{2.8}y^3. \tag{54}$$

The exact solution is given in [35] as

$$u(x, y, t) = x^{2.8}y^3(1 + t^2). \tag{55}$$

Example 5.2. *Consider the two–dimensional generalized space–time fractional diffusion equation with variable coefficients (see Zheng and Zhang[35])*

$$\frac{\partial^\alpha u}{\partial t^\alpha} = f(x, y)\frac{\partial^\beta u}{\partial x^\beta} + g(x, y)\frac{\partial^\gamma u}{\partial y^\gamma} + q(x, y, t), \quad 0 < \alpha \leq 1,\ 1 < \beta, \gamma \leq 2. \tag{56}$$

In this test problem, the explicit solution is chosen as

$$u(x, y, t) = x^2y^3t^2, \tag{57}$$

such that when we substitute the solution into the equation, we obtain the diffusion coefficients as

$$f(x, y) = \frac{(3 - 2x)\Gamma(3 - \beta)}{2}, \quad g(x, y) = \frac{(4 - y)\Gamma(4 - \gamma)}{6}, \tag{58}$$

and the source term as

$$q(x, y, t) = \frac{2}{\Gamma(3 - \alpha)}x^2y^3t^{2-\alpha} + y^3t^2(2x^{3-\beta} - 3x^{2-\beta}) + x^2t^2(y^{4-\gamma} - 4y^{3-\gamma}). \tag{59}$$

The FPDE is solved subject to the initial condition

$$u(x, y, 0) = 0, \tag{60}$$

and boundary conditions

$$u(0, y, t) = 0, \quad u(1, y, t) = y^3t^2, \quad 0 \leq y \leq 1,\ 0 < t \leq 1, \tag{61}$$
$$u(x, 0, t) = 0, \quad u(x, 1, t) = x^2t^2, \quad 0 \leq x \leq 1,\ 0 < t \leq 1. \tag{62}$$

Example 5.3. *We consider the nonlinear two–dimensional space–time Fisher's partial differential equations [35]*

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\beta u}{\partial x^\beta} + \frac{\partial^\gamma u}{\partial y^\gamma} + u(1 - u) + q(x, y, t), \quad 0 < \alpha \leq 1,\ 1 < \beta, \gamma \leq 2, \tag{63}$$

with exact solution

$$u(x, y, t) = x^2y^2t. \tag{64}$$

The equation is solved subject to the boundary conditions

$$u(0,y,t)=0, \quad u(1,y,t)=y^2t, \quad 0\le y\le 1,\ 0<t\le 1, \tag{65}$$
$$u(x,0,t)=0, \quad u(x,1,t)=x^2t, \quad 0\le x\le 1,\ 0<t\le 1, \tag{66}$$

and the initial condition

$$u(x,y,0)=0. \tag{67}$$

The function $q(x,y,t)$ *is defined as*

$$q(x,y,t)=\frac{1}{\Gamma(2-\alpha)}x^2y^2t^{1-\alpha}-\frac{2}{\Gamma(3-\beta)}x^{2-\beta}y^2t-\frac{2}{\Gamma(3-\gamma)}x^2y^{2-\gamma}t-x^2y^2t + x^4y^4t^2. \tag{68}$$

6. Result and Discussion

This section discusses the numerical results obtained by applying the method to Examples 5.1–5.3. We focus on the accuracy of the scheme and the ease of implementation. The numerical simulations were made using the PYTHON programming language run on a computer with *Intel Core i5–7200U, CPU @ 2.50 GHz*, and *8 GB DDR4* installed memory. The results show how the orders of the shifted Chebyshev polynomials in each variable affect the accuracy of the numerical method. In order to determine the accuracy of the method, we define the infinity error norm which measures the maximum of the absolute values of the difference between the exact and numerical solutions

$$\|\varepsilon\|_\infty=\|u(x_i,y_j,t_k)-U(x_i,y_j,t_k)\|_\infty, \tag{69}$$

where u and U are respectively the exact and numerical solutions. This is a good measure for accuracy because we expect that the difference between the exact and numerical solutions at every grid points to be close to zero.

Figure 1 shows the numerical and exact solutions for Example 5.1. It can be seen that both solutions are in agreement. The errors presented in Tables 1 and 2 are associated with Example 5.1. In Table 1, the number of grid points in both the spatial and temporal domains are varied. The result show accurate results for different combinations of the numbers of grid points and/or orders of the shifted Chebyshev polynomials. We observe that the numerical solutions become better as the orders of the polynomials increase. The table also shows the condition number of the matrix, as well as the computational time. The condition numbers are obtained using the PYTHON package "*numpy.linalg.cond*". Both the condition number and the computational time increase as the combination of the grid points increases. This is explained by the higher number of algebraic equations that need to be solved with an increase in number of grid points. For instance, in the case when $N_x=N_y=15$ and $N_t=10$, there are 2816 coupled linear algebraic equations to be solved, and when $N_x=N_y=N_t=7$, there are 512 linear algebraic equations. This well–conditioned system of equations was solved and accurate solutions obtained in less than 2 seconds, thus showing that the method is not only accurate but also efficient. The high number of coupled linear algebraic equations solved in few seconds establish the efficiency of the method. Given that fractional derivatives are non–local, we remark that this is a fairly good result. Table 2 shows the comparison between the exact solution and the approximate solution obtained at selected points (x, y, t). We observe that the magnitude of the error at these selected points is small.

The numerical solutions of Example 5.2 at $t = 0.5$ and $t = 1$ are shown in Figure 2. These solutions are in agreement with the exact solutions. In Tables 3 and 4, we present the error norm, condition number and computational time for Example 5.2 for different combinations of the numbers of terms in the polynomials and different combination of the fractional orders respectively. Again, we observe accuracy in the results which is evident from the magnitude of the error norms and efficiency, evident from the CPU time required to solve the system of algebraic equations. Table 4 shows the accuracy for different values of $\alpha \in (0, 1)$ and $\beta, \gamma \in (1, 2)$ for fixed numbers of grid points $N_x = N_y = N_t = 10$. In Table 5, we present the error norms for Example 5.3 which is a nonlinear space–time fractional differential equation. The results presented in Table 5 are obtained after the sixth iteration. A distinctive difference between the result obtained in Example 5.3 and that obtained in Examples 5.1 and 5.2 is the computational time to solve the resulting system of algebraic equation in Example 5.3. The CPU time in Example 5.3 is higher than that in Examples 5.1 and 5.2. This owes to the fact that Example 5.3 is solved through an iterative process. Figure 3 depicts the numerical solution and absolute error for Example 5.3 at $t = 0.5$ and $t = 1$ with $\alpha = 9/10$ and $\beta = \gamma = 1.9$. One obvious advantage of the method over many other methods that have been used to find solutions of FPDEs is the fact that it only requires small numbers of terms of the shifted Chebyshev polynomials, and as a result small numbers of grid points to achieve accuracy.

Although, we assumed that the solution $u(x, y, t)$ must be continuously differentiable for the approximation to be convergent. However, a closer inspection of the examples in the previous section shows that the exact solutions may not be continuously differentiable. Nevertheless, as it has been shown, the method converges and perform well. It is quite ubiquitous in literature to construct approximations for equations whose solutions are non–smooth.

Table 1: Error norms of Example 5.1 for different combinations of the values of N_x, Ny, N_t.

(N_x, N_y, N_t)	$\|\varepsilon\|_\infty$	condition number	CPU time(sec)
$(5, 5, 5)$	$1.7367e-04$	$1.0050e+03$	0.0304
$(7, 7, 5)$	$2.5291e-05$	$4.9732e+03$	0.0463
$(7, 7, 7)$	$2.0069e-05$	$5.6195e+03$	0.0688
$(10, 10, 7)$	$7.9423e-06$	$5.1577e+04$	0.2336
$(10, 10, 10)$	$3.3726e-06$	$5.4943e+04$	0.2863
$(15, 15, 10)$	$3.2808e-06$	$2.2740e+09$	1.8369

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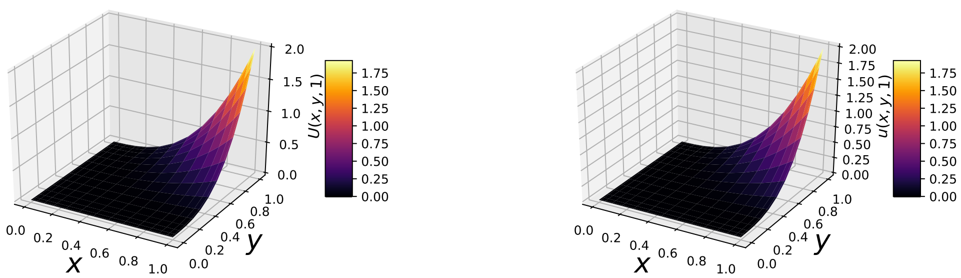
Table 2: Comparison of the numerical and exact solutions at some random points of x, y, t: Example 5.1 $N_x = 10, N_y = 10, N_t = 10$

(x, y, t)	$u(x, y, t)$	$U(x, y, t)$	$abs(u - U)$
$(0.50, 0.50, 0.10)$	0.0181	0.0181	$3.0437e-07$
$(0.50, 0.90, 0.10)$	0.1072	0.1072	$6.4668e-07$
$(0.90, 0.90, 0.10)$	0.5638	0.5638	$1.5162e-07$
$(0.50, 0.50, 0.50)$	0.0224	0.0224	$3.0558e-07$
$(0.90, 0.50, 0.50)$	0.1180	0.1180	$4.9251e-07$
$(0.50, 0.90, 0.50)$	0.1328	0.1328	$8.2658e-07$
$(0.90, 0.90, 0.50)$	0.6984	0.6984	$3.1169e-06$
$(0.90, 0.50, 0.90)$	0.1716	0.1716	$1.9308e-07$
$(0.50, 0.90, 0.90)$	0.1932	0.1932	$3.3049e-07$
$(0.90, 0.90, 0.90)$	1.0158	1.0158	$9.3025e-07$

Table 3: Absolute error norms of Example 5.2 for different values of N_x, N_y, N_t and $\alpha = 0.7, \beta = 1.5, \gamma = 1.6$

(N_x, N_y, N_t)	$\lVert\varepsilon\rVert_\infty$	condition number	CPU time(sec)
$(5, 5, 5)$	$1.4801e-04$	$1.5767e+03$	0.0248
$(7, 7, 5)$	$6.5902e-05$	$4.4287e+03$	0.0499
$(7, 7, 7)$	$6.2105e-05$	$4.7101e+03$	0.0713
$(10, 10, 7)$	$3.2852e-05$	$1.4321e+04$	0.2672
$(10, 10, 10)$	$3.2941e-05$	$1.4905e+04$	0.3124
$(15, 15, 10)$	$9.0358e-06$	$3.8865e+08$	1.7789



(a) Numerical solution (b) Exact solution

Figure 1: The numerical and exact solutions for Example 5.1 at $t = 1$.

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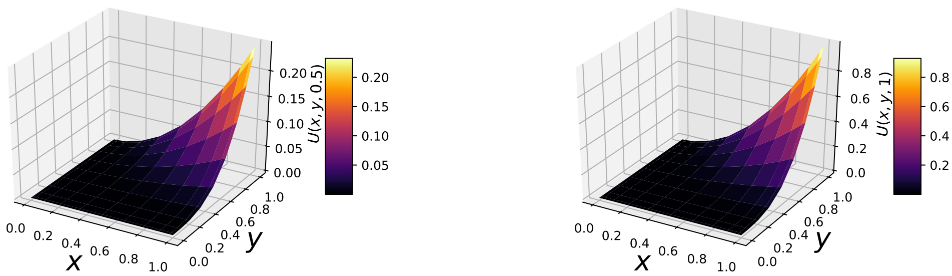
Table 4: Error norm values obtained when Example 5.2 is solved with $N_x = N_y = N_t = 10$ for different values of α, β, γ.

(α, β, γ)	$\|\varepsilon\|_\infty$	condition number	CPU time(sec)
$(0.1, 1.1, 1.2)$	$1.1259e-05$	$8.6909e+03$	0.2882
$(0.3, 1.3, 1.4)$	$2.7852e-05$	$9.1981e+03$	0.2972
$(0.5, 1.5, 1.6)$	$3.3015e-05$	$1.3738e+04$	0.3062
$(0.7, 1.7, 1.8)$	$3.1277e-05$	$2.1455e+04$	0.3032

Table 5: Absolute error norms obtained for Example 5.3 for different values of α, β, γ after the sixth iteration: $N_x = N_y = N_t = 10$.

(α, β, γ)	$\|\varepsilon\|_\infty$	condition number	CPU time(sec)
$(0.1, 1.1, 1.2)$	$3.4572e-04$	$1.1729e+04$	1.2437
$(0.3, 1.3, 1.4)$	$1.3217e-04$	$9.0201e+03$	1.2407
$(0.5, 1.5, 1.6)$	$1.2695e-04$	$1.2389e+04$	1.2377
$(0.7, 1.7, 1.8)$	$1.1264e-04$	$2.0124e+04$	1.2277



(a) $t = 1/2$ (b) $t = 1$

Figure 2: Numerical solutions of Example 5.2 for different t with $N_t = N_x = N_y = 10$, $\alpha = 1/2$ and $\beta = \gamma = 3/2$.

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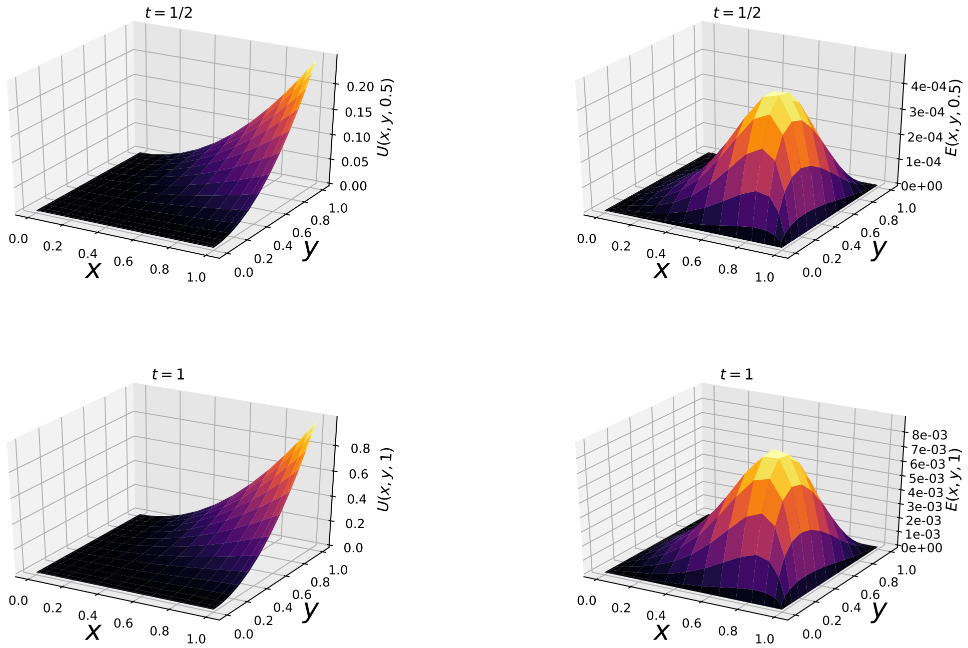

Figure 3: Surface plots of the numerical solution and absolute error for different values of t for Example 5.3 with $N_t = N_x = N_y = 15$ and $\alpha = 9/10$ and $\beta = \gamma = 1.9$.

7. Conclusion

In this study, we presented a numerical scheme for two–dimensional space–time fractional partial differential equations. The method is purely spectral and approximates the dependent variable and its fractional derivative (both spatial and temporal) using shifted Chebyshev polynomials and integrated using shifted Chebyshev–Gauss–Lobatto quadrature. In the case of nonlinear two–dimensional space–time FPDEs, we used quasilinearization to linearize the equation and then expanded the solution in terms of the shifted Chebyshev polynomials . The method was described and applied to some FPDEs of the initial–boundary value type. In the examples considered, we found that the method is accurate, reliable and efficient. The accuracy was confirmed through an analysis of the magnitude of the error norms. The accuracy of the method can be attributed to the fact that the approach is purely spectral, and spectral methods are non–local in nature, they generate an approximate solution over the entire grid points instead of using neighbouring points. The efficiency is evident in the computational time required to solve the system of algebraic equations resulting from the expansion in terms of shifted Chebyshev polynomials. In future, the method may be extended to a large time domain and irregular spatial domains.

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