CBE, Shanghai Business School, Shanghai, China


#### Abstract

In this note, we provide some effective treatments of a general linear model with adding-up restrictions via algebraic operations of given vectors and matrices in the model, including analytic expressions of the well-known ordinary least-squares estimator (OLSE) and the best linear unbiased estimator (BLUE) of the unknown parameters in the model.


Keywords: General linear model; adding-up restrictions; estimability; OLSE; BLUE
AMS Classifications: 62H12; 62J05.

## 1 Introduction

We begin with introducing the common matrix and vector notation and ready-made matrix manipulations that will be used in this note. Throughout, let $\mathbb{R}^{m \times n}$ stand for the collection of all $m \times n$ matrices with real numbers; $\mathbf{A}^{\prime}, r(\mathbf{A})$, and $\mathscr{R}(\mathbf{A})$ stand for the transpose, the rank, and the range (column space) of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, respectively; and let $\mathbf{I}_{m}$ denote the identity matrix of order $m$. The Moore-Penrose inverse of $\mathbf{A}$, denoted by $\mathbf{A}^{+}$, is defined to be the unique solution $\mathbf{G}$ satisfying the four matrix equations $\mathbf{A G A}=\mathbf{A}, \mathbf{G A G}=\mathbf{G},(\mathbf{A G})^{\prime}=\mathbf{A G}$, and $(\mathbf{G A})^{\prime}=\mathbf{G A}$. Further, let $\mathbf{P}_{\mathbf{A}}, \mathbf{E}_{\mathbf{A}}$, and $\mathbf{F}_{\mathbf{A}}$ stand for the three orthogonal projectors (symmetric idempotent matrices) $\mathbf{P}_{\mathbf{A}}=\mathbf{A} \mathbf{A}^{+}, \mathbf{E}_{\mathbf{A}}=\mathbf{A}^{\perp}=\mathbf{I}_{m}-\mathbf{A} \mathbf{A}^{+}$, and $\mathbf{F}_{\mathbf{A}}=\mathbf{I}_{n}-\mathbf{A}^{+} \mathbf{A}$. Two symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size are said to satisfy the inequality $\mathbf{A} \succcurlyeq \mathbf{B}$ in the Löwner partial ordering if $\mathbf{A}-\mathbf{B}$ is nonnegative definite.

Consider a general linear model

$$
\begin{equation*}
\mathscr{M}: \mathbf{y}=\mathbf{X} \beta+\varepsilon, \quad E(\varepsilon)=\mathbf{0}, \quad \operatorname{Cov}(\varepsilon)=\sigma^{2} \boldsymbol{\Sigma} \tag{1.1}
\end{equation*}
$$

where $\mathbf{y}$ is an $n \times 1$ observable random vector, $\mathbf{X}$ is an $n \times p$ known model matrix of arbitrary rank, $\boldsymbol{\beta}$ is a $p \times 1$ vector of fixed but known parameters, $\sigma^{2}$ is an arbitrary positive scaling factor, $\boldsymbol{\Sigma}$ is an $n \times n$ known nonnegative definite matrix of arbitrary rank.

In the parametric regression analysis, it is quite common to add certain restrictions on unknown parameters, such as, a system of linear matrix equations $\mathbf{B} \beta=\mathbf{c}$ in (1.1). In addition to imposing restrictions on unknown parameters in a given regression model, it is natural to take into account as well the situations of adding certain limitations and restrictions upon observable random variables in the model from theoretical and applied points of view. Under the model assumption in (1.1), we further assume that the observable random vector $\mathbf{y}$ satisfies a consistent linear matrix equation

$$
\begin{equation*}
\mathbf{A y}=\mathbf{b} \tag{1.2}
\end{equation*}
$$

where $\mathbf{A}$ is an $m \times n$ known matrix with $\operatorname{rank}(\mathbf{A})=k \leq \min \{m, n\}$ and $\mathbf{b}$ is an $m \times 1$ known vector with $\mathbf{b} \in \mathscr{R}(\mathbf{A})$. In this case, (1.1) together with (1.2) can be written as

$$
\begin{equation*}
\mathscr{N}: \mathbf{y}=\mathbf{X} \boldsymbol{\beta}+\varepsilon, \quad \mathbf{A} \mathbf{y}=\mathbf{b}, \quad E(\varepsilon)=\mathbf{0}, \quad \operatorname{Cov}(\varepsilon)=\sigma^{2} \boldsymbol{\Sigma} \tag{1.3}
\end{equation*}
$$

and the equation $\mathbf{A y}=\mathbf{b}$ is called adding-up restrictions to $\mathbf{y}$ in (1.1). This kind of restrictions were noticed and recognized in certain fields of applied statistics. For example, economists demonstrated certain appearance of adding-up restrictions and considered a number of estimation and inference problems concerning linear regression models with adding-up restrictions; see Haupt \& Oberhofer (2002, 2006); Ravikumar et al (2000).

Notice that the observable random vector $\mathbf{y}$ occurs in the two equations in (1.3), there are some alternative methods to approach estimation and inference problems of unknown parameters in the model. We next show how to convert (1.3) into common linear models with implicit and explicit restrictions to the unknown parametric vector $\beta$, respectively. Since $y$ is a random variable, the expectation and covariance matrix of both sides of the given equation $\mathbf{A y}-\mathbf{b}=\mathbf{0}$ with respect to $\mathbf{y}$ are given by

$$
\begin{equation*}
E(\mathbf{A} \mathbf{y}-\mathbf{b})=\mathbf{A} \mathbf{X} \boldsymbol{\beta}-\mathbf{b}=\mathbf{0} \text { and } \operatorname{Cov}(\mathbf{A} \mathbf{y}-\mathbf{b})=\sigma^{2} \mathbf{A} \boldsymbol{\Sigma} \mathbf{A}^{\prime}=\mathbf{0} \tag{1.4}
\end{equation*}
$$

E-mail address: yongge.tian@gmail.com

Note that the matrix $\boldsymbol{\Sigma}$ in (1.3) is positive semi-definite. Then it is easy to verify that the matrix equality $\mathbf{A} \boldsymbol{\Sigma} \mathbf{A}^{\prime}=\mathbf{0}$ is equivalent to $\boldsymbol{\Sigma}=\mathbf{F}_{\mathbf{A}} \boldsymbol{\Sigma} \mathbf{F}_{\mathbf{A}}$. Thus, the adding-up equation $\mathbf{A y}=\mathbf{b}$ in fact implies

$$
\begin{equation*}
\mathbf{A} \mathbf{X} \boldsymbol{\beta}=\mathbf{b} \text { and } \boldsymbol{\Sigma}=\mathbf{F}_{\mathbf{A}} \boldsymbol{\Sigma} \mathbf{F}_{\mathbf{A}} \tag{1.5}
\end{equation*}
$$

These two conditions arising from the adding-up equation enable us to merge the adding-up equation into (1.1) in certain feasible ways. For instance, substituting the first equation into the second equation in (1.3) and noting (1.5), we can equivalently rewrite (1.3) in the following implicitly restricted linear model

$$
\mathscr{N}_{a}:\left[\begin{array}{l}
\mathbf{y}  \tag{1.6}\\
\mathbf{b}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{X} \\
\mathbf{A X}
\end{array}\right] \beta+\left[\begin{array}{c}
\varepsilon \\
\mathbf{A} \varepsilon
\end{array}\right], \quad E\left[\begin{array}{c}
\varepsilon \\
\mathbf{A} \varepsilon
\end{array}\right]=\mathbf{0}, \quad \operatorname{Cov}\left[\begin{array}{c}
\varepsilon \\
\mathbf{A} \varepsilon
\end{array}\right]=\sigma^{2}\left[\begin{array}{cc}
\mathbf{F}_{\mathbf{A}} \boldsymbol{\Sigma} \mathbf{F}_{\mathbf{A}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{array}\right]
$$

Also replacing $\mathbf{A y}=\mathbf{b}$ with (1.5), we can rewrite (1.3) with probability 1 in the following explicitly restricted linear model

$$
\begin{equation*}
\mathscr{N}_{b}: \mathbf{y}=\mathbf{X} \beta+\varepsilon, \quad \mathbf{A} \mathbf{X} \beta=\mathbf{b}, \quad E(\varepsilon)=\mathbf{0}, \quad \operatorname{Cov}(\varepsilon)=\sigma^{2} \mathbf{F}_{\mathbf{A}} \boldsymbol{\Sigma} \mathbf{F}_{\mathbf{A}} \tag{1.7}
\end{equation*}
$$

The two alternative expressions in (1.6) and (1.7) make clear insights actionable into regression analysis of (1.3) via regular statistical inference. Recall as a classic topic in regression analysis that there has been some general discussion regarding estimation and inference problems of a linear model with implicit and explicit restrictions to unknown parameters in the model; see e.g., Tian et al (2007) and references therein. Thus, we are able to make statistical inference of (1.3) via the two alternative forms in (1.6) and (1.7) through use of common theory and methodology of dealing linear regression models under various assumptions.

In this note, we reconsider some basic estimation and inference problems regarding $\mathscr{N}$ in (1.3). The note is organized as follows. In Section 2, we introduce some basic formulas, results, and facts in matrix theory, as well as two groups of established results concerned with OLSEs and BLUEs under (1.1). In Sections 3 and 4, we present the description of the estimability of unknown parametric vector $\mathbf{K} \boldsymbol{\beta}$ under $\mathscr{N}_{a}$ and $\mathscr{N}_{b}$ in (1.6) and (1.7), and the derivations of analytical expressions of the OLSEs and BLUEs of $\mathbf{K} \boldsymbol{\beta}$ through $\mathscr{N}_{a}$ and $\mathscr{N}_{b}$, respectively. Section 5 gives some remarks regarding further research problems associated with general linear models with adding-up restrictions.

## 2 Some preliminaries

The theory of generalized inverses of matrices is a main source of the techniques that were brought into linear regression analysis in 1950s, and thus plays a key role for carrying out the estimation and inference in a wide variety of situations; see e.g., Bingham \& Krzanowski (2022); Puntanen et al (2011); Searle (1982). In order to simplify various matrix expressions involving generalized inverses of matrices, we need to use the following rank formulas.

Lemma 2.1 (Marsaglia \& Styan (1974)). Let $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{B} \in \mathbb{R}^{m \times k}$, and $\mathbf{C} \in \mathbb{R}^{l \times n}$. Then,
(a) $r[\mathbf{A}, \mathbf{B}]=r(\mathbf{A})+r\left(\mathbf{E}_{\mathbf{A}} \mathbf{B}\right)=r(\mathbf{B})+r\left(\mathbf{E}_{\mathbf{B}} \mathbf{A}\right)$.
(b) $r\left[\begin{array}{l}\mathbf{A} \\ \mathbf{C}\end{array}\right]=r(\mathbf{A})+r\left(\mathbf{C F}_{\mathbf{A}}\right)=r(\mathbf{C})+r\left(\mathbf{A F}_{\mathbf{C}}\right)$.

In particular,
(c) $r[\mathbf{A}, \mathbf{B}]=r(\mathbf{A}) \Leftrightarrow \mathbf{E}_{\mathbf{A}} \mathbf{B}=\mathbf{0} \Leftrightarrow \mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A})$.
(d) $r\left[\begin{array}{l}\mathbf{A} \\ \mathbf{C}\end{array}\right]=r(\mathbf{A}) \Leftrightarrow \mathbf{C F}_{\mathbf{A}}=\mathbf{0} \Leftrightarrow \mathscr{R}\left(\mathbf{C}^{\prime}\right) \subseteq \mathscr{R}\left(\mathbf{A}^{\prime}\right)$.

Lemma 2.2 (Penrose (1955)). Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times k}$, Then the linear matrix equation $\mathbf{A X}=\mathbf{B}$ is solvable for $\mathbf{X}$ if and only if $r[\mathbf{A}, \mathbf{B}]=r(\mathbf{A})$, or equivalently, $\mathbf{A A}^{+} \mathbf{B}=\mathbf{B}$. In this case, the general solution of the equation can be written in the following parametric form $\mathbf{X}=\mathbf{A}^{+} \mathbf{B}+\left(\mathbf{I}_{n}-\mathbf{A}^{+} \mathbf{A}\right) \mathbf{U}$, where $\mathbf{U} \in \mathbb{R}^{n \times k}$ is an arbitrary matrix.

We turn now to certain basic definitions, and best-known facts and results in linear model theory regarding estimability, OLSEs, and BLUEs of unknown parameters in (1.1); see e.g., Puntanen et al (2011); Tian (2013).

Definition 2.3. Let $\mathscr{M}$ be as given in (1.1) and let $\mathbf{K} \in \mathbb{R}^{k \times p}$ be given. The vector $\mathbf{K} \boldsymbol{\beta}$ of parametric functions is said to be estimable under $\mathscr{M}$ if there exists an $\mathbf{L} \in \mathbb{R}^{k \times n}$ such that $\mathrm{E}(\mathbf{L y}-\mathbf{K} \boldsymbol{\beta})=\mathbf{0}$ holds for all $\beta$ under $\mathscr{M}$.

Definition 2.4. Let $\mathscr{M}$ be as given in (1.1), and let $K \in \mathbb{R}^{k \times p}$ be given.
(a) The OLSE of the parametric vector $\boldsymbol{\beta}$ under (1.1), denoted by $\operatorname{OLSE}_{\mathscr{M}}(\boldsymbol{\beta})$, is defined to be

$$
\begin{equation*}
\widehat{\boldsymbol{\beta}}=\underset{\boldsymbol{\beta}}{\operatorname{argmin}}(\mathbf{y}-\mathbf{X} \boldsymbol{\beta})^{\prime}(\mathbf{y}-\mathbf{X} \boldsymbol{\beta}) . \tag{2.1}
\end{equation*}
$$

The OLSE of $\mathbf{K} \boldsymbol{\beta}$ under (1.1) is defined to be $\operatorname{OLSE}_{\mathscr{M}}(\mathbf{K} \boldsymbol{\beta})=\operatorname{KOLSE}_{\mathscr{M}}(\boldsymbol{\beta})$.
(b) The BLUE of the vector of parametric functions $\mathbf{K} \boldsymbol{\beta}$ under (1.1), denoted by $\operatorname{BLUE}_{\mathscr{M}}(\mathbf{K} \boldsymbol{\beta})$, is defined to be linear statistic $\mathbf{L y}$, where $\mathbf{L}$ is a matrix such that $\operatorname{Cov}(\mathbf{L y}-\mathbf{K} \boldsymbol{\beta})=\min$ in the Löwner partial ordering subject to $\mathrm{E}(\mathbf{L y}-\mathbf{K} \boldsymbol{\beta})=\mathbf{0}$.

The conventionality of OLSEs and BLUEs under linear regression models attracted statisticians' long attention in the historical development of parametric regression theory, and numerous formulas and facts regarding the OLSEs and BLUEs of $\boldsymbol{\beta}$ and $\mathbf{K} \boldsymbol{\beta}$ under (1.1) were established via various acceptable algebraic operations of the given vectors and matrices and their generalized inverses, including, e.g., the two groups of well-known results in the following two lemmas.

Lemma 2.5. Let $\mathscr{M}$ be as given in (1.1), and let $\mathbf{K} \in \mathbb{R}^{k \times p}$ be given. Then the general expression of OLSEs of $\boldsymbol{\beta}$ in $\mathscr{M}$ can be written as

$$
\begin{equation*}
\operatorname{OLSE}_{\mathscr{M}}(\beta)=\mathbf{X}^{+} \mathbf{y}+\mathbf{F}_{\mathbf{X}} \mathbf{v} \tag{2.2}
\end{equation*}
$$

where $\mathbf{v} \in \mathbb{R}^{p \times 1}$ is arbitrary; and the OLSE of $\mathbf{K} \boldsymbol{\beta}$ under $\mathscr{M}$ can be written as

$$
\begin{equation*}
\operatorname{OLSE}_{\mathscr{M}}(\mathbf{K} \boldsymbol{\beta})=\mathbf{K} \mathbf{X}^{+} \mathbf{y}+\mathbf{K} \mathbf{F}_{\mathbf{X}} \mathbf{v} \tag{2.3}
\end{equation*}
$$

Lemma 2.6. Let $\mathscr{M}$ be as given in (1.1), $\mathbf{K} \in \mathbb{R}^{k \times p}$, and suppose $\mathbf{K} \boldsymbol{\beta}$ is estimable under $\mathscr{M}$. Then the BLUE of $\mathbf{K} \boldsymbol{\beta}$ under $\mathscr{N}_{a}$ can be written as

$$
\begin{equation*}
\operatorname{BLUE}_{\mathscr{M}}(\mathbf{K} \boldsymbol{\beta})=\mathbf{P}_{\mathbf{K} ; \mathbf{X} ; \boldsymbol{\Sigma}} \mathbf{y} \tag{2.4}
\end{equation*}
$$

where $\mathbf{P}_{\mathbf{K} ; \mathbf{X} ; \mathbf{\Sigma}}$ is the solution of the matrix equation

$$
\begin{equation*}
\mathbf{G}\left[\mathbf{X}, \boldsymbol{\Sigma} \mathbf{X}^{\perp}\right]=[\mathbf{K}, \mathbf{0}] \tag{2.5}
\end{equation*}
$$

This equation is always solvable for $\mathbf{G}$, that is, $\mathscr{R}\left([\mathbf{K}, \mathbf{0}]^{\prime}\right) \subseteq \mathscr{R}\left(\left[\mathbf{X}, \boldsymbol{\Sigma} \mathbf{X}^{\perp}\right]^{\prime}\right)$. In this case, the general solution of (2.5) can be expressed as

$$
\begin{equation*}
\mathbf{P}_{\mathbf{K} ; \mathbf{X} ; \boldsymbol{\Sigma}}=[\mathbf{K}, \mathbf{0}]\left[\mathbf{X}, \boldsymbol{\Sigma} \mathbf{X}^{\perp}\right]^{+}+\mathbf{U E}_{\left[\mathbf{X}, \boldsymbol{\Sigma} \mathbf{X}^{\perp}\right]} \tag{2.6}
\end{equation*}
$$

where $\mathbf{U} \in \mathbb{R}^{k \times n}$ is arbitrary. Moreover, the following results hold.
(a) $r\left[\mathbf{X}, \boldsymbol{\Sigma} \mathbf{X}^{\perp}\right]=r[\mathbf{X}, \boldsymbol{\Sigma}]$ and $\mathscr{R}\left[\mathbf{X}, \boldsymbol{\Sigma} \mathbf{X}^{\perp}\right]=\mathscr{R}[\mathbf{X}, \boldsymbol{\Sigma}]$,
(b) The product $\mathbf{P}_{\mathbf{K} ; \mathbf{X} ; \boldsymbol{\Sigma}}$ can uniquely be written as $\mathbf{P}_{\mathbf{K} ; \mathbf{X} ; \boldsymbol{\Sigma}} \boldsymbol{\Sigma}=[\mathbf{K}, \mathbf{0}]\left[\mathbf{X}, \boldsymbol{\Sigma} \mathbf{X}^{\perp}\right]^{+} \boldsymbol{\Sigma}$.
(c) The expectation and covariance matrix of $\operatorname{BLUE}_{\mathscr{N}}(\mathbf{K} \boldsymbol{\beta})$ are given by

$$
E\left(\operatorname{BLUE}_{\mathscr{M}}(\mathbf{K} \boldsymbol{\beta})\right)=\mathbf{K} \boldsymbol{\beta} \quad \text { and } \quad \operatorname{Cov}\left(\operatorname{BLUE}_{\mathscr{M}}(\mathbf{K} \boldsymbol{\beta})\right)=[\mathbf{K}, \mathbf{0}]\left[\mathbf{X}, \boldsymbol{\Sigma} \mathbf{X}^{\perp}\right]^{+} \boldsymbol{\Sigma}\left([\mathbf{K}, \mathbf{0}]\left[\mathbf{X}, \boldsymbol{\Sigma} \mathbf{X}^{\perp}\right]^{+}\right)^{\prime} .
$$

## 3 Estimation results under $\mathscr{N}_{a}$

In what follows, we denote

$$
\widehat{\mathbf{y}}=\left[\begin{array}{l}
\mathbf{y} \\
\mathbf{b}
\end{array}\right], \quad \widehat{\mathbf{X}}=\left[\begin{array}{c}
\mathbf{X} \\
\mathbf{A X}
\end{array}\right], \quad \widehat{\boldsymbol{\Sigma}}=\left[\begin{array}{cc}
\mathbf{F}_{\mathbf{A}} \boldsymbol{\Sigma} \mathbf{F}_{\mathbf{A}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{array}\right]
$$

We first describe the consistency problems associated with $\mathscr{N}_{a}$ in (1.6). Note that $[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}][\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}]^{+}[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}]=$ $[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}]$ from the definition of the Moore-Penrose inverse. Hence, it turns out under the assumptions in (1.5) that

$$
\begin{aligned}
E\left([\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}][\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}]^{+} \widehat{\mathbf{y}}-\widehat{\mathbf{y}}\right) & =[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}][\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}]^{+} \widehat{\mathbf{X}} \boldsymbol{\beta}-\widehat{\mathbf{X}} \boldsymbol{\beta}=\mathbf{0}, \\
\operatorname{Cov}\left([\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}][\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}]^{+} \widehat{\mathbf{y}}-\widehat{\mathbf{y}}\right) & =\sigma^{2}\left([\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}][\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}]^{+}-\mathbf{I}\right) \widehat{\boldsymbol{\Sigma}}\left([\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}][\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}]^{+}-\mathbf{I}\right)^{\prime}=\mathbf{0} .
\end{aligned}
$$

These two equalities imply $[\widehat{\mathbf{X}}, \widehat{\mathbf{\Sigma}}][\widehat{\mathbf{X}}, \widehat{\mathbf{\Sigma}}]^{+} \widehat{\mathbf{y}}=\widehat{\mathbf{y}}$ holds with probability 1 , or equivalently,

$$
\begin{equation*}
\widehat{\mathbf{y}} \in \mathscr{R}[\widehat{\mathbf{X}}, \widehat{\Sigma}] \tag{3.1}
\end{equation*}
$$

holds with probability 1 . In view of this, we use the following definition.
Definition 3.1. The linear model in (1.6) is said to be consistent if (3.1) holds with probability 1.
We next introduce the definitions of the OLSEs and BLUEs of vectors of parametric functions, and then presents exact formulas for calculating BLUEs under (1.6) and (1.7).
Definition 3.2. Let $\mathscr{N}_{a}$ be as given in (1.6), and let $\mathbf{K} \in \mathbb{R}^{k \times p}$ be given. The vector $\mathbf{K} \boldsymbol{\beta}$ of parametric functions is said to be estimable under $\mathscr{N}_{a}$ if there exists an $\mathbf{L} \in \mathbb{R}^{k \times(n+m)}$ such that $\mathrm{E}(\mathbf{L} \widehat{\mathbf{y}}-\mathbf{K} \boldsymbol{\beta})=\mathbf{0}$ holds for all $\beta$ under $\mathscr{N}_{a}$.

Definition 3.3. Let $\mathscr{N}_{a}$ be as given in (1.6), and let $\mathbf{K} \in \mathbb{R}^{k \times p}$ be given.
(a) The OLSE of the parametric vector $\boldsymbol{\beta}$ under (1.6), denoted by $\operatorname{OLSE}_{\mathscr{N}_{a}}(\boldsymbol{\beta})$, is defined to be

$$
\begin{equation*}
\widehat{\boldsymbol{\beta}}=\underset{\boldsymbol{\beta}}{\operatorname{argmin}}(\widehat{\mathbf{y}}-\widehat{\mathbf{X}} \boldsymbol{\beta})^{\prime}(\widehat{\mathbf{y}}-\widehat{\mathbf{X}} \boldsymbol{\beta}) . \tag{3.2}
\end{equation*}
$$

The OLSE of $\mathbf{K} \boldsymbol{\beta}$ under (1.5) is defined to be $\operatorname{OLSE}_{\mathscr{N}_{a}}(\mathbf{K} \boldsymbol{\beta})=\mathbf{K O L S E}_{\mathscr{N}_{a}}(\boldsymbol{\beta})$.
(b) The BLUE of the vector of parametric functions $\mathbf{K} \boldsymbol{\beta}$ under $\mathscr{N}$, denoted by BLUE $\mathscr{N}_{a}(\mathbf{K} \boldsymbol{\beta})$, is defined to be linear statistic $\mathbf{L} \widehat{\mathbf{y}}$, where $\mathbf{L}$ is a matrix such that $\operatorname{Cov}(\mathbf{L} \widehat{\mathbf{y}}-\mathbf{K} \boldsymbol{\beta})=$ min in the Löwner partial ordering subject to $\mathrm{E}(\mathbf{L} \widehat{\mathbf{y}}-\mathbf{K} \boldsymbol{\beta})=\mathbf{0}$.
Applying the above definitions to $\mathscr{N}_{a}$ be as given in (1.6), we obtain the following results.
Theorem 3.4. Let $\mathscr{N}_{a}$ be as given in (1.6), and let $\mathbf{K} \in \mathbb{R}^{k \times p}$ be given. Then $\mathbf{K} \boldsymbol{\beta}$ is estimable under $\mathscr{N}_{a}$ $\Leftrightarrow \mathscr{R}\left(\mathbf{K}^{\prime}\right) \subseteq \mathscr{R}\left(\mathbf{X}^{\prime}\right)$. In particular, $\mathbf{X} \boldsymbol{\beta}$ is always estimable under $\mathscr{N}_{a}$.

Proof. It follows from $\left.\mathrm{E}(\mathbf{L} \widehat{\mathbf{y}}-\mathbf{K} \boldsymbol{\beta})=\mathbf{0} \Leftrightarrow \mathscr{R}\left(\mathbf{K}^{\prime}\right) \subseteq \mathscr{R}\left(\mathbf{X}^{\prime}\right)-\mathbf{K} \boldsymbol{\beta}\right)=\widehat{\mathbf{X}} \boldsymbol{\beta}-\mathbf{K} \boldsymbol{\beta}=\mathbf{0}$ for all $\boldsymbol{\beta} \Leftrightarrow \widehat{\mathbf{X}}=\mathbf{K}$ $\Leftrightarrow \mathscr{R}\left(\mathbf{K}^{\prime}\right) \subseteq \mathscr{R}\left(\widehat{\mathbf{X}}^{\prime}\right) \Leftrightarrow \mathscr{R}\left(\mathbf{K}^{\prime}\right) \subseteq \mathscr{R}\left(\mathbf{X}^{\prime}\right)$ by Lemma 2.2.

Applying Lemmas 2.5 and 2.6 to (1.6), we obtain the following two results.
Theorem 3.5. Let $\mathscr{N}_{a}$ be as given in (1.6) and suppose $\mathbf{K} \beta$ is estimable under $\mathscr{N}_{a}$. Then the OLSE of $\boldsymbol{\beta}$ under $\mathscr{N}_{a}$ can be written as

$$
\begin{equation*}
\operatorname{OLSE}_{\mathscr{N}_{a}}(\boldsymbol{\beta})=\widehat{\mathbf{X}}^{+} \widehat{\mathbf{y}}+\mathbf{F}_{\mathbf{X}} \mathbf{v} \tag{3.3}
\end{equation*}
$$

where $\mathbf{v} \in \mathbb{R}^{p \times 1}$ is arbitrary; and the OLSE of $\mathbf{K} \boldsymbol{\beta}$ under $\mathscr{N}_{a}$ can uniquely be written as

$$
\begin{equation*}
\operatorname{OLSE}_{\mathscr{N}_{a}}(\mathbf{K} \boldsymbol{\beta})=\mathbf{K} \widehat{\mathbf{X}}^{+} \widehat{\mathbf{y}} \tag{3.4}
\end{equation*}
$$

with

$$
\begin{equation*}
E\left(\operatorname{OLSE}_{\mathscr{N}_{a}}(\mathbf{K} \boldsymbol{\beta})\right)=\mathbf{K} \boldsymbol{\beta} \quad \text { and } \operatorname{Cov}\left(\operatorname{OLSE}_{\mathscr{N}_{a}}(\mathbf{K} \boldsymbol{\beta})\right)=\sigma^{2} \mathbf{K} \widehat{\mathbf{X}}^{+} \widehat{\boldsymbol{\Sigma}}\left(\mathbf{K} \widehat{\mathbf{X}}^{+}\right)^{\prime} \tag{3.5}
\end{equation*}
$$

Theorem 3.6. Let $\mathscr{N}_{a}$ be as given in (1.6) and suppose $\mathbf{K} \boldsymbol{\beta}$ is estimable under $\mathscr{N}_{a}$. Then the BLUE of $\mathbf{K} \boldsymbol{\beta}$ under $\mathscr{N}_{a}$ can be written as

$$
\begin{equation*}
\operatorname{BLUE}_{\mathscr{N}_{a}}(\mathbf{K} \boldsymbol{\beta})=\mathbf{P}_{\mathbf{K} ; \widehat{\mathbf{x}} ; \widehat{\boldsymbol{\Sigma}}} \widehat{\mathbf{y}}, \tag{3.6}
\end{equation*}
$$

where $\mathbf{P}_{\mathbf{K} ; \widehat{\mathbf{x}} ; \widehat{\boldsymbol{\Sigma}}}$ is the solution of the matrix equation

$$
\begin{equation*}
\mathbf{G}\left[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}} \widehat{\mathbf{X}}^{\perp}\right]=[\mathbf{K}, \mathbf{0}] \tag{3.7}
\end{equation*}
$$

This equation is always solvable for $\mathbf{G}$, that is, $\mathscr{R}\left([\mathbf{K}, \mathbf{0}]^{\prime}\right) \subseteq \mathscr{R}\left(\left[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}} \widehat{\mathbf{X}}^{\perp}\right]^{\prime}\right)$. In this case, the general solution of (3.7) can be expressed as

$$
\begin{equation*}
\mathbf{P}_{\mathbf{K} ; \widehat{\mathbf{x}} ; \widehat{\boldsymbol{\Sigma}}}=[\mathbf{K}, \mathbf{0}]\left[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}} \widehat{\mathbf{X}}^{\perp}\right]^{+}+\mathbf{U E}_{\left[\widehat{\mathbf{x}}, \widehat{\boldsymbol{\Sigma}} \widehat{\mathbf{x}}^{\perp}\right]} \tag{3.8}
\end{equation*}
$$

where $\mathbf{U} \in \mathbb{R}^{k \times(n+m)}$ is arbitrary. Moreover, the following results hold.
(a) $r\left[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}} \widehat{\mathbf{X}}^{\perp}\right]=r[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}]$ and $\mathscr{R}\left[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}} \widehat{\mathbf{X}}^{\perp}\right]=\mathscr{R}[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}]$,
(b) The product $\mathbf{P}_{\mathbf{K} ; \widehat{\mathbf{X}} ; \widehat{\boldsymbol{\Sigma}}}$ can uniquely be written as $\mathbf{P}_{\mathbf{K} ; \widehat{\mathbf{x}} ; \widehat{\boldsymbol{\Sigma}}} \widehat{\boldsymbol{\Sigma}}=[\mathbf{K}, \mathbf{0}]\left[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}}^{\widehat{\mathbf{X}}^{\perp}}\right]^{+} \widehat{\boldsymbol{\Sigma}}$.
(c) The expectation and covariance matrix of $\operatorname{BLUE}_{\mathscr{N}_{a}}(\mathbf{K} \boldsymbol{\beta})$ are given by

$$
E\left(\operatorname{BLUE}_{\mathscr{N}_{a}}(\mathbf{K} \boldsymbol{\beta})\right)=\mathbf{K} \boldsymbol{\beta} \text { and } \operatorname{Cov}\left(\operatorname{BLUE}_{\mathscr{N}_{a}}(\mathbf{K} \boldsymbol{\beta})\right)=[\mathbf{K}, \mathbf{0}]\left[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}} \widehat{\mathbf{X}}^{\perp}\right]^{+} \widehat{\boldsymbol{\Sigma}}\left([\mathbf{K}, \mathbf{0}]\left[\widehat{\mathbf{X}}, \widehat{\boldsymbol{\Sigma}} \widehat{\mathbf{X}}^{\perp}\right]^{+}\right)^{\prime}
$$

(d) $\mathrm{BLUE}_{\mathscr{N}_{a}}(\mathbf{K} \boldsymbol{\beta})$ is unique iff $\widehat{\mathbf{y}} \in \mathscr{R}[\widehat{\mathbf{X}}, \widehat{\mathbf{\Sigma}}]$ holds with probability 1.

## 4 Estimation results under $\mathscr{N}_{b}$

In what follows, we denote $\widetilde{\mathbf{A}}=\mathbf{A X}$ and $\widetilde{\boldsymbol{\Sigma}}=\mathbf{F}_{\mathbf{A}} \boldsymbol{\Sigma} \mathbf{F}_{\mathbf{A}}$. Note that $\mathbf{A X} \boldsymbol{\beta}=\mathbf{b}$ is solvable for $\beta$ iff $\mathbf{b} \in \mathscr{R}(\widetilde{\mathbf{A}})$. By Lemma 3.2, the general solution of $\boldsymbol{\beta}$ and the corresponding $\mathbf{K} \boldsymbol{\beta}$ can be written in the following parametric forms

$$
\begin{gather*}
\beta=\widetilde{\mathbf{A}}^{+} \mathbf{b}+\mathbf{F}_{\widetilde{\mathbf{A}}} \gamma,  \tag{4.1}\\
\mathbf{K} \beta=\mathbf{K} \widetilde{\mathbf{A}}^{+} \mathbf{b}+\mathbf{K F}_{\widetilde{\mathbf{A}}} \boldsymbol{\gamma}, \tag{4.2}
\end{gather*}
$$

where $\boldsymbol{\gamma} \in \mathbb{R}^{p \times 1}$ is arbitrary. Substituting (4.1) into (1.6) yields

$$
\begin{equation*}
\widetilde{\mathscr{N}_{b}}: \mathbf{z}=\mathbf{X F}_{\widetilde{\mathbf{A}}} \gamma+\boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon})=\mathbf{0}, \quad \operatorname{Cov}(\boldsymbol{\varepsilon})=\sigma^{2} \widetilde{\boldsymbol{\Sigma}} \tag{4.3}
\end{equation*}
$$

where $\mathbf{z}=\mathbf{y}-\mathbf{X} \widetilde{\mathbf{A}}^{+} \mathbf{b}$. This is a common linear model in form, thus the estimablility, OLSE, and BLUE of the vector of parametric functions $\mathbf{K F}_{\widetilde{\mathbf{A}}} \gamma$ can be obtained from various known results as follows.

Definition 4.1. Let $\mathscr{N}_{b}$ be as given in (1.7), and let $\mathbf{K} \in \mathbb{R}^{k \times p}$ be given. The vector $\mathbf{K} \boldsymbol{\beta}$ of parametric functions is said to be estimable under $\mathscr{N}_{b}$ if there exist $\mathbf{L} \in \mathbb{R}^{k \times n}$ and $\mathbf{c} \in \mathbb{R}^{k \times 1}$ such that $\mathrm{E}(\mathbf{L y}+\mathbf{c}-\mathbf{K} \boldsymbol{\beta})=$ 0 holds under $\mathscr{N}_{b}$.

Lemma 4.2. Let $\mathscr{N}_{b}$ be as given in (1.7), and let $\mathbf{K} \in \mathbb{R}^{k \times p}$ be given. Then $\mathbf{K} \boldsymbol{\beta}$ is estimable under $\mathscr{N}_{b}$ iff $\mathscr{R}\left(\mathbf{K}^{\prime}\right) \subseteq \mathscr{R}\left(\mathbf{X}^{\prime}\right)$.

Theorem 4.3. Let $\mathscr{N}_{b}$ be as given in (1.7), let $\mathbf{K} \in \mathbb{R}^{k \times p}$ be given, and suppose $\mathbf{K} \boldsymbol{\beta}$ is estimable under (1.7). Then the OLSE of $\beta$ under $\mathscr{N}_{b}$ can be written as

$$
\begin{equation*}
\operatorname{OLSE}_{\mathscr{N}_{b}}(\boldsymbol{\beta})=\left(\widetilde{\mathbf{A}}^{+}-\mathbf{F}_{\widetilde{\mathbf{A}}}\left(\mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}\right)^{+} \mathbf{X} \widetilde{\mathbf{A}}^{+}\right) \mathbf{b}+\mathbf{F}_{\widetilde{\mathbf{A}}}\left(\mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}\right)^{+} \mathbf{y}+\mathbf{F}_{\widetilde{\mathbf{A}}} \mathbf{F}_{\mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}} \mathbf{u} \tag{4.4}
\end{equation*}
$$

where $\mathbf{u} \in \mathbb{R}^{p \times 1}$ is arbitrary. The OLSE of $\mathbf{K} \boldsymbol{\beta}$ under $\mathscr{N}_{b}$ can uniquely be written as

$$
\begin{equation*}
\operatorname{OLSE}_{\mathcal{N}_{b}}(\mathbf{K} \boldsymbol{\beta})=\left(\mathbf{K} \widetilde{\mathbf{A}}^{+}-\mathbf{K} \mathbf{F}_{\widetilde{\mathbf{A}}}\left(\mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}\right)^{+} \mathbf{X} \widetilde{\mathbf{A}}^{+}\right) \mathbf{b}+\mathbf{K} \mathbf{F}_{\widetilde{\mathbf{A}}}\left(\mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}\right)^{+} \mathbf{y} \tag{4.5}
\end{equation*}
$$

In this case,

$$
E\left(\operatorname{OLSE}_{\mathscr{N}_{b}}(\mathbf{K} \boldsymbol{\beta})\right)=\mathbf{K} \boldsymbol{\beta} \text { and } \operatorname{Cov}\left(\operatorname{OLSE}_{\mathscr{N}_{b}}(\mathbf{K} \boldsymbol{\beta})\right)=\sigma^{2} \mathbf{K} \mathbf{F}_{\widetilde{\mathbf{A}}}\left(\mathbf{X F}_{\widetilde{\mathbf{A}}}\right)^{+} \widetilde{\boldsymbol{\Sigma}}\left(\mathbf{K F}_{\widetilde{\mathbf{A}}}\left(\mathbf{X F}_{\widetilde{\mathbf{A}}}\right)^{+}\right)^{\prime}
$$

Proof. According to Lemma 2.5, the OLSE of $\boldsymbol{\gamma}$ under (4.3) can be written as

$$
\widehat{\gamma}=\left(\mathbf{X F}_{\widetilde{\mathbf{A}}}\right)^{+} \mathbf{z}+\mathbf{F}_{\mathbf{X F}_{\tilde{\mathbf{A}}}} \mathbf{u},
$$

where $\mathbf{u} \in \mathbb{R}^{p \times 1}$ is arbitrary. Substituting this formula into (4.1) gives the OLSE of $\beta$ under (1.6):

$$
\begin{aligned}
\operatorname{OLSE}_{\mathscr{N}_{b}}(\beta) & =\widetilde{\mathbf{A}}^{+} \mathbf{b}+\mathbf{F}_{\widetilde{\mathbf{A}}}\left(\mathbf{X F}_{\widetilde{\mathbf{A}}}\right)^{+} \mathbf{z}+\mathbf{F}_{\widetilde{\mathbf{A}}} \mathbf{F}_{\mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}} \mathbf{u} \\
& =\left(\widetilde{\mathbf{A}}^{+}-\mathbf{F}_{\widetilde{\mathbf{A}}}\left(\mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}\right)^{+} \mathbf{X} \widetilde{\mathbf{A}}^{+}\right) \mathbf{b}+\mathbf{F}_{\widetilde{\mathbf{A}}}\left(\mathbf{X F}_{\widetilde{\mathbf{A}}}\right)^{+} \mathbf{y}+\mathbf{F}_{\widetilde{\mathbf{A}}} \mathbf{F}_{\mathbf{X F}_{\widetilde{\mathbf{A}}}} \mathbf{u}
\end{aligned}
$$

establishing (4.4) and (4.5).

Theorem 4.4. Let $\mathscr{N}_{b}$ be as given in (1.7) and suppose $\mathbf{K} \beta$ is estimable under (1.7). Then the BLUE of $\mathbf{K} \boldsymbol{\beta}$ under $\mathscr{N}_{b}$ can be written as

$$
\begin{equation*}
\operatorname{BLUE}_{\mathscr{N}_{b}}(\mathbf{K} \boldsymbol{\beta})=\left(\mathbf{I}-\mathbf{P}_{\mathbf{K F}_{\tilde{\mathbf{A}}} ; \mathbf{X} \mathbf{F}_{\tilde{\mathbf{A}}} ; \tilde{\mathbf{\Sigma}}}\right) \mathbf{X} \widetilde{\mathbf{A}}^{+} \mathbf{b}+\mathbf{P}_{\mathbf{K F}_{\tilde{\mathbf{A}}} ; \mathbf{X} \mathbf{F}_{\tilde{\mathbf{A}}} ; \tilde{\mathbf{\Sigma}}} \mathbf{y} \tag{4.6}
\end{equation*}
$$

where
and $\mathbf{U}_{1} \in \mathbb{R}^{k \times n}$ is arbitrary. Moreover,
(a) $r\left[\mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\Sigma} \mathbf{E}_{\mathbf{X F}_{\widetilde{\mathbf{A}}}}\right]=r\left[\mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\Sigma}\right]$ and $\mathscr{R}\left[\mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\Sigma} \mathbf{E}_{\mathbf{X F}_{\widetilde{\mathbf{A}}}}\right]=\mathscr{R}\left[\mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}, \widetilde{\Sigma}\right]$.

(c) The expectation and covariance matrix of $\mathrm{BLUE}_{\mathcal{N}_{b}}(\mathbf{K} \boldsymbol{\beta})$ are given by
(d) BLUE $_{\mathscr{N}_{b}}(\mathbf{K} \boldsymbol{\beta})$ is unique if and only if $\left[\begin{array}{l}\mathbf{y} \\ \mathbf{b}\end{array}\right] \in \mathscr{R}\left[\begin{array}{cc}\mathbf{X} & \widetilde{\boldsymbol{\Sigma}} \\ \widetilde{\mathbf{A}} & \mathbf{0}\end{array}\right]$ holds with probability 1.

Proof. According to Lemma 2.6, the BLUE of $\mathbf{K F}_{\widetilde{\mathbf{A}}} \boldsymbol{\gamma}$ under (4.3) is given by

$$
\operatorname{BLUE}_{\mathscr{N}_{b}}\left(\mathbf{K F}_{\widetilde{\mathbf{A}}} \boldsymbol{\gamma}\right)=\mathbf{P}_{\mathbf{K F}_{\widetilde{\mathbf{A}}} ; \mathbf{X} \mathbf{F}_{\tilde{\mathbf{A}}} ; \widetilde{\Sigma}^{\mathbf{Z}} . . . ~}
$$

Substituting this BLUE into the equality in (4.2) gives the BLUE of $\mathbf{K} \boldsymbol{\beta}$ under (1.7)

$$
\begin{equation*}
\operatorname{BLUE}_{\mathscr{N}_{b}}(\mathbf{K} \boldsymbol{\beta})=\mathbf{K} \widetilde{\mathbf{A}}^{+} \mathbf{b}+\mathrm{BLUE}_{\mathscr{N}_{b}}\left(\mathbf{K F}_{\widetilde{\mathbf{A}}} \boldsymbol{\gamma}\right)=\mathbf{K} \widetilde{\mathbf{A}}^{+} \mathbf{b}+\mathbf{P}_{\mathbf{K F}_{\widetilde{\mathbf{A}}} ; \mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}} ; \widetilde{\mathbf{\Sigma}}}\left(\mathbf{y}-\mathbf{X A}^{+} \mathbf{b}\right) \tag{4.8}
\end{equation*}
$$

as required for (4.6).
Result (a) follows from Lemma 2.6(a).
It can be seen from (4.7) that $\mathbf{P}_{\mathbf{K F}_{\tilde{\mathbf{A}}} ; \mathbf{X F}_{\tilde{\mathbf{A}}} ; \tilde{\boldsymbol{\Sigma}}}$ is unique if and only if $\mathbf{E}_{\left[\mathbf{X F}_{\tilde{\mathbf{A}}}, \widetilde{\mathbf{\Sigma}} \mathbf{E}_{\mathbf{X F _ { \tilde { \mathbf { A } } }}}\right]}=\mathbf{0}$, i.e., $r\left[\mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\boldsymbol{\Sigma}} \mathbf{E}_{\mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}}\right]=n$. Also from (a) and Lemma 2.1(b) that $r\left[\mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\boldsymbol{\Sigma}} \mathbf{E}_{\mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}}\right]=r\left[\mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\boldsymbol{\Sigma}}\right]=$ $r\left[\begin{array}{cc}\mathbf{X} & \widetilde{\boldsymbol{\Sigma}} \\ \widetilde{\mathbf{A}} & \mathbf{0}\end{array}\right]-r \widetilde{\mathbf{A}}$, so that $\operatorname{Result}$ (b) follows.

Result (c) follows from (4.6).
 i.e.,

$$
\begin{equation*}
r\left[\mathbf{y}-\mathbf{X} \widetilde{\mathbf{A}}^{+} \mathbf{b}, \mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\boldsymbol{\Sigma}} \mathbf{E}_{\mathbf{X F}_{\widetilde{\mathbf{A}}}}\right]=r\left[\mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\boldsymbol{\Sigma}}\right] \tag{4.9}
\end{equation*}
$$

holds with probability 1 by Lemma 2.1(c). It is easy to derive from Lemma 2.1(b) and elementary block matrix operations that

$$
\begin{aligned}
r\left[\mathbf{y}-\mathbf{X} \widetilde{\mathbf{A}}^{+} \mathbf{b}, \mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\boldsymbol{\Sigma}} \mathbf{E}_{\mathbf{X \mathbf { F } _ { \tilde { \mathbf { A } } }}}\right] & =r\left[\mathbf{y}-\mathbf{X} \widetilde{\mathbf{A}}^{+} \mathbf{b}, \mathbf{X} \mathbf{F}_{\widetilde{\mathbf{A}}}, \widetilde{\boldsymbol{\Sigma}}\right] \\
& =r\left[\begin{array}{ccc}
\mathbf{y}-\mathbf{X} \widetilde{\mathbf{A}}^{+} \mathbf{b} & \mathbf{X} & \widetilde{\boldsymbol{\Sigma}} \\
\mathbf{0} & \widetilde{\mathbf{A}} & \mathbf{0}
\end{array}\right]-r \widetilde{\mathbf{A}} \\
& =r\left[\begin{array}{ccc}
\mathbf{y} & \mathbf{X} & \widetilde{\boldsymbol{\Sigma}} \\
\mathbf{b} & \widetilde{\mathbf{A}} & \mathbf{0}
\end{array}\right]-r \widetilde{\mathbf{A}}, \\
r\left[\mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\boldsymbol{\Sigma}}\right] & =r\left[\mathbf{X F}_{\widetilde{\mathbf{A}}}, \widetilde{\boldsymbol{\Sigma}}\right]=r\left[\begin{array}{cc}
\mathbf{X} & \widetilde{\boldsymbol{\Sigma}} \\
\widetilde{\mathbf{A}} & \mathbf{0}
\end{array}\right]-r \widetilde{\mathbf{A}} .
\end{aligned}
$$

So that (4.9) is equivalent to $r\left[\begin{array}{ccc}\mathbf{y} & \mathbf{X} & \widetilde{\boldsymbol{\Sigma}} \\ \mathbf{b} & \widetilde{\mathbf{A}} & \mathbf{0}\end{array}\right]=r\left[\begin{array}{cc}\mathbf{X} & \widetilde{\boldsymbol{\Sigma}} \\ \widetilde{\mathbf{A}} & \mathbf{0}\end{array}\right]$, i.e., $\left[\begin{array}{c}\mathbf{y} \\ \mathbf{b}\end{array}\right] \in \mathscr{R}\left[\begin{array}{cc}\mathbf{X} & \widetilde{\boldsymbol{\Sigma}} \\ \widetilde{\mathbf{A}} & \mathbf{0}\end{array}\right]$ holds by Lemma 2.1(c).

## 5 Conclusions

The previous facts and results provide an available computation procedure to deal with general linear regression models with adding-up restrictions to observable random variables via a series of algebraic operations of the given vectors and matrices in the models. Note that the OLSEs and BLUEs are defined by different optimality criteria in mathematics and statistics. Hence their expressions and properties are not necessarily the same, and thus it is natural to seek possible connections between these estimation results. It is, in fact, a subject area in regression analysis is to characterize relationships between different estimation results, which has deep roots with strong statistical explanation and usefulness in the theory of linear statistical models and applications; see e.g., Markiewicz et al (2021); Tian (2010) and references therein for the background and study of this subject. Based on the analytic expressions of OLSEs and BLUEs obtained, we can consider more problems in the statistical inference of general linear regression models with adding-up restrictions. Especially, it is natural to propose the following five equalities between the OLSEs and BLUEs under the two models in (1.1) and (1.3):
(a) $\operatorname{OLSE}_{\mathscr{M}}(\mathbf{K} \boldsymbol{\beta})=\operatorname{OLSE}_{\mathscr{N}}(\mathbf{K} \boldsymbol{\beta})$,
(b) $\mathrm{OLSE}_{\mathscr{M}}(\mathbf{K} \boldsymbol{\beta})=\operatorname{BLUE}_{\mathscr{N}}(\mathbf{K} \boldsymbol{\beta})$,
(c) $\operatorname{BLUE}_{\mathscr{M}}(\mathbf{K} \boldsymbol{\beta})=\operatorname{OLSE}_{\mathscr{N}}(\mathbf{K} \boldsymbol{\beta})$,
(d) $\operatorname{BLUE}_{\mathscr{M}}(\mathbf{K} \boldsymbol{\beta})=\operatorname{BLUE}_{\mathscr{N}}(\mathbf{K} \boldsymbol{\beta})$.
(e) $\operatorname{OLSE}_{\mathscr{N}}(\mathbf{K} \boldsymbol{\beta})=\operatorname{BLUE}_{\mathscr{N}}(\mathbf{K} \boldsymbol{\beta})$.

This kind of equalities for different estimators have deep roots with strong statistical explanation and usefulness in the theory of linear statistical models and applications; see e.g., Markiewicz et al (2021); Tian (2010) and references therein for the background and study of this subject area. It is believed that this work will provide remarkable insights into intrinsic natures hidden behind the adding-up restrictions, so that the algebraic treatments presented in this note will prompt other similar studies regarding different kinds of regression models with adding-up restrictions to observable random variables under various assumptions.

## References

Bingham, H.H., Krzanowski, W.J. (2022). Linear algebra and multivariate analysis in statistics: development and interconnections in the twentieth century. British J. Hist. Math., DOI: 10.1080/26375451.2022.2045811.
Haupt, H., Oberhofer, W. (2002). Fully restricted linear regression: a pedagogical note. Economics Bull. 3, 1-7.
Haupt, H., Oberhofer, W. (2006). Generalized adding-up in systems of regression equations. Economics Lett. 92, 263-269.
Markiewicz, A., Puntanen, S., Styan, G.P.H. (2021). The legend of the equality of OLSE and BLUE: highlighted by C.R. Rao in 1967. In: B.C. Arnold, N. Balakrishnan, C.A. Coelho (eds), Methodology and Applications of Statistics, A Volume in Honor of C.R. Rao on the Occasion of his 100th Birthday, Springer, 2021, pp. 51-76.
Marsaglia, G., Styan, G.P.H. (1974). Equalities and inequalities for ranks of matrices. Linear and Multilinear Algebra 2, 269-292.
Penrose, R. (1955). A generalized inverse for matrices. Proc. Cambridge Philos. Soc. 51, 406-413.
Puntanen, S., Styan, G.P.H., Isotalo, J. (2011). Matrix Tricks for Linear Statistical Models: Our Personal Top Twenty. Springer, Berlin.
Ravikumar, B., Ray, S., Savin, N.E. (2000). Robust Wald tests in SUR systems with adding-up restrictions. Econometrica 68, 715-719.
Searle, S.R. (1982). Matrix Algebra Useful for Statistics. Wiley, New York.
Tian, Y, (2010). On equalities of estimations of parametric functions under a general linear model and its restricted models. Metrika 72, 313-330
Tian, Y. (2013). On properties of BLUEs under general linear regression models. J. Statist. Plann. Inference 143, 771-782.
Tian, Y., Beisiegel, M., Dagenais, E., Haines, C. (2007). On the natural restrictions in the singular Gauss-Markov model. Stat. Papers 49, 553-564.

